Mathematics for Computer Science
Self-evaluation exercises for Week 1

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Exercise 1.1

In Exercise session 1 we have seen a proof of the implication:

If $1 = -1$, then $2 = 1$.

Modify the argument to obtain a proof of the following implication:

If $2 + 2 = 5$, then I am the Pope.

(There are proofs available in the literature and on the Web, but it is good to try by yourselves first.)

Exercise 1.2

In Exercise session 1 we proved the following proposition:

There exist two irrational numbers a and b such that a^b is rational.

The proof we gave was *nonconstructive*, because we did not give a pair (a, b) such that a and b are irrational and a^b is rational. In this exercise, we will give a constructive proof, and learn a little more in the meantime.

1. Modify the argument from our solution of Problem 1.6 to show that, however chosen the prime p and the positive integer n, the number $\log_p n$ is either integer or irrational.

2. Use point 1 to construct a pair (a, b) such that a and b are both irrational, but a^b is rational. *Hint:* Recall the rules of logarithm in base $a > 0$, $a \neq 1$:

    - $\log_a(b^k) = k \log_a b$;

- If $c > 0$ and $c \neq 1$, then $\log_a b = (\log_a c)(\log_c b)$.

Note: you need not have already proved point 1 when you prove point 2, but you *must* use it.

Exercise 1.3 (cf. Problem 1.10(b) in the textbook)

Let w, x, y, z be nonnegative integers such that:

$$x^2 + y^2 + z^2 = w^2 \,. \tag{1}$$

Let P be the proposition "w is even" and let Q be the proposition "x, y, and z are all even". Prove that

$$P \textbf{ iff } Q \,,$$

that is, whatever is our choice of w, x, y, z such that (1) is satisfied, the proposition P is true if and only if the proposition Q is true.

Hint: What is the remainder of the division of m^2 by 4 if m is even? What is it if m is odd?

Exercise 1.4 (from "What Is the Name of This Book?" by Raymond Smullyan)

You meet two men, of whom you know that each one is either a *knight* who only makes true statements, or a *knave* who only makes false statements; however, you don't know whether they are knights or knaves.

1. You ask the men: "Are you knights or knaves?" One of the two remains silent, but the other says: "We are both knaves."

   What are they?

2. What if the second man had said "we are not both knaves" instead?

3. What if the second man had said "we are not both knaves", but then the first man had said of him "that man is a knave"?

Exercise 1.5 (cf. Problem 1.19)

An integer m is a *divisor* of an integer n if there exists an integer k such that $m \cdot k = n$. (Note that, with this definition, every integer is a divisor of 0; more on this in Lecture 9.)

In this exercise, we will discuss an important theorem of algebra.

Theorem (Rational root theorem)**.** *Let*

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_dx^d$$

be a polynomial of degree $d \geq 1$ *such that all the coefficients* $a_0, a_1, a_2, \ldots, a_d$ *are integers, and let* m *and* n *be two integers such that* $n \neq 0$ *and no prime number is a divisor of both* m *and* n*.*

If m/n *is a* root *of* $p(x)$ *(that is,* $p(m/n) = 0$*) then* m *is a divisor of the* constant term a_0 *and* n *is a divisor of the* leading coefficient a_d.

1. Prove the rational root theorem.

   You may use the following fact: given any three integers a, b, and c, if a is a divisor of bc and a and b don't have any common prime divisors, then a is a divisor of c.

2. Use the rational root theorem to prove that, if the integer k is not the rth power of some other integer, then the rth root of k is irrational. *Hint:* Prove the contrapositive.

Note: you *do not* need to have solved point 1 before you solve point 2, but you *must* use it.

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Solutions

Exercise 1.1

The following proof is due to the logician Bertrand Russell:

> Suppose that $2 + 2 = 5$. By subtracting 2 from each side we obtain $2 = 3$. By swapping the sides we obtain $3 = 2$. By subtracting 1 to each side we obtain $2 = 1$.
>
> Now, I and the Pope are two. Since $2 = 1$, I and the Pope are one. Then I am the Pope.

Exercise 1.2

Since we don't need to have proved point 1 when we prove point 2, we prove point 2 first. As 9 is larger than $8 = 2^3$ and smaller than $16 = 2^4$, $\log_2 9$ is not integer, so it must be irrational. By taking $a = \sqrt{2}$ and $b = \log_2 9$, we get:

$$a^b = \left(\sqrt{2}\right)^{\log_2 9} = \left(2^{1/2}\right)^{2\log_2 3} = 2^{\log_2 3} = 3\,.$$

Now for the proof of point 1. As we did in classroom, we will prove that if $\log_p n$ is rational, then it is integer. If $n = 1$ then the logarithm is zero, so we may assume $n \geq 2$.

Assume that $\log_p n = a/b$ with a and b both integer. As $p > 1$ and $n \geq 2$, we can choose a and b both positive. Raising p to the powers $\log_p n$ and a/b gives $n = p^{a/b}$, which is equivalent to $n^b = p^a$.

Now, the left-hand side is a product of b integers, all equal to n. As p is prime, if it is a divisor of a product, then it is a divisor of at least one of the factors: all this means that n is a multiple of p. Let $k \geq 1$ be such that n is divisible by p^k, but not by p^{k+1}: then $n = p^k m$ with m not divisible by p. Then:

$$p^a = n^b = (p^k\, m)^b = p^{kb}\, m^b\,.$$

The only prime divisor of the left-hand side is p, so it must be $m = 1$ and $a = kb$. Then $\log_p n = \frac{a}{b} = k$ is integer.

Exercise 1.3

Let's follow the hint. If $m = 2n$ is even, then $m^2 = 4n^2$, and its remainder in the division by 4 is 0. If $m = 2n + 1$ is odd, then:

$$m^2 = (2n + 1)^2 = 4n^2 + 4n + 1\,,$$

and the remainder in the division of m^2 by 4 is 1.

One possible selection of cases is made according to the number of summands on the left-hand side of (1). It can be done as follows:

1. If x, y, and y are all even, then on the one hand, Q is true; and on the other hand, as w^2 must be even as a sum of even summands, w must be even in the first place, so P is true. Summarizing, if x, y, and z are all even, then P and Q are both true.

2. If exactly one of x, y, and z is odd, then on the one hand, Q is false; and on the other hand, as the remainder of the division of the left-hand side by 4 is 1, w^2 must be odd, and w must be odd in the first place, so P is false too. Summarizing, if exactly one of x, y, and z is odd, then P and Q are both false.

3. If exactly two of x, y, and z are odd, then things get interesting. Indeed, in this case, the remainder of the division of the left-hand side by 4 is the sum of one 0 and two 1s, so it is 2; but then, the remainder of the division of w^2 by 4 must also be 2, and this is impossible, because such remainder can only be either 0 or 1. This means that, *if (1) is satisfied, then it is impossible that exactly two of x, y, and z are odd:* as we are assumeing that (1) is satisfied, this case simply never happens. There is nothing wrong with this: when we do a proof by cases, it is important that every instance of the problem belongs to at least one case, but not that every case includes some instance of the problem.

4. For reasons similar to the previous case, under the premise that (1) is satisfied, it is impossible that all of x, y, and z are odd.

Exercise 1.4

We will discuss the three points and see that, while in the first and third case we can say what both men are, in the second case we can only exclude one of four possibilities.

1. The statement "we are both knaves" has been made by someone who either only makes true statements or only makes false statements, so it has a definite truth value. It also cannot be true: otherwise, the man who made it must have been a knave, and knaves only make false statements. Then it is false, so the man who spoke must be a knave; as this man is a knave and they are not both knaves, the other man is a knight.

2. The statement "we are not both knaves" could have been made by a knight, regardless of the other man being a knight or a knave; but it could also have been made by a knave, if the other man was a knave too. However, if the speaker is a knave, then the other must be also be a knave.

   Let's give a more detailed proof by cases, according to whether the speaker is a knight or a knave:

   (a) First, consider the case where the speaker is a knight. If, of two people, one is a knight, then surely the two men are not both knaves, and it doesn't matter whether the other is a knight or a knave.

   (b) Next, consider the case where the speaker is a knave. Then his statement "we are not both knaves" is false, so the two men are indeed both knaves.

   Another way to reach this conclusion goes as follows. If the proposition "we are both knaves" means that it *must* be that the speaker is a knave and the other is a knight, then its negation "we are not both knaves" means that it *cannot* be that the speaker is a knave and the other is a knight. But we have three more options, not just one, so we cannot determine what each man is.

3. If of two men, each one being a knight or a knave, one says that the other is a knave, it doesn't necessarily mean that the other man is a knave: but it means that one of the two men is a knight, and the other is a knave. In fact:

   (a) If the man who says "the other man is a knave" is a knight, then his statement is true, so the other man is a knave.

   (b) If the man who says "the other man is a knave" is a knave, then his statement is false, so the other man is a knight.

   Now, we already know from point 2 that the two men are either two knights, or two knaves, or the man who said "we are not both knaves" is a knight and the other man is a knave. From this and the statement of the other man we can conclude that this third option is the correct one.

Exercise 1.5

1. By hypothesis,

$$a_0 + a_1 \cdot \frac{m}{n} + a_2 \cdot \frac{m^2}{n^2} + \ldots + a_d \cdot \frac{m^d}{n^d} = 0\,.$$

   By multiplying both sides by n^d (which is nonzero by hypothesis) we obtain the equivalent equality:

$$a_0 n^d + a_1 m n^{d-1} + \ldots + a_d m^d = 0\,. \tag{2}$$

   By moving $a_0 n^d$ on the right-hand side in (2), we obtain:

$$a_1 m n^{d-1} + \ldots + a_d m^d = -a_0 n^d\,.$$

   The left-hand side is clearly divisible by m, and so must be the right-hand side. But by hypothesis, m and n do not have prime divisors in common, so m must be a divisor of a_0.

   The proof that n is a divisor of a_d is similar.

2. We prove the contrapositive: if $\sqrt[r]{k}$ is rational, then k is the rth power of some integer. Write $\sqrt[r]{k} = m/n$ with $n \neq 0$ and m and n having no common prime divisors. Then m/n is a rational root of the polynomial $p(x) = x^r - k$: by the rational root theorem, n must be a divisor of 1, which is only possible if $n = 1$ or $n = -1$. But in this case, m/n is either m or $-m$, so it is an integer: then either $k = m^r$ or $k = (-m)^r$.