Mathematics for Computer Science
Self-evaluation exercises for Week 4

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Exercise 4.1 (from the midterm test of 7 October 2020)

Use the Well Ordering Principle to prove the following: if n is a positive integer and $A, B_1, B_2, \ldots, B_n$, are arbitrary sets, then

$$A \cap (B_1 \cup B_2 \cup \ldots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n) .$$

Hint: start with proving that, if m is the minimum counterexample, then $m \geq 3$.

Exercise 4.2

Recall the rules of the Subset Take-Away game:

1. The initial position is a finite nonempty set.
2. Taking turns, the players take away subsets of the initial set.
3. It is not permitted to take away the entire initial set as the first move.
4. Once a subset has been taken away, no subset which contains it can be taken away anymore.

   In particular: no subset can be taken more than once.
5. A player who cannot take away a nonempty subset on his or her turn, loses the game.

We have seen in classroom that if the initial set has either 1, 2, or 3 elements, then the second player has a winning strategy.

Prove that if the initial set has 4 elements, then the second player still has a winning strategy. *Hint:* consider the cases where the first player chooses a subset with one, three, or two elements, the last case being the trickiest one.

Exercise 4.3 (cf. Problem 4.19)

Before you try this exercise, you might want to revise what you know from your Calculus course.

For each of the following real-valued total functions on the real numbers, indicate whether it is a bijection, a surjection but not a bijection, an injection but not a bijection, or neither an injection nor a surjection.

1. $x \mapsto x + 2$.
2. $x \mapsto 2x$.
3. $x \mapsto x^2$.
4. $x \mapsto x^3$.
5. $x \to \sin x$.
6. $x \to x \sin x$. *Hint:* intermediate value theorem.
7. $x \to e^x$.

Exercise 4.4

We have seen in classroom that if A inj B, then there exists a total injective *function* from A to B.

1. Prove a similar, but not identical, fact for the "surject" relation: if A surj B *and in addition* B *is nonempty*, then there exists a *total* surjective function from A to B.
2. What happens to the previous point if B is empty?

Exercise 4.5 (cf. Problem 4.29)

Consider a basic Web search engine, which stores information on Web pages and processes queries to find pages satisfying conditions provided by users. At a high level, we can formalize the key information as:

- A set P of *pages* that the search engine knows about.
- A binary relation L (for *link*) over pages, defined such that $p_1 L p_2$ if and only if p_1 links to p_2.

- A set E of *endorsers*, people who have recorded their opinions about which pages are high-quality.
- A binary relation R (for *recommends*) between endorsers and pages, such that eRp iff person e has recommended page p.
- A set W of *words* that may appear on pages.
- A binary relation M (for *mentions*) between pages and words, where pMw iff word w appears on page p.

Then, for example, if the word "logic" belongs to W, then the set of pages in P where the word "logic" appears is:

$$\{p \in P \mid p\, M \text{ “logic”}\} = M^{-1}(\text{“logic”})\,.$$

More complex relations can also be constructed from these basic ones. For example, the relation $H : E \to W$ defined by eHw if and only if e has recommended a page that contains w, is $M \circ R$, because:

$$eHw \textbf{ iff } \exists p \in P\,.\,(eRp \textbf{ and } pMw) \textbf{ iff } e(M \circ R)w\,.$$

Use the specification above to express the following:

1. The set of pages containing the word "logic" but not the word "predicate".
2. The set of pages containing the word "set" that have been recommended by "Meyer".
3. The set of endorsers who have recommended pages containing the word "algebra".
4. The set of pages that have at least one incoming or outgoing link.
5. The relation that relates word w and page p iff w appears on a page that links to p.
6. The relation that relates word w and endorser e iff w appears on a page that links to a page that e recommends.
7. The relation that relates pages p_1 and p_2 iff p_2 can be reached from p_1 by following a sequence of exactly 3 links.

Exercise 4.6

We have seen during Lecture 4 that if A and B are finite sets, then $|A| = |B|$ if and only if A bij B. In this exercise, we will prove a little more.

First, a definition. For any real number of x, the *ceiling* of x is the smallest integer k such that $k \geq x$: we denote such smallest k as $\lceil x \rceil$. For example, $\lceil 17 \rceil = 17$, $\lceil \pi \rceil = 4$, and $\lceil -\pi \rceil = -3$.

Now, a fact:

Lemma (The pigeonhole principle)**.** *Let m and b be positive integers. If m objects are placed into b boxes, then in the end at least one box will contain at least $\lceil m/b \rceil$ objects.*

You are *not* required to prove the pigeonhole principle (though it is a good exercise). However, you might want to use it to prove the following:

Theorem (Theorem E4.6)**.** *Let A and B be finite sets with $|A| = |B| = n$, and let $f : A \to B$ be a total function. Then f is injective if and only if f is surjective.*

Hint: Consider the arrows in the graph of f. Also, you may assume $n \geq 2$.

Exercise 4.7 (cf. Problem 4.37)

Let A and B be sets, both having *two or more* elements. We know from lecture 4 and exercise session 4 that if A and B are also finite, then $|A \times B|$ is larger than both $|A|$ and $|B|$, so there cannot be a bijection from $A \times B$ to either A or B.

Let now $A = \{0, 1\}$ and let A^ω (read: A to the omega) the set of *infinite binary strings,* where we write $x \in A^\omega$ as $x_0 x_1 x_2 \ldots x_n \ldots$ with $n \in \mathbb{N}$. Construct a bijection from $A^\omega \times A^\omega$ to A^ω. *Hint:* even positions and odd positions.

(This is a small taste of what we will discuss in Lecture 8.)

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This page too.

Solutions

Exercise 4.1

Let C be the set of counterexamples:

$$C = \{c \geq 1 \mid \exists A, B_1, \ldots, B_c \,.\, A \cap (B_1 \cup \ldots \cup B_c) \neq (A \cap B_1) \cup \ldots \cup (A \cap B_c)\} \ .$$

By contradiction, assume that C is nonempty: by the Well Ordering Principle, C has a minimum m. Then it must be $m \geq 3$, because for $n = 1$ the equality is trivially satisfied, and for $n = 2$ we have:

$$\begin{array}{rll} x \in A \cap (B_1 \cup B_2) & \textbf{iff} & x \in A \textbf{ and } (x \in B_1 \textbf{ or } x \in B_2) \\ & \textbf{iff} & (x \in A \textbf{ and } x \in B_1) \textbf{ or } (x \in A \textbf{ and } x \in B_2) \\ & \textbf{iff} & (x \in A \cap B_1) \textbf{ or } (x \in A \cap B_2) \\ & \textbf{iff} & x \in (A \cap B_1) \cup (A \cap B_2) \,. \end{array}$$

Let then the sets $A, B_1, \ldots, B_m$ be such that:

$$A \cap (B_1 \cup B_2 \cup \ldots \cup B_m) \neq (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_m) \,.$$

As $m \geq 3$, $m - 1$ is still a positive integer, and as it is smaller than m, for the sets $A, B_1, \ldots, B_{m-1}$ the equality holds:

$$A \cap (B_1 \cup B_2 \cup \ldots \cup B_{m-1}) = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_{m-1}) \,.$$

But then,

$$\begin{array}{rcl} A \cap (B_1 \cup B_2 \cup \ldots \cup B_m) & = & A \cap ((B_1 \cup B_2 \cup \ldots \cup B_{m-1}) \cup B_m) \\ & = & (A \cap (B_1 \cup B_2 \cup \ldots \cup B_{m-1})) \cup (A \cap B_m) \\ & = & (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_{m-1}) \cup (A \cap B_m) \ : \end{array}$$

contradiction.

Exercise 4.2

If the initial set $\{a, b, c, d\}$ has four elements, then the first player can take away as the first move either a subset of cardinality 1, or a subset of cardinality 2, or a subset of cardinality 3.

1. Let's start by supposing that the first player takes away a subset of one element, say, $\{a\}$. If the second player chooses $\{b, c, d\}$, then any subset taken away in the next moves cannot contain a, so it will be a

subset of $\{b, c, d\}$. This means that the second player has turned the game on four objects into a new game on three objects, in which they are still the second player; and we know that the second player has a winning strategy if the initial set has three elements.

2. We now notice that the second player can reason similarly if the first player takes away as the first move a subset of cardinality 3, say, $\{a, b, c\}$. If the second player chooses $\{d\}$, then any subset taken away in the next moves cannot contain d, so it will be a subset of $\{a, b, c\}$. This means that the second player has once again turned the game on four objects into a new game on three objects, in which they are still the second player; and we know that the second player has a winning strategy if the initial set has three elements.

3. The last case, where the first move takes away a subset of cardinality 2, say, $\{a, b\}$, requires more care. For example, if the second player takes away $\{c, d\}$, then the moves $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, and $\{b, d\}$ are still allowed. However, if the first player take away a subset of cardinality 2 and the second player responds by taking away the complement, then *no subset with three elements can be chosen anymore:* for example, any subset of $\{a, b, c, d\}$ of cardinality 3 contains either both a and b, or both c and d. Also, there are six subsets of cardinality 2 of a set of cardinality 4, so while the first player keeps taking away subsets with two elements, the second player can always respond by taking the complement.

   Sooner or later, the first player will have to start taking singletons; let's say they take $\{a\}$. If the second player takes, to fix the ideas, $\{b\}$ (more in general, if they take a singleton $\{x\}$ such that $\{a, x\}$ was one of the previous moves) then the moves $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, and $\{b, d\}$ all become illegal. But now the game has become a new game on the two-elements set $\{c, d\}$, where the original second player is still the second player, and has a winning strategy.

Exercise 4.3

1. This is a bijection: $y = x + 2$ if and only if $x = y - 2$.

2. This is also a bijection: $y = 2x$ if and only if $x = y/2$.

3. This is not a bijection! This is actually neither surjective, not injective: it is not surjective, because if $y < 0$, then for no x it is $x^2 = y$; it is not injective, because for both $x = 1$ and $x = -1$ we have $x^2 = 1$.

4. This is a bijection: $y = x^3$ if and only if $x = \sqrt[3]{y}$, and the cubic root of a real number is always defined, and has the same sign as the number.

5. This is neither a surjection, because for no x it is $\sin x = 2$; nor an injection, because $\sin 0 = \sin \pi$. (Remember that trigonometric functions consider angles as measured in radians.)

6. This is not a bijection, because $0 \sin 0 = \pi \sin \pi = 0$, so it is not injective. However, it is surjective! The reason is that the function is continuous and takes value x whenever $x = \dfrac{\pi}{2} + 2k\pi$, where k is an arbitrary integer. By the intermediate value theorem, a continuous function defined in a closed and bounded interval takes every value between its minimum and its maximum: thus, for every $y \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $x \sin x = y$.

7. This function is not surjective, because it only takes positive values; however, it is injective as it is strictly increasing.

Exercise 4.4

1. Let B be a nonempty set and let $f : A \to B$ be a surjective function. We can then define a total surjective function $g : A \to B$ by fixing an element $b_0 \in B$, and putting $g(x) = f(x)$ if $f(x)$ is defined, and $g(x) = b_0$ if $f(x)$ is undefined.

2. If B is empty, then the empty relation is the only surjective function from A to B, but it is total if and only if A is also empty.

Exercise 4.5

1. We want the pages which mention "logic" but do not mention "predicate". This corresponds to the difference set of $M^{-1}(\text{“logic”})$ with $M^{-1}(\text{“predicate”})$. So the set we need is:

$$A ::= M^{-1}(\text{“logic”}) - M^{-1}(\text{“predicate”})\,.$$

2. We want the pages which not only contain the word "set", but are also recommended by Meyer. This corresponds to the intersection of $M^{-1}(\text{“set”})$ of the pages where the word "set" is mentioned with the set $R(\text{Meyer})$ of the pages which Meyer recommends. So the set we need is:

$$B ::= M^{-1}(\text{“set”}) \cap R(\text{Meyer})\,.$$

3. We have to make two steps here: first, identify the pages which contain the word "algebra"; then, identify the people who endorse those pages. We know that the set of the pages which contain a word w is $M^{-1}(w)$, and that the set of endorsers of a page p is $E^{-1}(p)$. Thus, to find the set of endorsers who have recommended pages containing the word "algebra" we first apply M^{-1} to "algebra", then E^{-1} to M^{-1}("algebra"). So the set we need is:

$$C ::= (E^{-1} \circ M^{-1})(\text{"algebra"}) .$$

4. A page p has an incoming link if and only if there exists a page q such that qLp, and has an outgoing link if and only if there exists a page r such that pLr. The set of the q's which satisfy qLp is $L^{-1}(p)$, and the set of the r's which satisfy pLr is $L(p)$. As at least one of these must happen, the relation we look for is:

$$D ::= L^{-1}(p) \cup L(p) .$$

5. Let F be the relation we are looking for. We require that pFw if and only if there exists a page q such that qLp and qMw; this happens if and only if $pL^{-1}q$ and qMp. Then $F = M \circ L^{-1}$.

6. Let G be the relation we are looking for. We want that wGe if and only if there exists a page p such that, for some page q, it happens that pMw, pLq, and eRq. This is the same as asking that p and q satisfy $wM^{-1}p$, pLq, and $qR^{-1}e$. Then:

$$G = R^{-1} \circ L \circ M^{-1} .$$

   There is no need to use parentheses because, as the reader can verify[1], composition of relations is associative: however given three relations $R : A \to B$, $S : B \to C$, and $T : C \to D$, calling $U = T \circ (S \circ R)$ and $V = (T \circ S) \circ R$, we have aUd if and only if aVd, whatever $a \in A$ and $d \in D$ are.

7. Let H be the relation we are looking for. We want that p_1Hp_2 if and only if there exist words q_1 and q_2 such that p_1Lq_1, q_2Lp_2, and q_2Lp_2. Then:

$$H = E \circ E \circ E = E^3 .$$

   Note the new notation that we have introduced: a composition of n instances of a relation R is denoted by R^n.

[1]And if you haven't, you should!

Exercise 4.6

As in the thesis of Theorem E4.6, let A and B be finite sets with $|A| = |B| = n$, and let $f : A \to B$ be a total function. This means that f has both the [≥ 1 **out**] (for being total) and the [≤ 1 **out**] (for being a function) property, so it has the [$= 1$ **out**] property: in particular, its relation graph has exactly n arrows.

To prove Theorem E4.6, we prove that the *negations* of the two statements in the "if and only if" are equivalent: that is, the total function f is *not* injective if and only if it is *not* surjective. We observe that we only need to do so if $n \geq 2$, because for $n = 1$ and $n = 0$ there is only one total function from A to B, and that function is a bijection. Following the hint, we reason in terms of arrows.

- First, assume that f is not injective. Then f does not have the [≤ 1 **in**] property, so there will be two arrows (maybe more, but we only need two) that will point to the same element b_0 of B. Then at most $n - 2$ *other* elements of B can have entering arrows, so at most $n - 1$ of the n elements of B will have entering arrows. Then f doesn't have the [≥ 1 **in**] property, so it is not surjective.

- Next, assume that f is not surjective. Then f does not have the [≥ 1 **in**] property, so there will be an element b_0 of B which has no arrows entering. Then we have $m = n$ objects, the arrows, to put into $b \leq n - 1$ boxes, the elements of B to which they point. By the pigeonhole principle, at least one element of B will have at least:

$$\left\lceil \frac{m}{b} \right\rceil \geq \left\lceil \frac{n}{n-1} \right\rceil = 2$$

  arrow pointing to it. (Note that the ceiling function is *weakly increasing:* if $x < y$, then $\lceil x \rceil \leq \lceil y \rceil$.) Then f does not have the [≤ 1 **in**] property, so it is not injective.

Exercise 4.7

Let $x = x_0x_1x_2\ldots$ and $y = y_0y_1y_2\ldots$ be infinite binary strings. Define $f(x, y)$ bit by bit as follows:

$$(f(x,y))_i = \begin{cases} x_{i/2} & \text{if } i \text{ is even,} \\ y_{(i-1)/2} & \text{if } i \text{ is odd.} \end{cases} \tag{1}$$

That is, $f(x, y) = x_0y_0x_1y_1x_2y_2\ldots$

- f is a function. As soon as the arguments x and y are given, the value $f(x, y)$ is determined once and for all.
- f is total. The definition given by (1) can be applied to any pair (x, y) of infinite binary strings.
- f is injective. Assume $f(x, y) = f(z, w)$: then $(f(x, y))_i = (f(z, w))_i$ for every $i \in \mathbb{N}$. Two cases are possible:
  1. i is even. Write $i = 2j$ for suitable $j \in \mathbb{N}$. Then:
     $$x_j = (f(x, y))_i = (f(z, w))_i = z_j \, .$$
     As this is true for every $j \in \mathbb{N}$, we conclude that $x = z$.
  2. i is odd. Write $i = 2j + 1$ for suitable $j \in \mathbb{N}$. Then:
     $$y_j = (f(x, y))_i = (f(z, w))_i = w_j \, .$$
     As this is true for every $j \in \mathbb{N}$, we conclude that $y = w$.

  We have thus proved that if $f(x, y) = f(z, w)$, then $(x, y) = (z, w)$. As this holds for every two pairs $(x, y), (z, w)$, f is injective.
- f is surjective. Let z be an infinite binary sequence. Define:
  $$\begin{aligned} x_i &= z_{2i} \text{ for every } i \in \mathbb{N} \, , \\ y_i &= z_{2i+1} \text{ for every } i \in \mathbb{N} \, . \end{aligned}$$
  It is straightforward to prove that $f(x, y) = z$. As this can be done with any infinite binary string z, f is surjective.

The string $z = f(x, y)$ is called the *interleaving* of the strings x and y (in this order).