ITB8832 Mathematics for Computer Science
First midterm test: 4 October 2024

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Exercise 1 (3 points)

Use the Well Ordering Principle to prove that:

$$\sum_{k=0}^{n} k \cdot 2^k = (n-1) \cdot 2^{n+1} + 2 \text{ for every } n \in \mathbb{N} . \tag{1}$$

Important: Any solution which does not use the Well Ordering Principle will receive zero points.

Exercise 2 (3 points)

1. (2 points) Prove that the formula:

   $$(P \textbf{ implies } Q) \textbf{ implies } ((P \textbf{ implies } R) \textbf{ implies } (P \textbf{ implies } (Q \textbf{ and } R))) \tag{2}$$

   is valid.

2. (1 point) Prove that the formula:

   $$((P \textbf{ implies } Q) \textbf{ implies } (P \textbf{ implies } R)) \textbf{ implies } (P \textbf{ implies } (Q \textbf{ and } R)) \tag{3}$$

   is *not* valid.

Exercise 3 (3 points)

Let A and B be sets. Denote by B^A (read: "B to the A") the set of total functions from A to B.

1. (2 points) Prove that, if A and B are finite and nonempty, then $\left|B^A\right| = |B|^{|A|}$.

   Hint: Independent choices.

2. (1 point) Explain why the formula of the previous point is still true if A is empty.

Exercise 4 (6 points)

For each of the following questions, mark the only correct answer:

1. Which one of the following numbers is irrational?

   (a) $\log_5 125$.

   (b) $\log_{25} 125$.

   (c) $\log_{29} 125$.

2. Which one of the following sets is well ordered?

   (a) $U ::= \left\{ \frac{n}{n+1} \,\middle|\, n \in \mathbb{N} \right\}$.

   (b) $V ::= \left\{ \frac{1}{n+1} \,\middle|\, n \in \mathbb{N} \right\}$.

   (c) $W ::= \left\{ \frac{x}{x+1} \,\middle|\, x \in \mathbb{R}^+ \right\}$.

3. Indicate which one of the following formulas is equivalent to:

   $$((P \textbf{ implies } Q) \textbf{ implies } R) \textbf{ implies } (P \textbf{ implies } (Q \textbf{ implies } R))$$

   Hint: This is a "don't panic" question, and finding the correct answer is the same as determining the two wrong ones.

   (a) $((P \textbf{ implies } Q) \textbf{ implies } P) \textbf{ implies } (P \textbf{ implies } Q)$.

   (b) $(P \textbf{ implies } (Q \textbf{ implies } P)) \textbf{ implies } (P \textbf{ implies } Q)$.

   (c) $(P \textbf{ implies } (Q \textbf{ implies } P)) \textbf{ implies } Q$.

4. True or false: The predicate formula:

   $$(\forall x \,.\, (P(x) \textbf{ implies } Q(x))) \textbf{ or } (\forall x \,.\, (Q(x) \textbf{ implies } P(x)))$$

   has a counter-model.

5. Which one of the following relations is a bijection?

   (a) $R : \mathbb{N} \to \mathbb{N}$, xRy if and only if $x = 100 - y$.
   (b) $S : \mathbb{Z} \to \mathbb{Z}$, xSy if and only if $x = 100 - y$.
   (c) $T : \mathbb{Z} \to \mathbb{Z}$, xTy if and only if $x = 100y$.

6. True or false: There exists a finite set A such that $|A| = |\text{pow}(A)|$.

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Solutions

Exercise 1

Let C be the set of counterexamples to (1):

$$C ::= \left\{ c \in \mathbb{N} \,\middle|\, \sum_{k=0}^{c} k \cdot 2^k \neq (c-1) \cdot 2^{c+1} + 2 \right\} .$$

By contradiction, assume that C is nonempty. Let m be the smallest element of C. It cannot be $m = 0$, because for $n = 0$ the left-hand side of (1) is $0 \cdot 2^0 = 0$ and the right-hand side is $-1 \cdot 2^1 + 2 = 0$. Then $m \geq 1$, so $m - 1$ is still a natural number, and as it is smaller than the minimum counterexample to (1), we have:

$$\sum_{k=0}^{m-1} k \cdot 2^k = (m-2) \cdot 2^m + 2$$

But then,

$$\begin{aligned} \sum_{k=0}^{m} k \cdot 2^k &= \left(\sum_{k=0}^{m-1} k \cdot 2^k \right) + m \cdot 2^m \\ &= (m-2) \cdot 2^m + 2 + m \cdot 2^m \\ &= (2m-2) \cdot 2^m + 2 \\ &= (m-1) \cdot 2^{m+1} + 2 \ : \end{aligned}$$

which is precisely (1) with m in the role of n. We conclude that if there are any counterexamples to (1), then the smallest such counterexample is not a counterexample. This is a contradiction, so there are no counterexamples, and the identity (1) holds for every $n \in \mathbb{N}$.

Exercise 2

1. There are several ways to solve this point; we survey two.

(a) Truth table:

P	Q	R	$(P \longrightarrow Q)$	$\longrightarrow$	$((P \longrightarrow R)$	$\longrightarrow$	$(P \longrightarrow$	$(Q \wedge R)))$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**F**	**T**	**F**	**F**
T	**F**	**T**	**F**	**T**	**T**	**F**	**F**	**F**
T	**F**	**F**	**F**	**T**	**F**	**T**	**F**	**F**
F	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**T**	**T**	**T**	**F**
F	**F**	**T**	**T**	**T**	**T**	**T**	**T**	**F**
F	**F**	**F**	**T**	**T**	**T**	**T**	**T**	**F**

(b) Proof by cases:

i. First, assume that P is false. Then P **implies** Q, P **implies** R, and P **implies** (Q **and** R) are all true, so (2) is true.

ii. Now, assume that P is true. If Q is false, then P **implies** Q is false, so (2) is true. If Q is true, then P **implies** R and P **implies** (Q **and** R) both have the same truth value as R, so the implication from the former to the latter is true, and (2) is also true.

2. If P is true and Q is false, then P **implies** Q and P **implies** (Q **and** R) are both false, so regardless of the truth value of R, the implication (P **implies** Q) **implies** (P **implies** R) is true, and the main implication (3) is false.

Exercise 3

1. Let $|A| = k \geq 1$ and $|B| = n \geq 1$ for brevity. A total function from A to B is a collection of independent choices of elements of B, exactly one for every element of A. There are exactly n^k ways of making k independent choices between n options.

2. If A is empty, then the empty relation is the only total function from A to B, so $\left|B^A\right| = 1 = |B|^0 = |B|^{|A|}$.

Exercise 4

1. (a) **No:** $125 = 5^3$, so $\log_5 125 = 3$.

   (b) **No:** $125 = 5^3$ and $25 = 5^2$, so $\log_{25} 125 = 3/2$.

   (c) **Yes:** 29 is prime and not a divisor of 125, so $\log_{29} 125$ is irrational.

2. (a) **Yes:** this is the set $\mathbb{F}$ from Week 2.
   (b) **No:** V contains the infinite decreasing sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
   (c) **No:** if $n \in \mathbb{N}$ and $x = \frac{1}{n+1}$, then $\frac{x}{x+1} = \frac{1}{n+2}$, so W contains the infinite decreasing sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

3. (a) **Yes:** both formulas are valid. The one in the question is an implication with the weakening law as its conclusion, and as the latter is valid, so is the entire formula. The one in the answer is known as *Peirce's law*, and is equivalent to the law of excluded middle.
   (b) **No:** this formula is equivalent to P **implies** Q.
   (c) **No:** this formula is equivalent to Q.

4. **True:** we discussed this formula in Exercise session 3.

5. (a) **No:** R is not total, because for $x = 101$ there is no $y \in \mathbb{N}$ such that $x = 100 - y$.
   (b) **Yes:** S is a total function which is its own inverse, so it is a bijection.
   (c) **No:** T is not total, because for $x = 17$ there is no $y \in \mathbb{Z}$ such that $x = 100y$.

6. **False:** if $|A| = n$, then $|\text{pow}(A)| = 2^n$, and for every $n \in \mathbb{N}$ it is $n < 2^n$.