Mathematics for Computer Science
Self-evaluation exercises for Week 2

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Exercise 2.1 (from the midterm test of 03.10.2018, tweaked)

1. Prove by contradiction that $\log_{20} 50$ is irrational.
2. Let $a > 1$ and $b > 1$ be integers. Can $\log_a b$ be rational, but not integer?

Exercise 2.2 (from the midterm test of 07.10.2019)

Let a be a real number, different from 1. Use the Well Ordering Principle to prove that, for every nonnegative integer n,

$$1 + a + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a} . \tag{1}$$

Important: solutions which do not use the Well Ordering Principle will receive zero points.

Exercise 2.3 (cf. Problem 2.21(d))

Let G be the set of the rational numbers of the form m/n where $m, n > 0$ and $n \leq g$, where g is a *googol* 10^{100}.

Hint: Let $g!$ (read: "g factorial") be the product of all the positive integers from 1 to g. What can we say about the denominators of the fractions that can represent the elements of G?

Exercise 2.4 (cf. Problem 2.23)

Prove that a set R of real numbers is well ordered iff there is no infinite decreasing sequence of numbers in R. In other words: R is well ordered if and only if there is no set of numbers $r_i \in R$ such that

$$r_0 > r_1 > r_2 > \dots \tag{2}$$

Hint: A set is well ordered if and only if all its subsets are well ordered. Also, if $m \in S$ is not the minimum of S, then there is some $x \in S$ such that $x < m$.

Exercise 2.5 (from Raymond Smullyan's "The Gödelian Puzzle Book")

You meet a man whom you know to be either a *knight* who only makes true statements, or a *knave* who only makes false statements (but you don't know which of the two). The man makes the following statement:

"Today is not the first day on which I make this statement."

Is he a knight or a knave? *Hint:* choose a "good" subset of the set of natural numbers and use the Well Ordering Principle.

Exercise 2.6

Profe that the formulas

$$(P \textbf{ and } Q) \textbf{ implies } R \tag{3}$$

and

$$P \textbf{ implies } (Q \textbf{ implies } R) \tag{4}$$

are equivalent:

1. first, with a truth table;
2. then, with a proof by cases.

The equivalence between (3) and (4) is a special case of an important theorem of predicate logic, called the *deduction theorem*.

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Solutions

Exercise 2.1

1. By contradiction, assume $\log_{20} 50 = \dfrac{m}{n}$ with m and n integers. As both the base and the argument are larger than 1, the logarithm is positive, so we may suppose m and n positive; also, we can assume that $\gcd(m, n) = 1$.

   By hypothesis, $20^{m/n} = 50$, that is, $20^m = 50^n$. But $20 = 2^2 \cdot 5$ and $50 = 2 \cdot 5^2$, so the equality becomes:

   $$2^{2m} \cdot 5^m = 2^n \cdot 5^{2n} \, .$$

   This is only possible is $n = 2m$ and $m = 2n$: but then, $n = 4n$, which is only possible if $m = n = 0$ against the fact that n is the denominator in a fraction.

2. Yes, it is: it is sufficient that a and b are both powers of the same integer c. For example, if $a = 4 = 2^2$ and $b = 8 = 2^3$, then:

   $$\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2} \, .$$

Exercise 2.2

Let C be the set of counterexamples to (1):

$$C = \left\{ n \in \mathbb{N} \mid 1 + a + \ldots + a^n \neq \frac{1 - a^{n+1}}{1 - a} \right\} .$$

By contradiction, assume that C is nonempty: by the Well Ordering Principle, it has a smallest element c_0. This smallest element must be positive, because for $n = 0$ the sum on the left-hand side of (1) is 1 and the fraction on the right-hand side is $\dfrac{1 - a^{0+1}}{1 - a} = 1$. But if c_0 is positive, then $c_0 - 1$ is nonnegative, and as it is smaller than c_0, it satisfies (1), that is:

$$1 + a + \ldots + a^{c_0 - 1} = \frac{1 - a^{c_0}}{1 - a} \, .$$

But by adding a^{c_0} to both sides of the equality we get:

$$\begin{aligned}1 + a + \ldots + a^{c_0} &= \frac{1 - a^{c_0}}{1 - a} + a^{c_0} \\ &= \frac{1 - a^{c_0} + (1 - a)a^{c_0}}{1 - a} \\ &= \frac{1 - a^{c_0} + a^{c_0} - a^{c_0+1}}{1 - a} \\ &= \frac{1 - a^{c_0+1}}{1 - a} \,;\end{aligned}$$

that is, the minimum counterexample is not a counterexample after all. We have reached this contradiction because we had supposed that C is nonempty: therefore, C is empty, and (1) is true for every nonnegative integer n.

Exercise 2.3

Let $a = g!$. Then, since $n \leq g$ when $x = m/n \in G$, for every such x the number ax is a positive integer; also, if $x \leq y$, then $ax \leq ay$. So, however given a nonempty subset S of G, the set $T = \{ax \,|\, x \in S\}$ is a nonempty subset of positive integers: if m is the minimum of T, then m/a is the minimum of S.

Exercise 2.4

If a sequence such as in (2) exists, then the set of its terms does not have a minimum. However given an element, there will be another element (for example, the next one in the sequence) which is strictly smaller. In this case, R has a subset which is not well ordered, so it is not well ordered.

If R is not well ordered, take a nonempty subset S of R which has no minimum. Choose $r_0 \in S$: as r_0 is not the minimum of S, there exists $r_1 \in S$ which is strictly smaller than r_0. Similarly, as r_1 is not the minimum of S, there exists $r_2 \in S$ which is strictly smaller than r_1. Iterating the procedure, we obtain a sequence of elements of R such as in (2). More in detail:

1. We choose the starting element $r_0 \in S$ as we want.
2. For every $n \in \mathbb{N}$, after we have chosen $r_n \in S$, we choose $r_{n+1} \in S$ so that it is smaller than r_n. This is always possible, because S has no minimum, so in particular r_n is not the minimum of S.

Note that a set of numbers *can* have a minimum without being well ordered. For example, the set of nonnegative real numbers has 0 as its minimum and contains the infinite decreasing sequence $1 > \frac{1}{2} > \frac{1}{3} > \cdots$

Exercise 2.5

Even if the statement is self-referential, we know that it has been made by either a knight or a knave, so it must have a truth value.

Count the days since the birth of the man, starting with day 0. Since there is a day (namely, today) when he made that statement, by the Well Ordering Principle there must have been a *first* day when he made it. But on that first day, the statement was false! Since knights only make true statements, the man is a knave.

Exercise 2.6

1. Constructing a truth table:

P	Q	R	(P **and** Q)	**implies** R	P **implies**	(Q **implies** R)
T	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**	**F**	**F**
T	**F**	**T**	**F**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**T**	**T**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**T**	**F**	**F**	**T**	**T**	**F**
F	**F**	**T**	**F**	**T**	**T**	**T**
F	**F**	**F**	**F**	**T**	**T**	**T**

   Here, the fifth column contains the truth values of (3) given those of P, Q, and R, and the sixth one contains those of (4). The two columns are equal, so the two formulas are equivalent.

2. There are several ways to do a proof by cases. For example, we could consider the truth values of the variable R. Then:

   (a) First, assume that R is true. Then (P **and** Q) **implies** R is an implication with a true conclusion, so it is true; for the same reason, Q **implies** R is true, and so is P **implies** (Q **implies** R). We conclude that, if R is true, then the formulas (3) and (4) are both true.

   (b) Now, assume that R is false. Then (P **and** Q) **implies** R is false if and only if P **and** Q is true, that is, if and only if P and Q are both true; as for P **implies** (Q **implies** R), it is false if and only if P is true and Q **implies** R is false, which in turn only happens if Q to be true. We conclude that if R is false, then (3) and (4) are both false if P and Q are both true, and both true if either P is true, or Q is true, or both P and Q are true.

In either case, whatever the truth values of P, Q, and R are, the formulas (3) and (4) are either both true, or both false: thus, the two formulas are equivalent.