Mathematics for Computer Science
Exercise session 2: 11 September 2024

Silvio Capobianco

Last update: 11 September 2024

Problems for Section 1.8

Problem 1.17.

Prove that $\log_4 6$ is irrational.

Problem 1.18.

Prove by contradiction that $\sqrt{3} + \sqrt{2}$ is irrational.

Hint: $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$.

Problem 1.22.

A familiar proof that $\sqrt[3]{7^2}$ is irrational depends on the fact that a certain equation among those below is unsatisfiable by integers $a, b > 0$. (Note that more than one is unsatisfiable, but only one of them is relevant.) Indicate the equation that would appear in the proof, and explain why it is unsatisfiable. (Do *not* assume that $\sqrt[3]{7^2}$ is irrational.)

1. $a^2 = 7^2 + b^2$.
2. $a^3 = 7^2 + b^3$.
3. $a^2 = 7^2 b^2$.
4. $a^3 = 7^2 b^3$.
5. $a^3 = 7^3 b^3$.
6. $(ab)^3 = 7^2$.

Problems from Section 2.2

Problem 2.2 (with some small changes).

The *Fibonacci numbers* $F(0), F(1), F(2), \ldots$ are defined as follows:

$$F(n) = \begin{cases} 0 & \text{if } n = 0\,, \\ 1 & \text{if } n = 1\,, \\ F(n-1) + F(n-2) & \text{if } n > 1\,, \end{cases}$$

Exactly which sentence(s) in the following bogus proof contain logical errors? Explain.

Theorem (Bogus theorem)**.** *Every Fibonacci number is even.*

Bogus proof. Let all the variables n, m, k mentioned below take value in the nonnegative integers.

1. Let $EF(n)$ mean that $F(n)$ is even.
2. Let C be the set of counterexamples to the assertion that $EF(n)$ holds for all $n \in \mathbb{N}$, namely,
$$C ::= \{n \in \mathbb{N} \mid \textbf{not}(EF(n))\}\ .$$
3. Assume C is nonempty. By WOP, it has a minimum m.
4. Then $m > 0$, since $F(0) = 0$ is an even number.
5. Since m is a minimum counterexample, $F(k)$ is even for all $k < m$.
6. In particular, $F(m-1)$ and $F(m-2)$ are both even.
7. But $F(m) = F(m-1) + F(m-2)$, and the right-hand side is even.
8. That is, $EF(m)$ is true, and m is not a true counterexample.
9. Then C is empty, and $F(n)$ is even for all $n \in \mathbb{N}$.

Problem 2.4.

Use the *Well Ordering Principle* to prove that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

for all nonnegative integers n.

Problem 2.5

Use the Well Ordering Principle to prove that there is no solution over the positive integers to the equation:

$$4a^3 + 2b^3 = c^3 .$$

Problems for Section 2.4

Problem 2.21(a)-(c),(e).

Indicate which of the following sets of numbers have a minimum element and which are well ordered. For those that are not well ordered, give an example of a subset with no minimum element.

(a) The integers $\geq -\sqrt{2}$.

(b) The rational numbers $\geq \sqrt{2}$.

(c) The set of the rational numbers of the form $\frac{1}{n}$, where n is a positive integer.

(e) The set $\mathbb{F}$ of fractions of the form $\frac{n}{n+1}$ for $n \in \mathbb{N}$.

Problem 2.23.

Prove that a set R of real numbers is well ordered iff there is no infinite decreasing sequence of numbers in R. In other words: R is well ordered if and only if there is no set of numbers $r_i \in R$ such that

$$r_0 > r_1 > r_2 > \ldots \tag{2}$$

Hint: A set is well ordered if and only if all its subsets are well ordered. Also, if $m \in S$ is not the minimum of S, then there is some $x \in S$ such that $x < m$.

Problems for Section 3.1

Problem 3.2.

Your class has a textbook and a final exam. Let P, Q, and R be the following propositions:

- P ::= "You get an A on the final exam."
- Q ::= "You do every exercise in the book."
- R ::= "You get an A in the class."

Translate following assertions into propositional formulas using P, Q, R, and the propositional connectives **and**, **not**(), **implies**.

(a) You get an A in the class, but you do not do every exercise in the book.

(b) You get an A on the final exam, you do every exercise in the book, and you get an A in the class.

(c) To get an A in the class, it is necessary for you to get an A on the final.

(d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

Problem 3.5.

Sloppy Sam is trying to prove a certain proposition P. He defines two related propositions Q and R, and then proceeds to prove three implications:

$$P \textbf{ implies } Q, \quad Q \textbf{ implies } R, \quad R \textbf{ implies } P.$$

He then reasons as follows:

> If Q is true, then since I proved Q **implies** R, I can conclude that R is true. Now, since I proved R **implies** P, I can conclude that P is true. Similarly, if R is true, then P is true and so Q is true. Likewise, if P is true, then so are Q and R. So any way you look at it, all three of P, Q and R are true.

(a) Exhibit truth tables for

$$(P \textbf{ implies } Q) \textbf{ and } (Q \textbf{ implies } R) \textbf{ and } (R \textbf{ implies } P) \tag{3}$$

and for

$$P \textbf{ and } Q \textbf{ and } R. \tag{4}$$

Use these tables to find a truth assignment for P, Q, R so that (3) is **T** and (4) is **F**.

(b) You show these truth tables to Sloppy Sam and he says "OK, I'm wrong that P, Q and R all have to be true, but I still don't see the mistake in my reasoning. Can you help me understand my mistake?" How would you explain to Sammy where the flaw lies in his reasoning?

This page intentionally left blank.

This page too.

Solutions

Problem 1.17.

By contradiction, assume $\log_4 6 = m/n$ for suitable positive integers m, n. Then $6^n = 4^{n \log_4 6} = 4^m$, and as $n > 0$, the left-hand side is divisible by 3, which the right-hand side is not: contradiction.

Problem 1.19.

We follow the hint and perform the multiplication:

$$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 3 - 2 = 1\,.$$

This means that $\sqrt{3} - \sqrt{2}$ is the multiplicative inverse of $\sqrt{3} + \sqrt{2}$. By contradiction, assume $\sqrt{3} + \sqrt{2} = m/n$ is rational. Then $\sqrt{3} - \sqrt{2} = n/m$ is rational too, and so is their difference $2\sqrt{2}$. But then, so is $\sqrt{2}$: contradiction.

If, instead of the difference $2\sqrt{2}$, we consider the sum $2\sqrt{3}$, we reach a similar contradiction. Indeed, an argument similar to our proof of the irrationality of $\sqrt{2}$ leads us to the conclusion that $\sqrt{3}$ is irrational.

Problem 1.22.

1. $a^2 = 7^2 + b^2$ is *satisfiable* in the positive integers, so it won't help us prove that $\sqrt[3]{7^2}$ is irrational *because* a certain equation is *un*satisfiable in the positive integers! Specifically, $(7, 24, 25)$ is a *Pythagorean triple*, that is, a triple of positive integers (k, m, n) such that $k^2 + m^2 = n^2$. We could then put $a = 25$ and $b = 24$.

   And even if it had been unsatisfiable, it involves a sum instead of a ratio or a product, so it cannot help in our case.

2. $a^3 = 7^2 + b^3$ is unsatisfiable. However, it gives us no information about the existance of two positive integers a, b such that $7^2 = a^3/b^3$, so this is not the equation we are looking for.

   To see why the equality is unsatisfiable, observe that it is equivalent to $a^3 - b^3 = 7^2$. But $a^3 - b^3 = (a - b)(a^2 + ab + b^2) = 7 \cdot 7$ with a and b both positive integers is only possible if $a - b = 1$ and $a^2 + ab + b^2 = 7^2$, because if a and b are positive integers, then:

   $$a^2 + ab + b^2 > a^2 + b^2 \geq a + b > a - b\,.$$

Then $a = b + 1$ and:

$$a^2 + ab + b^2 = b^2 + 2b + 1 + b^2 + b + b^2 = 3b^2 + 3b + 1\,;$$

but $3b^2 + 3b + 1 = 7^2$ is equivalent to $b^2 + b - 16 = 0$, which has no integer solutions.

3. $a^2 = 7^2 b^2$ is satisfiable by choosing $b > 0$ arbitrarily and $a = 7b$.

4. $a^3 = 7^2 b^3$ is the equation we are looking for: it means that there are no two positive integers such that $\frac{a^3}{b^3} = 7^2$, that is, $\frac{a}{b} = \sqrt[3]{7^2}$. The reason why this equation has no positive integer solution is that the exponent of 7 in the prime factorization of the left-hand side of $a^3 = 7^2 b^3$ is a multiple of 3, but the one of the right-hand side isn't.

5. $a^3 = 7^3 b^3$ is satisfiable by choosing $b > 0$ arbitrarily and $a = 7b$.

6. $(ab)^3 = 7^2$ is unsatisfiable for the same reason why $a^3 = 7^2 b^3$ is unsatisfiable. However, a and b appear in a product instead of a ratio, so this is not the equation we are looking for.

Problem 2.2 (with some small changes).

The problem is with point 6. Until now, we only know that m is positive: it could well be 1. (It is so indeed, but that's not the point.) But if $m = 1$, then $m - 2 = -1$ is not a natural number; and we have only defined the Fibonacci numbers as a function on the naturals, not on all integers! For what we know, $F(-1)$ might not exist.[1]

Problem 2.4.

Let C be the set of counterexamples to (1), namely,

$$C ::= \left\{ n \in \mathbb{N} \;\middle|\; \sum_{k=0}^{n} k^2 \neq \frac{n(n+1)(2n+1)}{6} \right\}.$$

If C is nonempty, then it has a minimum element m. Such m must be positive, because for $n = 0$ both sides of (1) are zero. Since m is the minumum of

[1]As a curiosity: it is possible to define the Fibonacci numbers on negative integers, and it turns out that it must be $F(-1) = 1$. More in general, if n is a positive integer, then $F(-n) = (-1)^{n-1} F(n)$.

C, $m-1$, which is still a natural number as m is positive, *does* satisfy (1). Then:

$$\sum_{k=0}^{m-1} k^2 = \frac{(m-1)m(2(m-1)+1)}{6}\,.$$

But then,

$$\begin{aligned}
\sum_{k=0}^{m} k^2 &= \sum_{k=0}^{m-1} k^2 + m^2 \\
&= \frac{(m-1)m(2(m-1)+1)}{6} + m^2 \\
&= \frac{(m^2-m)(2m-1)+6m^2}{6} \\
&= \frac{2m^3-3m^2+m+6m^2}{6} \\
&= \frac{2m^3+3m^2+m}{6} \\
&= \frac{m(2m^2+3m+1)}{6} \\
&= \frac{m(m+1)(2m+1)}{6}\,:
\end{aligned}$$

that is, m *does* satisfy (1) after all. The contradiction stems from our hypothesis that C be nonempty: hence, C is empty, and (1) holds for every nonnegative integer m.

Problem 2.5

Let c_0 be the smallest positive integer such that positive integers a_0 and b_0 exist such that $4a_0^3+2b_0^3=c_0^3$. We observe that $c_0>1$, because the left-hand side must be even: indeed, c_0 itself must be even, so it must be $c_0=2c_1$ for some positive integer c_1. We then have:

$$4a_0^3+2b_0^3=8c_1^3\,,$$

which, dividing by 2, yields:

$$2a_0^3+b_0^3=4c_1^3\,.$$

Now, the right-hand side is even, so both summands on the left-hand side must be even. Then b_0 must be even too: let $b_0=2b_1$ for a suitable positive integer h. Again, we get, first, $2a_0^3+8b_1^3=4c_1^3$, then, dividing by 2,

$$a_0^3+4b_1^3=2c_1^3\,.$$

This time, with the same logic, $a_0 = 2a_1$ for a suitable positive integer a_1. Substituting and simplifying, we find:

$$4a_1^3 + 2b_1^3 = c_1^3 \,,$$

which is a solution over the positive integers with $c_1 < c_0$. (Note that we *must* have proved that $c_0 > 0$; otherwise, $c_1 = c_0/2$ could have been zero as well.) We have thus discovered that the smallest counterexample c_0 was not the smallest: then there was no c_0 in the first place, and the equation does not have a solution on the positive integers.

Problem 2.21(a)-(c),(e).

(a) This set is well ordered. We have seen during Lecture 2 that every subset of the set of integer numbers which has a lower bound is well ordered.

(b) This set is not well ordered. The simplest way to see that it is so, might be to show that this set has an infinite strictly decreasing sequence, then apply our solution to Problem 2.23 (see the self-evaluation exercises for Week 2). Well, 2 and 3 are rational numbers both larger than $\sqrt{2}$. Also, if a and b are rational numbers and $a < b$, then their *average* $\frac{a+b}{2}$ is also a rational number, is larger than a, and is smaller than b. We can then construct a strictly decreasing sequence of rational numbers larger than $\sqrt{2}$ by setting $x_0 = 3$ and $x_n = \frac{2 + x_{n-1}}{2}$ for every $n \geq 1$.

(c) This set is not well ordered: no point $x = 1/n$ can be the minimum, because $1/(n+1) < 1/n$ if n is a positive integer.

(e) This set is well ordered, for the following reason. Rewrite:

$$\frac{n}{n+1} = 1 - \frac{1}{n+1} \,.$$

This tells us that the larger n is, the larger $\frac{n}{n+1}$ is; and vice versa, the smaller n is, the smaller $\frac{n}{n+1}$ is. Then the smallest element of a nonempty subset S of $\mathbb{F}$ is $\frac{n_0}{n_0+1}$ where n_0 is the smallest element of:

$$T = \left\{ n \in \mathbb{N} \,\middle|\, \frac{n}{n+1} \in S \right\} \,.$$

Problem 2.23.

If a sequence such as in (2) exists, then the set of its terms does not have a minimum. However given an element, there will be another element (for example, the next one in the sequence) which is strictly smaller. In this case, R has a subset which is not well ordered, so it is not well ordered.

If R is not well ordered, take a nonempty subset S of R which has no minimum. Choose $r_0 \in S$: as r_0 is not the minimum of S, there exists $r_1 \in S$ which is strictly smaller than r_0. Similarly, as r_1 is not the minimum of S, there exists $r_2 \in S$ which is strictly smaller than r_1. Iterating the procedure, we obtain a sequence of elements of R such as in (2). More in detail:

1. We choose the starting element $r_0 \in S$ as we want.

2. For every $n \in \mathbb{N}$, after we have chosen $r_n \in S$, we choose $r_{n+1} \in S$ so that it is smaller than r_n. This is always possible, because S has no minimum, so in particular r_n is not the minimum of S.

Note that a set of numbers *can* have a minimum without being well ordered. For example, the set of nonnegative real numbers has 0 as its minimum and contains the infinite decreasing sequence $1 > \frac{1}{2} > \frac{1}{3} > \cdots$

Problem 3.2.

(a) In this case, R is verified, Q is not, and P is irrelevant: the assertion translates as R **and not**(Q).

(b) Here P, Q, and R are all verified, so this assertion can be rewritten as: P **and** Q **and** R. Recall that **and** is associative, so (P **and** Q) **and** R is equivalent to P **and** (Q **and** R).

(c) Here we have a clear implication, and a causal one too! What the assertion says, is that if you get an A in the class, it means that you had gotten an A in the final. The translation into mathematical language is then R **implies** P.

(d) In this case, P is true, Q is false, and R is true: the assertion translates as P **and not**(Q) **and** R.

At the end of the exercise, observe how, in mathematical language, "but" and "nevertheless" mean the same as "and". The differences in the *tone* of the three words are lost in translation.

Problem 3.5.

(a) We first construct the truth table for P **and** Q **and** R, as it is almost immediate:

P	Q	R	P **and** Q **and** R
T	**T**	**T**	**T**
T	**T**	**F**	**F**
T	**F**	**T**	**F**
T	**F**	**F**	**F**
F	**T**	**T**	**F**
F	**T**	**F**	**F**
F	**F**	**T**	**F**
F	**F**	**F**	**F**

For the formula

$$S ::= (P \textbf{ implies } Q) \textbf{ and } (Q \textbf{ implies } R) \textbf{ and } (R \textbf{ implies } P)$$

we proceed in two steps: first, we construct the truth values for each of the implications; then, we compute those for their conjunction.

P	Q	R	P **implies** Q	Q **implies** R	R **implies** P	S
T	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**	**T**	**F**
T	**F**	**T**	**F**	**T**	**T**	**F**
T	**F**	**F**	**F**	**T**	**T**	**F**
F	**T**	**T**	**T**	**T**	**F**	**F**
F	**T**	**F**	**T**	**F**	**T**	**F**
F	**F**	**T**	**T**	**T**	**F**	**F**
F	**F**	**F**	**T**	**T**	**T**	**T**

We then see that, if P, Q and R are all **F**, then (3) is **T** and (4) is **F**.

Alternatively, we can observe that if P, Q, and R are all false, then P **implies** Q, Q **implies** R, and R **implies** P are all true, so S is true, while **F and F and F** is clearly false.

(b) Sam is silently assuming that *some* of P, Q and R are true. But why should it be so? All he has proved is that they are *equivalent:* either they all all true, or all false. To check which case it is, he must find a proof or disproof of any of the three (no matter which) which does *not* depend on the others, but only on other things which he *knows*, not just assumes, to be true.