Mathematics for Computer Science
Exercise session 4: 25 September 2024

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Problems from Section 4.1

Problem 4.3.

(a) Verify that the propositional formula (P **and** $\overline{Q}$) **or** (P **and** Q) is equivalent to P.

(b) Prove that

$$A = (A - B) \cup (A \cap B)$$

for all sets A, B, by showing

$$x \in A \textbf{ iff } x \in (A - B) \cup (A \cap B)$$

for all elements x using the equivalence of part (a) in a chain of **iff** 's.

Problem 4.5.

Prove De Morgan's Law for set equality

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \tag{1}$$

by showing with a chain of **iff** 's that $x \in$ the left-hand side of (1) iff $x \in$ the right-hand side. You may assume the propositional version (3.14) of De Morgan's Law.

Problem 4.6.

Let A and B be sets.

(a) Prove that

$$\mathrm{pow}(A \cap B) = \mathrm{pow}(A) \cap \mathrm{pow}(B)\,.$$

(b) Prove that

$$\mathrm{pow}(A) \cup \mathrm{pow}(B) \subseteq \mathrm{pow}(A \cup B)\,,$$

with equality holding iff one of A or B is a subset of the other.

Problem 4.7.

The game of *Subset Take-Away* is played between two players with the following rules:

1. The initial position is a finite nonempty set.
2. Taking turns, the players take away subsets of the initial set.
3. It is not permitted to take away the entire initial set as the first move.
4. Once a subset has been taken away, no subset which contains it can be taken away anymore.

   In particular: no subset can be taken more than once.
5. A player who cannot take away a nonempty subset on his or her turn, loses the game.

Prove that the second player has a winning strategy in the following cases:

(a) A has one element.

(b) A has two elements.

(c) A has three elements.

Note that the book claims that it is is unsolved whether the second player has a winning strategy for any initial position. However, this has been disproved in 2017: if the initial set has 7 elements, then the first player has a winning strategy. See `arXiv:1702.03018`, which however uses a terminology different from that of the textbook.

Problems for Section 4.2

Problem 4.14.

Prove that for any sets A, B, C and D, if the Cartesian products $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.

Problem 4.15.

(a) Give a simple example where the following result fails, and briefly explain why:

False Theorem. *For sets A, B, C and D, let*

$$\begin{aligned} L &::= (A \cup B) \times (C \cup D)\,, \\ R &::= (A \times C) \cup (B \times D)\,. \end{aligned}$$

Then $L = R$.

(b) Identify the mistake in the following proof of the False Theorem.

Bogus proof. Since L and R are both sets of pairs, it is sufficient to prove that $(x, y) \in L \longleftrightarrow (x, y) \in R$ for all x, y. The proof will be a chain of **iff** implications:

$$\begin{aligned} (x,y) \in R \quad &\textbf{iff} \quad (x,y) \in (A \times C) \cup (B \times D) \\ &\textbf{iff} \quad (x,y) \in A \times C \ \textbf{or}\ (x,y) \in B \times D \\ &\textbf{iff} \quad (x \in A \ \textbf{and}\ y \in C) \ \textbf{or}\ (x \in B \ \textbf{and}\ y \in D) \\ &\textbf{iff} \quad (x \in A \ \textbf{or}\ x \in B) \ \textbf{and}\ (y \in C \ \textbf{or}\ y \in D) \\ &\textbf{iff} \quad (x \in A \cup B) \ \textbf{and}\ (y \in C \cup D) \\ &\textbf{iff} \quad (x,y) \in L\,. \end{aligned}$$

□

Problem 4.17.

Describe a total injective function—that is, a relation which has the [$= 1$ **out**] and [≤ 1 **in**] properties—from $\mathbb{R} \to \mathbb{R}$ that is not a bijection.

Problem 4.29(a)

Consider a basic Web search engine, which stores information on Web pages and processes queries to find pages satisfying conditions provided by users. At a high level, we can formalize the key information as:

- A set P of *pages* that the search engine knows about.
- A binary relation L (for *link*) over pages, defined such that $p_1 L p_2$ if and only if p_1 links to p_2.

- A set E of *endorsers*, people who have recorded their opinions about which pages are high-quality.
- A binary relation R (for *recommends*) between endorsers and pages, such that eRp iff person e has recommended page p.
- A set W of *words* that may appear on pages.
- A binary relation M (for *mentions*) between pages and words, where pMw iff word w appears on page p.

Then, for example, if the word "logic" belongs to W, then the set of pages in P where the word "logic" appears is:

$$\{p \in P \mid p\, M \text{ ``logic''}\} = M^{-1}(\text{``logic''})\,.$$

Use the specification above to express the set of pages that contain the word "logic", but not the word "predicate".

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Solutions

Problem 4.3.

(a) We can use either truth tables, or Boolean algebra, or proof by cases.

- With truth tables:

P	Q	(P **and**	**not**(Q))	**or**	(P **and** Q)
T	**T**	**F**	**F**	**T**	**T**
T	**F**	**T**	**T**	**T**	**F**
F	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**T**	**F**	**F**

- With Boolean algebra:

$$\begin{aligned}(P \textbf{ and } \overline{Q}) \textbf{ or } (P \textbf{ and } Q) \quad & \textbf{iff} \quad P \textbf{ and } (\overline{Q} \textbf{ or } Q) \\ & \textbf{iff} \quad P \textbf{ and } \mathbf{T} \\ & \textbf{iff} \quad P\,.\end{aligned}$$

- By cases:
  (a) If P is false, then P **and not**(Q) and P **and** Q are both false, so (P **and not**(Q)) **or** (P **and** Q) is false.
  (b) If P is true and Q is false, then P **and not**(Q) is true, so (P **and not**(Q)) **or** (P **and** Q) is true.
  (c) If P is true and Q is true, then P **and** Q is true, so (P **and not**(Q)) **or** (P **and** Q) is true.

(b) Let $P ::= x \in A$ and $Q ::= x \in B$. Then:

$$\begin{aligned}x \in A \quad & \textbf{iff} \quad (x \in A \textbf{ and not}(x \in B)) \textbf{ or } (x \in A \textbf{ and } x \in B) \\ & \textbf{iff} \quad (x \in A - B) \textbf{ or } (x \in A \cap B) \\ & \textbf{iff} \quad x \in (A - B) \cup (A \cap B)\,.\end{aligned}$$

Problem 4.5.

Let D be the domain. As $\overline{A} = D - A$ and so on, our discussion should start with:

$$x \in \overline{A \cap B} \quad \textbf{iff} \quad x \in D \textbf{ and not}(x \in A \cap B)$$

However, as D is the domain, everything which is an element of some set is also an element of D, so *the part "$x \in D$" gives no new information.* We can then simplify the discussion a little bit by omitting "$x \in D$":

$$
\begin{aligned}
x \in \overline{A \cap B} \quad & \textbf{iff} \quad \textbf{not}(x \in A \cap B) \\
& \textbf{iff} \quad \textbf{not}(x \in A \textbf{ and } x \in B) \\
& \textbf{iff} \quad (\textbf{not}(x \in A) \textbf{ or } \textbf{not}(x \in B)) \\
& \textbf{iff} \quad \textbf{not}(x \in A) \textbf{ or } \textbf{not}(x \in B)) \\
& \textbf{iff} \quad x \in \overline{A} \textbf{ or } x \in \overline{B} \\
& \textbf{iff} \quad x \in (\overline{A} \cup \overline{B})\,.
\end{aligned}
$$

Note how the third and fourth **iff** 's exploit De Morgan's Law and distributivity, respectively.

Problem 4.6.

(a) Let S be a set. We must prove:

$$S \subseteq A \cap B \textbf{ iff } S \subseteq A \textbf{ and } S \subseteq B \tag{2}$$

We can do it through the following sequence of equivalences:

$$
\begin{aligned}
S \subseteq A \quad & \longleftrightarrow \quad \forall x\,.\,(x \in S \longrightarrow x \in A \cap B) \\
& \longleftrightarrow \quad \forall x\,.\,(x \in S \longrightarrow (x \in A \wedge x \in B)) \\
& \longleftrightarrow \quad \forall x\,.\,((x \in S \longrightarrow x \in A) \wedge (x \in S \longrightarrow x \in B)) \\
& \longleftrightarrow \quad (\forall x\,.\,(x \in S \longrightarrow x \in A)) \wedge (\forall x\,.\,(x \in S \longrightarrow x \in B)) \\
& \longleftrightarrow \quad S \subseteq A \wedge S \subseteq B
\end{aligned}
$$

This chain of implications, however, has two weak links: the third equivalence, and the fourth one. Of these, one can be proved easily by using the equivalence $(P \longrightarrow Q) \longleftrightarrow (\overline{P} \vee Q)$ and the distributive law. However, we have seen in Exercise session 3 that, in general, we cannot transform a predicate formula of the form $\forall x\,.\,P(x) \diamond Q(x)$, where $\diamond$ is a logical connective, into one of the form $(\forall x\,.\,P(x)) \diamond (\forall x\,.\,Q(x))$. This, however, works for conjunction, in the following way. First, we observe that $\forall x\,.\,(P(x) \wedge Q(x))$ and $(\forall x\,.\,P(x)) \wedge (\forall x\,.\,Q(x))$ are equivalent if and only if their negations $\exists x\,.\,(\overline{P(x)} \vee \overline{Q(x)})$ and $(\exists x\,.\,\overline{P(x)}) \vee (\exists x\,.\,\overline{Q(x)})$ imply each other. (Exercise: prove that these are, indeed, the negations of the original formulas.) We will prove that this is the case.

1. Assume that $\exists x \, . \, (\overline{P(x)} \vee \overline{Q(x)})$ is true. Choose x_0 such that $\overline{P(x_0)} \vee \overline{Q(x_0)}$ is true. If $P(x_0)$ is false, then $\exists x \, . \, \overline{P(x)}$ is true, and so is $(\exists x \, . \, \overline{P(x)}) \vee (\exists x \, . \, \overline{Q(x)})$; if $Q(x_0)$ is false, then $\exists x \, . \, \overline{Q(x)}$ is true, and so is $(\exists x \, . \, \overline{P(x)}) \vee (\exists x \, . \, \overline{Q(x)})$.
2. Assume now that $(\exists x \, . \, \overline{P(x)}) \vee (\exists x \, . \, \overline{Q(x)})$ is true. If $\exists x \, . \, \overline{P(x)}$ is true, choose x_0 so that $P(x_0)$ is false: then $\overline{P(x_0)} \vee \overline{Q(x_0)}$ is true, so $\exists x \, . \, (\overline{P(x)} \vee \overline{Q(x)})$ is true; if $\exists x \, . \, \overline{Q(x)}$ is true, choose x_0 so that $Q(x_0)$ is false: then $\overline{P(x_0)} \vee \overline{Q(x_0)}$ is true, so $\exists x \, . \, (\overline{P(x)} \vee \overline{Q(x)})$ is true.

We could also prove the original equivalence as a double implication:

$$(S \subseteq A \cap B \longrightarrow S \subseteq A \wedge S \subseteq B) \wedge (S \subseteq A \wedge S \subseteq B \longrightarrow S \subseteq A \cap B) \quad (3)$$

Remember the convention on precedence between connectives, so that the above means:

$$((S \subseteq A \cap B) \longrightarrow (S \subseteq A \wedge S \subseteq B)) \wedge ((S \subseteq A \wedge S \subseteq B) \longrightarrow (S \subseteq A \cap B))$$

Suppose the left-hand side of (2) holds. Let x be an arbitrary object. If $x \in S$, then $x \in A \cap B$, so both $x \in A$ and $x \in B$ by definition of intersection. We have thus proved that, if $S \subseteq A \cap B$, then $S \subseteq A$ **and** $S \subseteq B$: that is, $\mathrm{pow}(A \cap B) \subseteq \mathrm{pow}(A) \cap \mathrm{pow}(B)$.

Suppose now that the right-hand side of (2) holds. Recall that such intersection is never empty, because the empty set is a subset of every set, thus an element of every power set. Let $S \subseteq A$ and $S \subseteq B$. If S is empty, then $S \subseteq A \cap B$ for sure; if S is not empty, then every element of S is an element of both A and B, thus of $A \cap B$, and this shows $S \subseteq A \cap B$. We have thus proved that, if $S \subseteq A$ **and** $S \subseteq B$, then $S \subseteq A \cap B$: that is, $\mathrm{pow}(A) \cap \mathrm{pow}(B) \subseteq \mathrm{pow}(A \cap B)$. Double inclusion means equality.

(b) Let S be a set. We must prove:

$$S \subseteq A \textbf{ or } S \subseteq B \textbf{ implies } S \subseteq A \cup B \quad (4)$$

But this is easy to see: if $S \subseteq A$, then for every $x \in S$ it is also $x \in A$, thus $x \in A \cup B$ as well, and as x is arbitrary, $S \subseteq A \cup B$. Similarly, if $S \subseteq B$, then $S \subseteq A \cup B$.

Now, if for some $x \in A$ it is $x \notin B$, then any subset of $A \cup B$ which has x as an element cannot be a subset of B. It might still be, however, that every element of B is also an element of A: in this case, $A \cup B = A$ and $S \subseteq B$ **implies** $S \subseteq A$, so:

$$\mathrm{pow}(A) \cup \mathrm{pow}(B) = \mathrm{pow}(A) = \mathrm{pow}(A \cup B) \, .$$

That is: if $B \subseteq A$, then the inclusion at (b) is an equality. The same holds, with the roles of A and B swapped, if $A \subseteq B$. However, if neither $A \subseteq B$ nor $B \subseteq A$, then there exist $x \in A$ and $y \in A$ such that $x \notin B$ and $y \notin A$: in this case, $\{x, y\}$ is a subset of $A \cup B$, but not a subset of A nor of B, and the inclusion is strict.

Problem 4.7.

(a) If the initial set has only one element, then the first player can take no subset at all, and loses the game.

(b) If the initial set $\{a, b\}$ has two elements, then the first player can only take away a subset with one element. Then the second player can take away *the other* subset with one element, and win the game.

(c) If the initial set $\{a, b, c\}$ has three elements, then the first player can take away either a subset with one element, or a subset with two elements.

In the first case, let's say that the first player takes away $\{a\}$. This eliminates the moves $\{a, b\}$ and $\{a, c\}$, so any other move must be a subset of $\{b, c\}$. If the second player chooses $\{b, c\}$, they reduce the original game to a game starting from a set with two elements, *where they are still the second player,* so they have a winning strategy.

In the second case, let's say that the first player takes away $\{a, b\}$. If the second player takes away $\{c\}$, they make the moves $\{b, c\}$ and $\{a, c\}$ impossible, so any further move must be a subset of $\{a, b\}$. Again the second player has reduced the original game to a game starting from a set with two elements, for which they have a winning strategy.

Problem 4.14.

We prove the contrapositive: if $A \cap C \neq \emptyset$ and also $B \cap D \neq \emptyset$, then $(A \times B) \cap (C \times D) \neq \emptyset$.

Assume $A \cap C \neq \emptyset$ and also $B \cap D \neq \emptyset$. Take $x \in A \cap C$ and $y \in B \cap D$. Then the pair (x, y) is an element of both $A \times B$ and $C \times D$.

Problem 4.15.

(a) Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$.

Then $L = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ but $R = \{(1, 3), (2, 4)\}$.

Here is a more dramatic counterexample. If A and D are empty, but B and C are not, then L is not empty, but R is. There is no such thing as a pair without a first element, or without a second element.

The problem here is that the choices for the left and right component are *independent* in L, but not in R. In L, if we have chosen the first component from A, then we still have the option of choosing the second component from either C or D; but in R, we are forced to choose it from C.

(b) The problem is in the fourth passage, which *looks like* an application of the distributivity law for disjunction, but is not: it is a swap of **and** with **or** and vice versa, which is *not* allowed by the rules of Boolean algebra, together with a swap between the right-hand side in the first parentheses and the left-hand side ofthe other parentheses. What we can conclude from

$$(x \in A \textbf{ and } y \in C) \textbf{ or } (x \in B \textbf{ and } y \in D)$$

is not $(x \in A \textbf{ or } x \in B) \textbf{ and } (y \in C \textbf{ or } y \in D)$, but, for example,

$$(x \in A \textbf{ or } (x \in B \textbf{ and } y \in D)) \textbf{ and } (y \in C \textbf{ or } (x \in B \textbf{ and } y \in D)),$$

which we can further split into:

$$\begin{array}{l} (x \in A \textbf{ or } x \in B) \\ \textbf{ and } (x \in A \textbf{ or } y \in D) \\ \textbf{ and } (y \in C \textbf{ or } x \in B) \\ \textbf{ and } (y \in C \textbf{ or } y \in D) \end{array}$$

This formula is not the one on the fourth line of the bogus proof! And while the first and fourth clause are harmless, the second and third are not: if $x \in A$ but $x \notin B$, then it must be $y \in C$; similarly, if $y \in C$ but $y \notin D$, then it must be $x \in A$. The set $(A \cup B) \times (C \cup D)$ has no such constraints.

Problem 4.17.

The function $f(x) = 2^x$ works just fine:

1. It is total, because 2^x is defined for every $x \in \mathbb{R}$.
2. It is injective, because it is *strictly increasing*, that is, if $x < y$ then $2^x < 2^y$.
3. It is a function, because given $x \in \mathbb{R}$, the value 2^x is unique.
4. It is not surjective, because there is no $x \in \mathbb{R}$ such that $2^x = -17$.

Problem 4.29(a).

The text of the exercise explains that the set of pages which contain the word "logic" is M^{-1} ("logic"). For the same reason, the set of pages which contain the word "predicate" is M^{-1} ("predicate"). Then the set of pages which contain the word "logic" but not the word "predicate" is the difference set M^{-1} ("logic") $-$ M^{-1} ("predicate").