Mathematics for Computer Science
Exercise session 4, 27 September 2023

Silvio Capobianco

Last update: 27 September 2023

Problems from Section 4.1

Problem 4.3.

(a) Verify that the propositional formula $(P \textbf{ and } \overline{Q}) \textbf{ or } (P \textbf{ and } Q)$ is equivalent to P.

(b) Prove that
$$A = (A - B) \cup (A \cap B)$$
for all sets A, B, by showing
$$x \in A \textbf{ iff } x \in (A - B) \cup (A \cap B)$$
for all elements x using the equivalence of part (a) in a chain of **iff** 's.

Problem 4.5.

Prove De Morgan's Law for set equality
$$\overline{A \cap B} = \overline{A} \cup \overline{B} \tag{1}$$
by showing with a chain of **iff** 's that $x \in$ the left-hand side of (1) iff $x \in$ the right-hand side. You may assume the propositional version (3.14) of De Morgan's Law.

Problem 4.6.

Let A and B be sets.

(a) Prove that

$$\mathrm{pow}(A \cap B) = \mathrm{pow}(A) \cap \mathrm{pow}(B)\,.$$

(b) Prove that

$$\mathrm{pow}(A) \cup \mathrm{pow}(B) \subseteq \mathrm{pow}(A \cup B)\,,$$

with equality holding iff one of A or B is a subset of the other.

Problem 4.7.

The game of *Subset Take-Away* is played between two players with the following rules:

1. The initial position is a finite nonempty set.
2. Taking turns, the players take away subsets of the initial set.
3. It is not permitted to take away the entire initial set as the first move.
4. Once a subset has been taken away, no subset which contains it can be taken away anymore.

   In particular: no subset can be taken more than once.
5. A player who cannot take away a nonempty subset on his or her turn, loses the game.

Prove that the second player has a winning strategy in the following cases:

1. A has one element.
2. A has two elements.
3. A has three elements.

Note that the book claims that it is is unsolved whether the second player has a winning strategy for any initial position. However, this has been disproved in 2017: if the initial set has 7 elements, then the first player has a winning strategy. See `arXiv:1702.03018`, which however uses a terminology different from that of the textbook.

Problems for Section 4.2

Problem 4.14.

Prove that for any sets A, B, C and D, if the Cartesian products $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.

Problem 4.15.

(a) Give a simple example where the following result fails, and briefly explain why:

False Theorem. *For sets A, B, C and D, let*

$$\begin{aligned} L &::= (A \cup B) \times (C \cup D), \\ R &::= (A \times C) \cup (B \times D). \end{aligned}$$

Then $L = R$.

(b) Identify the mistake in the following proof of the False Theorem.

Bogus proof. Since L and R are both sets of pairs, it is sufficient to prove that $(x, y) \in L \longleftrightarrow (x, y) \in R$ for all x, y. The proof will be a chain of **iff** implications:

$$\begin{aligned} (x, y) \in R \quad &\textbf{iff} \quad (x, y) \in (A \times C) \cup (B \times D) \\ &\textbf{iff} \quad (x, y) \in A \times C \textbf{ or } (x, y) \in B \times D \\ &\textbf{iff} \quad (x \in A \textbf{ and } y \in C) \textbf{ or } (x \in B \textbf{ and } y \in D) \\ &\textbf{iff} \quad (x \in A \textbf{ or } x \in B) \textbf{ and } (y \in C \textbf{ or } y \in D) \\ &\textbf{iff} \quad (x \in A \cup B) \textbf{ and } (y \in C \cup D) \\ &\textbf{iff} \quad (x, y) \in L. \end{aligned}$$

□

Problem 4.16.

The *inverse* R^{-1} of a binary relation R from A to B is the relation from B to A defined by:

$$bR^{-1}A \textbf{ iff } aRb$$

In other words, you get the diagram for R^{-1} from R by "reversing the arrows" in the diagram describing R. Now many of the relational properties of R correspond to different properties of R^{-1} . For example, R is *total* iff R^{-1} is a *surjection.*

Fill in the remaining entries in this table:

R is	iff R^{-1} is
total	a surjection
a function	
a surjection	
an injection	
a bijection	

Hint: Explain what's going on in terms of "arrows" from A to B in the diagram for R.

Problem 4.17.

Describe a total injective function—that is, a relation which has the [$= 1$ **out**] and [≤ 1 **in**] properties—from $\mathbb{R} \to \mathbb{R}$ that is not a bijection.

Problem 4.22

(a) Prove that if A surj B and B surj C, then A surj C.

(b) Explain why A surj B if and only if B inj A.

(c) Conclude from (a) and (b) that if A inj B and B inj C, then A inj C.

(d) According to Definition 4.5.2, A inj B requires a total injective *relation.* Explain why A inj B iff there is a total injective *function* from A to B.

Problems for Section 4.5

Problem 4.39

Let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be a set of size n, and $B = \{b_0, b_1, \ldots, b_{m-1}\}$ a set of size m. Prove that $|A \times B| = mn$ by defining a simple bijection from $A \times B$ to the nonnegative integers from 0 to $mn - 1$.

This page intentionally left blank.

This page too.

Solutions

Problem 4.3.

(a) By using distributivity:

$$\begin{aligned}(P \textbf{ and } \overline{Q}) \textbf{ or } (P \textbf{ and } Q) \quad &\textbf{iff} \quad P \textbf{ and } (\overline{Q} \textbf{ or } Q) \\ &\textbf{iff} \quad P \textbf{ and } \mathbf{T} \\ &\textbf{iff} \quad P\,.\end{aligned}$$

(b) Let $P ::= x \in A$ and $Q ::= x \in B$: then,

$$\begin{aligned}x \in A \quad &\textbf{iff} \quad (x \in A \textbf{ and } \textbf{not}(x \in B)) \textbf{ or } (x \in A \textbf{ and } x \in B) \\ &\textbf{iff} \quad (x \in A - B) \textbf{ or } (x \in A \cap B) \\ &\textbf{iff} \quad x \in (A - B) \cup (A \cap B)\,.\end{aligned}$$

Problem 4.5.

Let D be the domain. As $\overline{A} = D - A$ and so on, our discussion should start with:

$$x \in \overline{A \cap B} \quad \textbf{iff} \quad x \in D \textbf{ and } \textbf{not}(x \in A \cap B)$$

However, as D is the domain, everything which is an element of some set is also an element of D, so *the part "$x \in D$" gives no new information.* We can then simplify the discussion a little bit by omitting "$x \in D$":

$$\begin{aligned}x \in \overline{A \cap B} \quad &\textbf{iff} \quad \textbf{not}(x \in A \cap B) \\ &\textbf{iff} \quad \textbf{not}(x \in A \textbf{ and } x \in B) \\ &\textbf{iff} \quad (\textbf{not}(x \in A) \textbf{ or } \textbf{not}(x \in B)) \\ &\textbf{iff} \quad \textbf{not}(x \in A) \textbf{ or } \textbf{not}(x \in B)) \\ &\textbf{iff} \quad x \in \overline{A} \textbf{ or } x \in \overline{B} \\ &\textbf{iff} \quad x \in (\overline{A} \cup \overline{B})\,.\end{aligned}$$

Note how the third and fourth **iff** 's exploit De Morgan's Law and distributivity, respectively.

Problem 4.6.

(a) Let S be a set. We must prove:

$$S \subseteq A \cap B \textbf{ iff } S \subseteq A \textbf{ and } S \subseteq B \tag{2}$$

We can better do this[1] by proving the equivalence as a double implication:

$$(S \subseteq A \cap B \longrightarrow S \subseteq A \wedge S \subseteq B) \wedge (S \subseteq A \wedge S \subseteq B \longrightarrow S \subseteq A \cap B) \tag{3}$$

Suppose the left-hand side of (2) holds. Let x be an arbitrary element: if $x \in S$, then $x \in A \cap B$, so both $x \in A$ and $x \in B$ by definition of intersection. We have thus proved that, if $S \subseteq A \cap B$, then $S \subseteq A$ **and** $S \subseteq B$: that is, $\mathrm{pow}(A \cap B) \subseteq \mathrm{pow}(A) \cap \mathrm{pow}(B)$.

Suppose now that the right-hand side of (2) holds. Recall that such intersection is never empty, because the empty set is a subset of every set, thus an element of every power set. Let $S \subseteq A$ and $S \subseteq B$: if S is empty, then $S \subseteq A \cap B$ for sure; if S is not empty, then every element of S belongs to both A and B, thus to $A \cap B$, and this shows $S \subseteq A \cap B$. We have thus proved that, if $S \subseteq A$ **and** $S \subseteq B$, then $S \subseteq A \cap B$: that is, $\mathrm{pow}(A) \cap \mathrm{pow}(B) \subseteq \mathrm{pow}(A \cap B)$. Double inclusion means equality.

(b) Let S be a set. We must prove:

$$S \subseteq A \textbf{ or } S \subseteq B \textbf{ implies } S \subseteq A \cup B \tag{4}$$

But this is easy to see: if $S \subseteq A$, then for every $x \in S$ it is also $x \in A$, thus $x \in A \cup B$ as well, and as x is arbitrary, $S \subseteq A \cup B$. Similarly, if $S \subseteq B$, then $S \subseteq A \cup B$.

Now, if for some $x \in A$ it is $x \notin B$, then any subset of $A \cup B$ which has x as an element cannot be a subset of B. It might still be, however, that every element of B is also an element of A: in this case, $A \cup B = A$ and $S \subseteq B$ **implies** $S \subseteq A$, so:

$$\mathrm{pow}(A) \cup \mathrm{pow}(B) = \mathrm{pow}(A) = \mathrm{pow}(A \cup B)\,.$$

That is: if $B \subseteq A$, then the inclusion at (b) is an equality. The same holds, with the roles of A and B swapped, if $A \subseteq B$. However, if neither $A \subseteq B$ nor $B \subseteq A$, then there exist $x \in A$ and $y \in A$ such that $x \notin B$ and $y \notin A$: in this case, $\{x, y\}$ is a subset of $A \cup B$, but not a subset of A nor of B, and the inclusion is strict.

[1] The classroom discussion depends on a passage which is not immediate to justify.

Problem 4.7.

1. If the initial set has only one element, then the first player can take no subset at all, and loses the game.

2. If the initial set $\{a, b\}$ has two elements, then the first player can only take away a subset with one element. Then the second player can take away *the other* subset with one element, and win the game.

3. If the initial set $\{a, b, c\}$ has three elements, then the first player can take away either a subset with one element, or a subset with two elements.

   In the first case, let's say that the first player takes away $\{a\}$. This eliminates the moves $\{a, b\}$ and $\{a, c\}$, so any other move must be a subset of $\{b, c\}$. If the second player chooses $\{b, c\}$, they reduce the original game to a game starting from a set with two elements, *where they are still the second player,* so they have a winning strategy.

   In the second case, let's say that the first player takes away $\{a, b\}$. If the second player takes away $\{c\}$, they make the moves $\{b, c\}$ and $\{a, c\}$ impossible, so any further move must be a subset of $\{a, b\}$. Again the second player has reduced the original game to a game starting from a set with two elements, for which they have a winning strategy.

Problem 4.14.

We prove the contrapositive: if $A \cap C \neq \emptyset$ and also $B \cap D \neq \emptyset$, then $(A \times B) \cap (C \times D) \neq \emptyset$.

Assume $A \cap C \neq \emptyset$ and also $B \cap D \neq \emptyset$. Take $x \in A \cap C$ and $y \in B \cap D$: then the pair (x, y) belongs to both $A \times B$ and $C \times D$.

Problem 4.15.

(a) If $A = \{1\}, B = \{2\}, C = \{3\}, D = \{4\}$, then $L = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ but $R = \{(1, 3), (2, 4)\}$.

Here is a more dramatic counterexample. If A and D are empty, but B and C are not, then L is not empty and R is. There is no such thing as a pair without a first element, or without a second element.

The problem here is that the choices for the left and right component are independent in L, but not in R. In L, if we have chosen the first component from A, then we still have the option of choosing the second component from either C or D: but in R, we are forced to choose it from C.

(b) The problem is in the fourth passage, which *looks like* an application of the distributivity law for disjunction, but is not: it is a swap of **and** with **or** and vice versa, which *is not* allowed by the rules of Boolean algebra, plus a swap between the right-hand side in the first parentheses and the left-hand side ofthe other parentheses. What we can conclude from

$$(x \in A \textbf{ and } y \in C) \textbf{ or } (x \in B \textbf{ and } y \in D)$$

is not $(x \in A \textbf{ or } x \in B) \textbf{ and } (y \in C \textbf{ or } y \in D)$, but, for example,

$$(x \in A \textbf{ or } (x \in B \textbf{ and } y \in D)) \textbf{ and } (y \in C \textbf{ or } (x \in B \textbf{ and } y \in D)),$$

which we can further split into:

$$\begin{array}{l} (x \in A \textbf{ or } x \in B) \\ \textbf{ and } (x \in A \textbf{ or } y \in D) \\ \textbf{ and } (y \in C \textbf{ or } x \in B) \\ \textbf{ and } (y \in C \textbf{ or } y \in D) \end{array}$$

This formula is not the one on the fourth line of the bogus proof! And while the first and fourth clause are harmless, the second and third are not: if $x \in A$ but $x \notin B$, then it must be $y \in C$; similarly, if $y \in C$ but $y \notin D$, then it must be $x \in A$. The set $(A \cup B) \times (C \cup D)$ has no such constraints.

Problem 4.16.

We preliminarily observe that $(R^{-1})^{-1} = R$, as:

$$a\,(R^{-1})^{-1}\,b \textbf{ iff } b\,R^{-1}\,a \textbf{ iff } a\,R\,b$$

Then we can immediately fill:

R is	iff R^{-1} is
total	a surjection
a function	
a surjection	total
an injection	
a bijection	

To fill the rest of the table, we observe that the relation diagram of R^{-1} is obtained from that of R by first reflecting it along a vertical line which cuts the arrows in half, then reversing the direction of each arrow. This leads to the following important observation:

R has the $\star n$ **in** property if and only if R^{-1} has the $\star n$ **out** property

where $\star$ is either $\leq$, $\geq$, or $=$. As the inverse of the inverse relation is the original relation, the observation above also holds with the roles of R and R^{-1} swapped.

We can now go on:

$$\begin{array}{lll} R \text{ is a function} & \textbf{iff} & R \text{ has the } \leq 1 \textbf{ out} \text{ property} \\ & \textbf{iff} & R^{-1} \text{ has the } \leq 1 \textbf{ in} \text{ property} \\ & \textbf{iff} & R^{-1} \text{ is an injection} \end{array}$$

To conclude, we recall that a bijection is a total function which is both injective and surjective: in this case, R^{-1} is a surjective and injective relation which is both a function and total, so it is also a bijection. And vice versa. The final table is thus:

R is	iff R^{-1} is
total	a surjection
a function	an injection
a surjection	total
an injection	a function
a bijection	a bijection

Problem 4.17.

The function $f(x) = 2^x$ works just fine:

1. it is total, because 2^x is defined for every $x \in \mathbb{R}$;

2. it is injective, because it is *strictly increasing*, that is, if $x < y$ then $2^x < 2^y$;

3. it is a function, because given $x \in \mathbb{R}$, the value 2^x is unique;

4. but it is not surjective, because there is no $x \in \mathbb{R}$ such that $2^x = -17$.

Another example which I, as instructor, like a lot is the *arctangent*, defined for $x \in \mathbb{R}$ as the unique $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan\theta = x$. Note that the *range* (image of the domain) of the arctangent function is bounded.

Problem 4.22

(a) Let $f : A \to B$ and $g : B \to C$ be surjective functions. A good candidate for a surjective function from A to C is $g \circ f$: let's put it to the test.

- $g \circ f$ is surjective. Let $z \in C$: as g is surjective, there exists $y \in B$ such that $g(y) = z$. But f is surjective too, so there exists $x \in A$ such that $f(x) = y$. Then $(g \circ f)(x) = g(f(x)) = g(y) = z$.
- $g \circ f$ is a function. Let $x \in A$: as f is a function, there exists at most one $y \in B$ such that $f(x) = y$. But g is a function, so there exists at most one $z \in C$ such that $g(y) = z$. Consequently, if there is any element w of C at all such that $(g \circ f)(x) = w$, it must be $w = z$.

(b) By our solution of Problem 4.16, $R : A \to B$ is a surjective function if and only if $R^{-1} : B \to A$ is a total injective relation.

(c) If A inj B and B inj C, then by point (b), C surj B and B surj A too; from point (a) then follows C surj A, from which again A inj B thanks to point (b).

(d) Let $R : A \to B$ be a total injective relation. Then R has the [≥ 1 **out**] and the [≤ 1 **in**] properties. We can then construct a relation which has the [$= 1$ **out**] and [≤ 1 **in**] properties—that is, a total injective function—by choosing, for every $a \in A$, exactly one $b \in B$ such that aRb, and defining $f(a)$ as *that* b. This relation has the [$= 1$ **out**] property by construction, and still has the [≤ 1 **in**] property, because we cannot add entering arrows by removing arrows.

Problem 4.39

We observe that we can order the elements of $A \times B$ into a *matrix* with n rows and m columns:

$$\begin{pmatrix} (a_0, b_0) & (a_0, b_1) & (a_0, b_2) & \dots & (a_0, b_{m-1}) \\ (a_1, b_0) & (a_1, b_1) & (a_1, b_2) & \dots & (a_1, b_{m-1}) \\ (a_2, b_0) & (a_2, b_1) & (a_2, b_2) & \dots & (a_2, b_{m-1}) \\ \vdots & & & & \vdots \\ (a_{n-1}, b_0) & (a_{n-1}, b_1) & (a_{n-1}, b_2) & \dots & (a_{n-1}, b_{m-1}) \end{pmatrix} \tag{5}$$

But we can do the same with the natural numbers smaller than mn:

$$\begin{pmatrix} 0 & 1 & 2 & \dots & m-1 \\ m & m+1 & m+2 & \dots & 2m-1 \\ 2m & 2m+1 & 2m+2 & \dots & 3m-1 \\ \vdots & & & & \vdots \\ (n-1)m & (n-1)m+1 & (n-1)m+2 & \dots & mn-1 \end{pmatrix} \tag{6}$$

(The last number is $(n-1)m + m - 1 = nm - 1$.) Each possible pair (a_i, b_j) appears exactly once in the matrix (5). Each possible natural number smaller than mn appears exactly once in the matrix (6). Then we can obtain a bijection between $A \times B$ and $\{0, \dots, mn-1\}$ by *superimposing the matrices*. If we do so, we notice that the pair (a_i, b_j) corresponds to the number $mi + j$: this is the bijection we were looking for.