ITB8832 Mathematics for Computer Science
Second midterm test: 3 November 2023

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Exercise 1 (3 points)

Let A and B be nonempty sets. Prove that if A surj B, then A^* surj B^*.

Hint: Use a surjective function $f : A \to B$ to define recursively a surjective function $g : A^* \to B^*$.

Exercise 2 (3 points)

Four men and four women have the following lists of preferences:

Aragorn	Arwen, Hermione, Chani, Catelyn
Eddard	Arwen, Catelyn, Hermione, Chani
Paul	Chani, Catelyn, Hermione, Arwen
Ron	Catelyn, Chani, Arwen, Hermione

Arwen	Aragorn, Paul, Ron, Eddard
Catelyn	Aragorn, Eddard, Ron, Paul
Chani	Eddard, Paul, Ron, Aragorn
Hermione	Eddard, Ron, Aragorn, Paul

Prove that the set of matchings:

(Aragorn, Arwen), (Eddard, Catelyn), (Paul, Chani), (Ron, Hermione)

matches every woman to her optimal husband.

Exercise 3 (3 points)

A subset L of the set ASCII* is called *decidable* if there exists a program P such that for every $s \in$ ASCII*, when P receives s in input:

- if $s \in L$, then P returns 1 and halts; and

- if $s \notin L$, then P returns 0 and halts.

We then say that P *decides* L.

Prove that L is decidable if and only if L and $\overline{L}$ are both recognizable.

Exercise 4 (6 points overall)

For each of the following questions, mark the only correct answer.

1. Which one of the following predicates is a preserved invariant for the Die Hard machine with jugs of 12 and 15 liters?

   (a) b and ℓ are both multiples of 4.

   (b) b and ℓ are both multiples of 3.

   (c) ℓ is a multiple of 3.

2. True or false: if the Mating Ritual with men as suitors returns the same stable set of matchings which it returns with women as suitors, then there is a unique stable set of matchings.

3. Which one of the following Nim games is a win for the first player?

   (a) $\mathrm{Nim}_{(5,9,12)}$.

   (b) $\mathrm{Nim}_{(5,9,14)}$.

   (c) $\mathrm{Nim}_{(7,9,14)}$.

4. Which one of the following functions from $\{0,1\}^*$ into itself is correctly defined by recursion?

   (a) $f(\langle 0\,,\, s\rangle) = \langle 1\,,\, f(s)\rangle$, $f(\langle 1\,,\, s\rangle) = \langle 0\,,\, f(s)\rangle$.

   (b) $g(\lambda) = \lambda$, $g(\langle 0\,,\, s\rangle) = \langle 0\,,\, g(\langle 1\,,\, s\rangle)\rangle$, $g(\langle 1\,,\, s\rangle) = \langle 1\,,\, g(\langle 0\,,\, s\rangle)\rangle$.

   (c) $h(\lambda) = \lambda$, $h(\langle 0\,,\, s\rangle) = \langle 0\,,\, \langle 1\,,\, h(s)\rangle\rangle$, $h(\langle 1\,,\, s\rangle) = \langle 1\,,\, \langle 0\,,\, h(s)\rangle\rangle$.

5. True or false: for any two sets A and B, if A strict B then A^* strict B^*.

6. Which one of the following sets is unrecognizable?

   (a) $L ::= \{s \in \mathrm{ASCII}^* \mid s \in \mathrm{lang}(P_s)\}$.

   (b) $M ::= \{s \in \mathrm{ASCII}^* \mid$ "antidisestablishmentarianism" $\in \mathrm{lang}(P_s)\}$.

   (c) $N ::= \{s \in \mathrm{ASCII}^* \mid \mathrm{lang}(P_s)$ is finite$\}$.

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Solutions

Exercise 1

Let $f : A \to B$ be a surjective function. As B is nonempty, we may assume, as we have seen in Lecture 8, that f is total. Define then $g : A^* \to B^*$ recursively as follows:

- **Base case:** $s = \lambda$. Then $g(s) = \lambda$.
- **Constructor case:** $g(\langle a\,,\, s\rangle) = \langle f(a)\,,\, g(s)\rangle$ for every $a \in A$ and $s \in A^*$.

Then g is a function, because so is f. We will now show by structural induction on $t \in B^*$ that g is surjective.

- **Base case:** $t = \lambda$. Then $t = g(\lambda)$.
- **Constructor case:** $t = \langle b\,,\, t'\rangle$ for some $b \in B$ and $t' \in B^*$. Assume that there exists $s' \in A^*$ such that $g(s') = t'$. As f is surjective, there exists $a \in A$ such that $f(a) = b$. Then $s = \langle a\,,\, s'\rangle$ is such that $g(s) = t$.

Exercise 2

The set of matchings that matches every woman with her optimal husband is the one produced by the Mating Ritual with women as suitors, so one way to solve the exercise is to run it and see if it produces the set of matchings in the text. Said, done:

- On the first day:
  - In the morning, Arwen and Catelyn become suitors of Aragorn, and Chani and Hermione become suitors of Eddard.
  - In the afternoon, Aragorn sends Catelyn away, and Eddard sends Chani away.
  - In the evening, Catelyn erases Aragorn from her list, and Chani erases Eddard from her list.
- On the second day:
  - In the morning, Catelyn becomes a suitor of Eddard, and Chani becomes a suitor of Paul.
  - In the afternoon, Eddard sends Hermione away.

    - In the evening, Hermione erases Eddard from her list.

- On the third day:
    - In the morning, Hermione becomes a suitor of Ron.
    - Every man has exactly one suitor, and the pairs are formed:

$$(\text{Aragorn, Arwen}), (\text{Eddard, Catelyn}),$$
$$(\text{Paul, Chani}), (\text{Ron, Hermione}).$$

Another way to solve the exercise, and even prove something more, goes as follows. As Aragorn and Arwen are each other's first choice, they are each other's optimal spouse: in fact, they are each other's only feasible spouse. As Eddard and Catelyn are each other's second choice and are not the first choices of their own first choices, they are each other's only feasible (thus also optimal) spouse too. Then we only need to match Chani and Hermione to their optimal husband between Paul and Ron according to the following set of preferences:

- Chani: Paul, Ron;
- Hermione: Ron, Paul;
- Paul: Chani, Hermione;
- Ron: Chani, Hermione.

But this set of preferences has (Paul, Chani), (Ron, Hermione) as its *unique* stable set of matchings. Then not only the set of matchings in the text of the exercise is the one that matches every woman to her optimal husband: it is the unique stable set of matchings for that set of preferences!

Exercise 3

First, assume that L is decidable. Let P be a program that decides L. Then a program Q that recognizes L can be obtained as follows:

- Run P on s.
- If P returns 1, then halt.
- If P returns 0, then run forever.

Similarly, a program R that recognizes $\overline{L}$ can be obtained as follows:

- Run P on s.
- If P returns 1, then run forever.
- If P returns 0, then halt.

Now, assume that L and $\overline{L}$ are both recognizable. Let P_1 and P_0 be two programs which recognize L and $\overline{L}$, respectively. Then a program Q that decides L can be obtained as follows:

- At every time $n \in \mathbb{Z}^+$:
  - Run the nth step of P_1 on s.
  - If P_1 halts, then return 1 and halt.
  - Run the nth step of P_0 on s.
  - If P_0 halts, then return 0 and halt.

Sooner or later, one between P_1 and P_0 will halt, and so will Q.

Exercise 4

1. (a) **No:** from $(12, 0)$ we can reach $(12, 15)$.
   (b) **Yes.**
   (c) **No:** from $(3, 1)$ we can reach $(4, 0)$.

2. **True:** in this situation, everyone is at the same time the optimal and pessimal spouse of their match, so no other spouse is feasible.

3. (a) **No:** $5 \text{ XOR } 9 \text{ XOR } 12 = 0$, so $\text{Nim}_{(5,9,12)}$ is a win for the *second* player.
   (b) **Yes:** $5 \text{ XOR } 9 \text{ XOR } 14 = 2$, so $\text{Nim}_{(5,9,14)}$ is a win for the first player.
   (c) **No:** $7 \text{ XOR } 9 \text{ XOR } 14 = 0$, so $\text{Nim}_{(7,9,14)}$ is a win for the *second* player.

4. (a) **No:** the base case $s = \lambda$ is missing.
   (b) **No:** on the right-hand side of the constructor cases, g is evaluated on a string which is not s, and as a result, the calculation never terminates. For example, $f(0) = 0f(1) = 01f(0) = 010f(1) = \ldots$.
   (c) **Yes:** this is called the *Thue-Morse function*, from the mathematicians Axel Thue and Marston Morse.

5. **False:** if $A = \{0, 1\}$ and $B = \mathbb{Z}^+$, then A strict B, but A^* and B^* are both countably infinite.

6. (a) **No:** the program Q which, given s in input, compiles P_s and runs it on s, halts if and only if $s \in L$.

   (b) **No:** the program Q which, given the string s in input, compiles P_s and runs it on "antidisestablishmentarianism", halts if and only if $s \in M$.

   (c) **Yes:** $N =$ Finite-halt, and we know from Exercise session 8 that Finite-halt is unrecognizable.