ITB8832 Mathematics for Computer Science
First midterm test: 6 October 2022

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Exercise 1 (3 points)

A certain function $H : \mathbb{N} \to \mathbb{N}$ is defined recursively as follows:

- $H(0) = 1$;
- $H(n) = 2H(n-1) + 1$ for every $n \geq 1$.

Use the Well Ordering Principle to prove that:

$$H(n) = 2^{n+1} - 1 \text{ for every } n \in \mathbb{N} . \tag{1}$$

Important: Any solution which does not use the Well Ordering Principle will receive zero points.

Exercise 2 (2 points)

Prove that the formulas:

$$(P \textbf{ or } Q) \textbf{ implies } (R \textbf{ implies } S) \tag{2}$$

and:

$$((P \textbf{ or } Q) \textbf{ and } R) \textbf{ implies } S \tag{3}$$

are equivalent.

Exercise 3 (4 points)

Let A, B, and C be sets and let $f : A \to B$ and $g : B \to C$ be functions. We have seen in Exercise session 4 that if both f and g are surjective, then their composition $g \circ f$ is surjective.

1. (1 point) Can $g \circ f$ be surjective if g is not surjective?
2. (2 points) If you answered yes to the previous point, give an example where g is not surjective but $g \circ f$ is.

   If you answered no, prove that whatever A, B, C, and f are, if g is not surjective, then $g \circ f$ is not surjective.
3. (1 point) Give an example where f is not surjective, but $g \circ f$ is surjective. *Hint:* Use finite sets.

Important: Point 1 will be considered solved even if you *guess* the correct answer, without proving it in Point 2.

Exercise 4 (6 points)

For each of the following questions, mark the only correct answer:

1. Which one of the following numbers is irrational?
   (a) $\log_9 27$.
   (b) $\log_9 36$.
   (c) $\log_9 81$.
2. Which one of the following sets is well ordered?
   (a) $A ::= \{x \in \mathbb{Z} \mid x \leq -17\}$.
   (b) $B ::= \{x \in \mathbb{Z} \mid x \geq -17\}$.
   (c) $C ::= \{x \in \mathbb{R} \mid x \geq -17\}$.
3. Which one of the following formulas is equivalent to ((P **implies** Q) **and** P) **implies** Q?
   (a) P **implies** ((P **implies** Q) **implies** Q).
   (b) ((P **implies** Q) **implies** Q) **implies** P.
   (c) (P **implies** (Q **implies** P)) **implies** Q.
4. True or false: the predicate formula

   $$(\forall x \,.\, \exists y \,.\, P(x,y)) \textbf{ implies } (\exists y \,.\, \forall x \,.\, P(x,y))$$

   is valid.
5. Which one of the following relations is a bijection?

(a) $R : \mathbb{R} \to \mathbb{R}$, xRy if and only if $y = x^2$.

(b) $S : \mathbb{R} \to \mathbb{R}$, xSy if and only if $x = y^2$.

(c) $T : \mathbb{R} \to \mathbb{R}$, xTy if and only if $x = y^3$.

6. True or false: for any two finite sets A and B, $|A| = |B|$ if and only if A inj B **and** A surj B.

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Solutions

Exercise 1

Let
$$C ::= \left\{ n \in \mathbb{N} \mid H(n) \neq 2^{n+1} - 1 \right\}$$
be the set of counterexamples to (1). By contradiction, assume C is nonempty: by the Well Ordering Principle, C has a smallest element m. It cannot be $m = 0$, because:
$$H(0) = 1 = 2 - 1 = 2^{0+1} - 1 \, .$$
Then $m \geq 1$, so $m - 1$ is still a nonnegative integer, and as it is smaller than m, it is not a counterexample to (1). But then:
$$\begin{array}{rcl} H(m) & = & 2H(m-1) + 1 \\ & = & 2 \cdot (2^m - 1) + 1 \\ & = & 2^{m+1} - 2 + 1 = 2^{m+1} - 1 \, . \end{array}$$
Then the minimum counterexample m is not a counterexample after all. This is a contradiction, so C is empty, and $H(n) = 2^{n+1} - 1$ for every $n \in \mathbb{N}$.

Exercise 2

There are several ways to solve the exercise.

1. Comparison of truth values:

   The propositions (2) and (3) are implications, so it might be easier to prove that (2) is *false* if and only if (3) is false. Let's see:

   (a) The formula (P **or** Q) **implies** (R **implies** S) is false if and only if P **or** Q is true and R **implies** S is false. This only happens if at least one between P and Q is true, R is true, and S is false.

   (b) The formula ((P **or** Q) **and** R) **implies** S) is false if and only if (P **or** Q) **and** R is true and S is false. This only happens if at least one between P and Q is true, R is true, and S is false.

   Alternatively, we could do a proof by cases according to the truth value of S:

   (a) If S is true, then so is R **implies** S, so both (2) and (3) are true as implications with a true conclusion.

(b) Assume S is false. Let's split this case into two subcases, according to the truth value of R.

   i. If R is true, then on the one hand, R **implies** S is false, so (2) is true if and only if P **or** Q is false, that is, if P and Q are both false; and on the other hand, (3) is true if and only if (P **or** Q) **and** R is false, which, as R is true, is only possible if both P and Q are both false.

   ii. If R is false, then on the one hand, R **implies** S is true, and so is (2); and on the other hand, (P **or** Q) **and** R is false, so (3) is true.

2. Boolean algebra:

   We must first rewrite the implications as disjunctions. So:

$$(P \vee Q) \longrightarrow (R \longrightarrow S) \quad \longleftrightarrow \quad \overline{P \vee Q} \vee (\overline{R} \vee S)$$

   and

$$((P \vee Q) \wedge R) \longrightarrow S \quad \longleftrightarrow \quad \overline{(P \vee Q) \wedge R} \vee S$$

   Then associativity and de Morgan's laws give us:

$$\begin{aligned} \overline{P \vee Q} \vee (\overline{R} \vee S) \quad &\longleftrightarrow \quad (\overline{P \vee Q} \vee \overline{R}) \vee S \\ &\longleftrightarrow \quad \overline{(P \vee Q) \wedge R} \vee S\,. \end{aligned}$$

Exercise 3

1. No, it cannot.

2. We prove the contrapositive: if $g \circ f$ is surjective, then g is surjective.

   Assume $g \circ f : A \to C$ is surjective. Fix $z \in C$: there exists $x \in A$ such that $g(f(x)) = (g \circ f)(x) = z$. Then, if we want $y \in B$ such that $g(y) = z$, we only need to choose $y = f(x)$ where x is the same that we had found a little ago. Note that we require no special properties from f, except to be defined on x: which it must be, otherwise $g \circ f$ would not be defined on x.

3. There are plenty of options. For example, let $A = \{1\}$, $B = \{2, 3\}$, $C = \{4\}$, $f(1) = 2$, $g(2) = 4$, $g(3)$ undefined. Then f is not surjective, but $g \circ f$ is.

Exercise 4

1. (a) **No:** $9 = 3^2$ and $27 = 3^3$, so $\log_9 27 = 3/2$.

   (b) **Yes:** if $\log_9 36 = m/n$ for two integers m and n, which we may assume both positive as 9 and 36 are both larger than 1, then $9^m = 36^n$, which is impossible, because the left-hand side is odd and the right-hand side is even.

   (c) **No:** $81 = 9^2$, so $\log_9 81 = 2$.

2. (a) **No:** A itself is nonempty and doesn't have a minimum.

   (b) **Yes:** every subset of $\mathbb{Z}$ which is bounded from below is well ordered.

   (c) **No:** C contains the nonempty subset $\left\{\frac{1}{n+1} \mid n \in \mathbb{N}\right\}$, which doesn't have a minimum.

3. (a) **Yes:** both formulas express the modus ponens.

   (b) **No:** ((P **implies** Q) **implies** Q) **implies** P is false if P is false and Q is true, while the formula in the question is always true.

   (c) **No:** (P **implies** (Q **implies** P)) **implies** Q is false if P is true and Q is false, while the formula in the question is always true.

4. **False:** choosing as the domain the arithmetics of natural numbers, as type for the variables the set of natural numbers, and interpreting $P(x, y)$ as $x < y$, the formula means "if for every natural number there is a larger natural number, then there is a natural number which is larger than every natural number", which is false.

5. (a) **No:** there is no $x \in \mathbb{R}$ such that $x^2 = -1$, so R is not surjective; also, $1R1$ and $(-1)R1$, so R is not injective.

   (b) **No:** S is the inverse of R, which is not a bijection, so S is not a bijection either. More precisely, as R is neither surjective nor injective, S is neither total nor a function.

   (c) **Yes:** T is the inverse of the function $f(x) = x^3$, which is a bijection of $\mathbb{R}$ into itself.

6. **True:** for finite sets, the second formula is equivalent to $|A| \leq |B|$ **and** $|A| \geq |B|$.