ITB8832 Mathematics for Computer Science
Second midterm test: 1^{st} November 2024

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Exercise 1 (3 points)

Let A and B be sets and let $f : A \to B$ be a total function. The *mapping* of f onto $s \in A^*$ is the string $f^*(s) \in B^*$ defined recursively as follows:

- **Base case:** $f^*(\lambda) = \lambda$.
- **Constructor case:** $f^*(\langle a , s \rangle) = \langle f(a) , f^*(s) \rangle$ for every $a \in A$ and $s \in A^*$.

Let now A, B, and C be sets and let $f : A \to B$ and $g : B \to C$ be total functions. Prove by structural induction that:

$$(g \circ f)^*(s) = g^*(f^*(s)) \quad \text{for every } s \in A^* . \tag{1}$$

Exercise 2 (3 points)

Four men and four women have the following lists of preferences:

Beren	Lúthien, Cymoril, Rosie, Yennefer
Elric	Yennefer, Lúthien, Cymoril, Rosie
Geralt	Lúthien, Yennefer, Rosie, Cymoril
Sam	Rosie, Cymoril, Lúthien, Yennefer

Cymoril	Elric, Beren, Sam, Geralt
Lúthien	Beren, Geralt, Elric, Sam
Rosie	Elric, Beren, Sam, Geralt
Yennefer	Sam, Geralt, Elric, Beren

Prove that the set of matchings:

(Beren, Lúthien), (Elric, Cymoril), (Geralt, Yennefer), (Sam, Rosie)

matches every man to his optimal wife.

Exercise 3 (3 points)

Let M be a program that halts on at least one input $t \in \text{ASCII}^*$. Define:

$$H_M ::= \{s \in \text{ASCII}^* \mid \text{lang}(P_s) = \text{lang}(M)\} \ . \tag{2}$$

Prove that the complement $\overline{H_M}$ of H_M is not recognizable. *Hint:* Use an effective reduction from the Halting Problem and exploit the hypothesis that $\text{lang}(M) \neq \emptyset$.

Exercise 4 (6 points)

For each of the following questions, mark the only correct answer.

1. Which one of the following predicates is a preserved invariant for the Die Hard machine with jugs of 12 and 18 liters?
   (a) b and ℓ are both multiples of 3.
   (b) b and ℓ are both multiples of 4.
   (c) ℓ is a multiple of 3.
2. True or false: In the Mating Ritual, if a woman has a suitor on a given evening, then she will have the same suitor on the next evening.
3. Which one of the following Nim games is a win for the first player?
   (a) $\text{Nim}_{(6,12,18)}$.
   (b) $\text{Nim}_{(2,16,18)}$.
   (c) $\text{Nim}_{(12,16,28)}$.
4. Which one of the following functions is correctly defined by recursion?
   (a) $f(0) = 1$; $f(n) = 2f(n+1) + 1$ for every $n \in \mathbb{N}$.
   (b) $g(r, 0) = 1$ for every $r \in \mathbb{R}$; $g(r, n+1) = \dfrac{r}{n+1} \cdot g(r-1, n)$ for every $r \in \mathbb{R}$ and $n \in \mathbb{N}$.
   (c) $h(n+1) = 2h(n) + 1$ for every $n \in \mathbb{N}$.
5. True or false: The complement of the set No-halt is recognizable.
6. Which one of the following sets is countable?
   (a) $L ::= \text{pow}(\mathbb{N})$.
   (b) $M ::= \mathbb{N}^*$.
   (c) $N ::= \mathbb{N}^\omega$.

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Solutions

Exercise 1

By structural induction on $s \in A^*$:

- **Base case:** $s = \lambda$. Then:

$$(g \circ f)^* (\lambda) = \lambda = g^* (\lambda) = g^* (f^* (\lambda)) \ .$$

- **Constructor case:** Let $s = \langle a \, , \, t \rangle$ for some $a \in A$ and $t \in A^*$. Assume the thesis to be true for t, that is,

$$(g \circ f)^* (t) = g^* (f^* (t)) \ .$$

  Then:

$$\begin{array}{rcl} (g \circ f)^* (s) & = & (g \circ f)^* (\langle a \, , \, t \rangle) \\ & = & \langle (g \circ f)(a) \, , \, (g \circ f)^* (t) \rangle \\ & = & \langle g(f(a)) \, , \, g^* (f^* (t)) \rangle \\ & = & g^* (\langle f(a) \, , \, f^* (t) \rangle) \\ & = & g^* (f^* (\langle a \, , \, t \rangle)) \\ & = & g^* (f^* (s)) \ . \end{array}$$

Exercise 2

The matching which pairs every man with his optimal wife is produced by the Mating Ritual with the men as suitors.

1. On the first day:
   - In the morning, Beren and Geralt become suitors of Lúthien, Elric becomes a suitor of Yennefer, and Sam becomes a suitor of Rosie.
   - In the afternoon, Lúthien dismisses Geralt.
   - In the evening, Geralt erases Lúthien from his list.
2. On the second day:
   - In the morning, Geralt becomes a suitor of Yennefer.
   - In the afternoon, Yennefer dismisses Elric.
   - In the evening, Elric erases Yennefer from his list.

3. On the third day:

    - In the morning, Elric becomes a suitor of Lúthien.
    - In the afternoon, Lúthien dismisses Elric.
    - In the evening, Elric erases Lúthien from his list.

4. On the fourth day:

    - In the morning, Elric becomes a suitor of Cymoril.
    - Every woman have exactly one suitor, and the couples are formed:

      (Beren, Lúthien), (Elric, Cymoril),
      (Geralt, Yennefer), (Sam, Rosie).

The discussion can be simplified a bit by observing that Beren and Lúthien are each other's first choice, thus also each other's unique feasible spouse.[1] Then the set of matching which pairs every man to his optimal wife can be determined by only considering the simplified set of preferences:

Elric	Yennefer, Cymoril, Rosie	Cymoril	Elric, Sam, Geralt
Geralt	Yennefer, Rosie, Cymoril	Rosie	Elric, Sam, Geralt
Sam	Rosie, Cymoril, Yennefer	Yennefer	Sam, Geralt, Elric

and running the Mating Ritual with men as suitors:

1. On the first day:

    - In the morning, Elric and Geralt become suitors of Yennefer, and Sam becomes a suitor of Rosie.
    - In the afternoon, Yennefer dismisses Elric.
    - In the evening, Elric erases Yennefer from his list.

2. On the second day:

    - In the morning, Elric becomes a suitor of Cymoril.
    - Every woman have exactly one suitor, and the couples are formed:

      (Elric, Cymoril), (Geralt, Yennefer), (Sam, Rosie).

[1] This was expected, since Beren and Lúthien are the Middle-Earth counterparts of J.R.R. Tolkien and his wife!

Exercise 3

For every $s \in \text{ASCII}^*$ construct a string $r \in \text{ASCII}^*$ which is the source code for a program R that performs the following steps:

1. Take a string $t \in \text{ASCII}^*$ in input.

2. Run P_s on s.

3. Run M on t.

Then $s \in$ No-halt if and only if $r \notin H_M$:

- First, assume $s \in$ No-halt. Then R never reaches step 3, so $\text{lang}(R) = \emptyset$. As $\text{lang}(M)$ is nonempty by hypothesis, $r \notin H_M$.

- Now, assume $s \notin$ No-halt. Then, whatever the input string t is, R halts on t if and only if M halts on t: that is, $\text{lang}(R) = \text{lang}(M)$. By definition, $r \in H_M$.

By contradiction, assume that $\overline{H_M} = \text{lang}(Q)$ for some program Q. Then a program K which, when given in input any $s \in \text{ASCII}^*$, constructs r and runs Q on r, will halt on s if and only if $s \in$ No-halt: that is, it would be No-halt $= \text{lang}(K)$. But this is impossible, because we know from Lecture 8 that No-halt is not recognizable.

Exercise 4

1. (a) **Yes.**

   (b) **No:** from $(12, 16)$ we can reach $(10, 18)$.

   (c) **No:** from $(12, 17)$ we can reach $(11, 18)$.

2. **False:** on the next evening, she will still have a suitor, but he may be different from the one of the previous evening.

3. (a) **Yes:** $6 \text{ XOR } 12 \text{ XOR } 18 = 24$, so $\text{Nim}_{(6,12,18)}$ is a win for the first player.

   (b) **No:** $2 \text{ XOR } 16 \text{ XOR } 18 = 0$, so $\text{Nim}_{(2,16,18)}$ is a win for the *second* player.

   (c) **No:** $12 \text{ XOR } 16 \text{ XOR } 28 = 0$, so $\text{Nim}_{(12,16,28)}$ is a win for the *second* player.

4. (a) **No:** the recursive step is going the wrong way around.

(b) **Yes:** the function g is called the *generalized binomial coefficient*, usually written $\binom{r}{n}$ and read "r choose n".

(c) **No:** the base case $n = 0$ is missing.

5. **True:** the program P which, given in input a string $s \in \text{ASCII}^*$, compiles P_s and runs it on s, halts on s if and only if $s \notin$ No-halt.

6. (a) **No:** $\mathbb{N}$ strict pow($\mathbb{N}$) by Cantor's second theorem.

   (b) **Yes:** for the same reason why $(\mathbb{Z}^+)^*$ is countable.

   (c) **No:** clearly $\mathbb{N}^\omega$ surj $\{0,1\}^\omega$, and we can modify the proof of Theorem 4.5.5 to prove that $\{0,1\}^\omega$ bij pow($\mathbb{N}$).