Mathematics for Computer Science
Self-evaluation exercises for Lecture 3

Silvio Capobianco

Last update: 19 September 2024

Exercise 3.1 (from the classroom test of 3 October 2018)

Find a disjunctive normal form for the following formula:

$$(P \textbf{ or } Q) \textbf{ implies not}(R \textbf{ and } P)$$

Use either a truth table, or logical equivalences.

Exercise 3.2 (cf. Problem 1.16(d))

A finite set of propositional formulas $X = \{P_1, \dots, P_n\}$ is *consistent* if there exists an assignment of truth values to *all* the variables which appear in *any* formulas in which *all* propositions are true. For example:

- The set $\{P \textbf{ and not}(Q), Q \textbf{ or } R\}$ is consistent, because setting $P = \mathbf{T}$, $Q = \mathbf{F}$, and $R = \mathbf{T}$ makes both P **and not**(Q) and Q **or** R true.

- The set $\{A \textbf{ and not}(A)\}$ is not consistent, because A **and not**(A) is unsatisfiable.

Construct a formula S such that S is valid if and only if X is *not* consistent.

Exercise 3.3 (from the midterm test of 30 September 2022)

Determine a disjunctive normal form for the following propositional formula:

$$(P \vee Q) \wedge (P \vee (\overline{Q} \wedge R)) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \tag{1}$$

Any DNF for (1) will be accepted as a solution; it doesn't need to be full.

Exercise 3.4 (cf. Problem 3.28)

Express each of the following statements using quantifiers, logical connectives, and/or the following predicates:

- $P(x)$ ::= 'x is a monkey"
- $Q(x)$ ::= 'x is a 6.042 TA"
- $R(x)$ ::= 'x comes from the 23rd century"
- $S(x)$ ::= 'x likes to eat pizza"

where x ranges over all living things.

(a) No monkey likes to eat pizza.

(b) Nobody from the 23rd century dislikes eating pizza.

(c) All 6.042 TAs are monkeys.

(d) No 6.042 TA comes from the 23rd century.

(e) Does part (d) follow from parts (a), (b), and (c)? If so, give a proof. If not, give a counterexample.

Hint: Contradiction.

(f) Translate into English: $\forall x \,.\, (R(x) \textbf{ or } S(x) \textbf{ implies } Q(x))$

(g) Translate into English:

$$\exists x \,.\, (R(x) \textbf{ and not}(Q)(x)) \textbf{ implies } \forall x \,.\, (P(x) \textbf{ implies } S(x))$$

Exercise 3.5 (cf. Problems 3.29, 3.30, and 3.31)

Find counter-models for the following predicate formulas:

1. $(\forall x \,.\, \exists y \,.\, P(x, y)) \textbf{ implies } \forall z \,.\, P(z, z)$
2. $\exists x \,.\, P(x) \textbf{ implies } \forall x \,.\, P(x)$
3. $(\exists x \,.\, P(x) \textbf{ and } \exists x \,.\, Q(x)) \textbf{ implies } \exists x \,.\, (P(x) \textbf{ and } Q(x))$

Hint: use arithmetics of nonnegative integers as the environment and the set of natural numbers as the type of all variables.

Exercise 3.6 (from the classroom test of 3 October 2018)

Find a counter-model for the following predicate formula:

$$(\exists x\,.\,\forall y\,.\,(P(x)\ \textbf{implies}\ Q(y)))\ \ \textbf{implies}\ \ (\forall x\,.\,(P(x)\ \textbf{implies}\ \exists y\,.\,Q(y)))\ .$$

Exercise 3.7 (from the midterm test of 1st October 2021, expanded)

Let F be a propositional formula depending on the propositional variables $P_1, P_2, \ldots, P_n$. Let now $G(x)$ be the predicate formula obtained by starting from F and replacing, for every i from 1 to n, every occurrence of the propositional variable P_i with a predicate $Q_i(x)$, where the variable x is the same for all predicates. For example:

- If $F ::= P_1$ **and** $(P_2$ **or** $P_3)$, then $G(x) ::= Q_1(x)$ **and** $(Q_2(x)$ **or** $Q_3(x))$.
- If $F ::= P_1$ **implies** $(P_2$ **implies** $P_1)$, then

  $G(x) ::= Q_1(x)$ **implies** $(Q_2(x)$ **implies** $Q_1(x))$.

Your tasks for this exercise:

1. Prove that if the propositional formula F is valid, then the predicate formula $\forall x\,.\,G(x)$ is also valid, in the sense that it doesn't have any counter-models.
2. Prove that if the predicate formula $\forall x\,.\,G(x)$ is valid (again, in the sense that it doesn't have any counter-models) then the propositional formula F is valid.

Hint: proof by contraposition.

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Solutions

Exercise 3.1

For a truth table:

P	Q	R	(P **or** Q)	**implies**	**not**(R **and** P)
T	**T**	**T**	**T**	**F**	**F**
T	**T**	**F**	**T**	**T**	**T**
T	**F**	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**T**	**T**
F	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**F**	**T**	**T**
F	**F**	**F**	**F**	**T**	**T**

Choosing the lines where the formula is true, we reach the full disjunctive normal form:

$$\begin{aligned}(P \textbf{ and } Q \textbf{ and } \overline{R}) \quad & \textbf{or} \quad (P \textbf{ and } \overline{Q} \textbf{ and } \overline{R}) \\ & \textbf{or} \quad (\overline{P} \textbf{ and } Q \textbf{ and } R) \\ & \textbf{or} \quad (\overline{P} \textbf{ and } Q \textbf{ and } \overline{R}) \\ & \textbf{or} \quad (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } R) \\ & \textbf{or} \quad (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } \overline{R})\end{aligned}$$

For logical equivalences:

1. First, we rewrite the implication:

   $$((P \textbf{ or } Q) \textbf{ implies not}(R \textbf{ and } P)) \textbf{ iff } (\textbf{not}(P \textbf{ or } Q) \textbf{ or not}(R \textbf{ and } P))$$

2. Next, we apply de Morgan's laws to only have negation on single variables:

   $$\begin{aligned}\textbf{not}(P \textbf{ or } Q) \quad & \textbf{iff} \quad \overline{P} \textbf{ and } \overline{Q} \\ \textbf{not}(R \textbf{ and } P) \quad & \textbf{iff} \quad \overline{R} \textbf{ or } \overline{P}\end{aligned}$$

   and by applying associativity we get the following formula, equivalent to the original one:

   $$(\overline{P} \textbf{ and } \overline{Q}) \textbf{ or } \overline{R} \textbf{ or } \overline{P}$$

3. The formula above is a disjunction of conjunctions, so we can apply distributivity and the equivalence A **iff** A **and** $(B$ **or** $\mathbf{not}(B))$ to rewrite each term of the disjunction as a conjunction so that P, Q and R, or their negations, appear exactly once:

$$\begin{aligned}\overline{P} \textbf{ and } \overline{Q} \quad &\textbf{iff} \quad \overline{P} \textbf{ and } \overline{Q} \textbf{ and } (R \textbf{ or } \overline{R}) \\ &\textbf{iff} \quad (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } R) \textbf{ or } (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } \overline{R})\,;\end{aligned}$$

$$\begin{aligned}\overline{R} \quad &\textbf{iff} \quad (P \textbf{ or } \overline{P}) \textbf{ and } \overline{R} \\ &\textbf{iff} \quad (P \textbf{ and } \overline{R}) \textbf{ or } (\overline{P} \textbf{ and } \overline{R}) \\ &\textbf{iff} \quad (P \textbf{ and } Q \textbf{ and } \overline{R}) \textbf{ or } (P \textbf{ and } \overline{Q} \textbf{ or } \overline{R}) \\ &\qquad \textbf{ or } (\overline{P} \textbf{ and } Q \textbf{ and } \overline{R}) \textbf{ or } (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } \overline{R})\,;\end{aligned}$$

$$\begin{aligned}\overline{P} \quad &\textbf{iff} \quad \overline{P} \textbf{ or } (Q \textbf{ or } \overline{Q}) \\ &\textbf{iff} \quad (\overline{P} \textbf{ and } Q) \textbf{ or } (\overline{P} \textbf{ and } \overline{Q}) \\ &\textbf{iff} \quad (\overline{P} \textbf{ and } Q \textbf{ and } R) \textbf{ or } (\overline{P} \textbf{ and } Q \textbf{ and } \overline{R}) \\ &\qquad \textbf{ or } (\overline{P} \textbf{ and } Q \textbf{ and } \overline{R}) \textbf{ or } (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } \overline{R})\,.\end{aligned}$$

4. By substituting equivalent formulas and applying commutativity and absorption, we reach precisely the disjunctive normal form we have found earlier.

Exercise 3.2

We first consider a "dual" form of the problem by considering a formula T which is *satisfiable* (instead of valid) if and only if X is consistent. Such formula is clearly the conjunction of the finitely many formulas that appear in X:

$$T ::= P_1 \textbf{ and } P_2 \textbf{ and } \ldots \textbf{ and } P_n\,.$$

Such formula is also *un*satisfiable if and only if X is *not* consistent. But we know that a formula is unsatisfiable if and only if its negation is valid. Then the formula S that we are looking for is simply the negation of T:

$$\begin{aligned}S ::= \mathbf{not}(T) \quad &= \quad \mathbf{not}(P_1 \textbf{ and } P_2 \textbf{ and } \ldots \textbf{ and } P_n) \\ &\longleftrightarrow \quad \mathbf{not}(P_1) \textbf{ or } \mathbf{not}(P_2) \textbf{ or } \ldots \textbf{ or } \mathbf{not}(P_n)\,.\end{aligned}$$

Exercise 3.3

We examine a solution with truth tables, and one with Boolean algebra.

- Truth table:

P	Q	R	$((P \vee Q)$	$\wedge$	$(P\vee$	$(\overline{Q} \wedge R)))$	$\wedge$	$(P\vee$	$(\overline{Q} \wedge \overline{R}))$
T	**T**	**T**	**T**	**T**	**T**	**F**	**T**	**T**	**F**
T	**T**	**F**	**T**	**T**	**T**	**F**	**T**	**T**	**F**
T	**F**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**F**
T	**F**	**F**	**T**	**T**	**T**	**F**	**T**	**T**	**T**
F	**T**	**T**	**T**	**F**	**F**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**	**F**	**F**
F	**F**	**T**	**F**	**F**	**T**	**T**	**F**	**F**	**F**
F	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**T**	**T**

  This gives the full DNF:

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \overline{R}) \vee (P \wedge \overline{Q} \wedge R) \vee (P \wedge \overline{Q} \wedge \overline{R})$$

  It can also be observed from the truth table that the formula (1) is true if and only if P is true, so it is equivalent to P, which is already a DNF.

- Boolean algebra:

$$\begin{aligned}
& (P \vee Q) \wedge (P \vee (\overline{Q} \wedge R)) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & (P \wedge (Q \vee (\overline{Q} \vee R))) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & (P \wedge ((Q \vee \overline{Q}) \vee R)) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & (P \wedge (\mathbf{T} \vee R)) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & (P \wedge \mathbf{T}) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & P \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & P
\end{aligned}$$

Exercise 3.4

(a) $\forall x \,.\, (P(x)$ **implies not**$(S(x)))$.

(b) $\forall x \,.\, (R(x)$ **implies** $S(x))$.

(c) $\forall x \,.\, (Q(x)$ **implies** $P(x))$.

(d) $\forall x \,.\, (R(x)$ **implies not**$(Q(x)))$.

(e) Yes, it does. Suppose parts (a), (b), and (a) are all true. By contradiction, assume that (d) is false. Then there exists an x_0 which is a 6.042 TA and comes from the 23rd century. On the one hand, as x_0 comes from the 23rd century, by (b,) they like eating pizza. On the other hand, as x_0 is a 6.042 TA, by (c), they are a monkey. But then, x_0 is a monkey who likes eating pizza, which contradicts (a).

(f) Anyone who either comes from the 23rd century or likes to eat pizza is a 6.042TA.

(g) If there is someone who comes from the 23rd century but is not a 6.042TA, then every monkey likes to eat pizza.

Exercise 3.5

1. Interpret $P(x, y)$ as "$x < y$". Then the formula means "if for every natural number there exists a larger natural number, then every natural number is smaller than itself", which is false.

2. Interpret $P(x)$ as "$x = 2$". Then the formula means "if there is a natural number equal to 2, then all natural numbers are equal to 2", which is false.

3. Interpret $P(x)$ as "$x > 17$" and $Q(x)$ as "$x < 17$". Then the formula means "if there exists a natural number larger than 17 and there exists a natural number smaller than 17, then there exists a natural number that is larger er and smaller than 17 at the same time", which is false.

Exercise 3.6

We want the main implication to be false, so the premise must be true and the conclusion must be false. Now, if $P(x)$ is false for some x, then for *that* x and for *every* y the formula $P(x)$ **implies** $Q(y)$ is true; on the other hand, if $P(x)$ is true for some x but $Q(y)$ is false for every y, then for *that* x the formula $P(x)$ **implies** $\exists y \,.\, Q(y)$ is false.

Let then x and y take values in the set $\mathbb{N}$ of nonnegative integers; let $P(x) ::= x = 0$ and $Q(y) ::= x < 0$. Then the premise of the main implication becomes:

$$\exists x \in \mathbb{N} \,.\, \forall y \in \mathbb{N} \,.\, (x = 0 \textbf{ implies } y < 0)\,,$$

which is true, because we can set $x = 1$; but the conclusion becomes:

$$\forall x \in \mathbb{N} \,.\, (x = 0 \textbf{ implies } \exists y \in \mathbb{N} \,.\, y < 0)\,,$$

which is false, because for $x = 0$ the implication has a true antecedent and a false consequent.

Another solution, suggested in classroom during a previous edition of this course, goes as follows. As we only need two values for x and one for y, we could choose a domain where $x \in X = \{x_{\mathbf{T}}, x_{\mathbf{F}}\}$, $y \in Y = \{y_{\mathbf{F}}\}$, $P(x_{\mathbf{T}}) = \mathbf{T}$, $P(x_{\mathbf{F}}) = \mathbf{F}$, and $Q(y_{\mathbf{F}}) = \mathbf{F}$. Indeed, we could just choose $x_{\mathbf{T}} = \mathbf{T}$, $x_{\mathbf{F}} = y_{\mathbf{F}} = \mathbf{F}$, $P(x) ::= x$, and $Q(y) ::= y$. Then $P(x_{\mathbf{F}})$ **implies** $\forall y \,.\, Q(y)$ is interpreted as $\mathbf{F}$ **implies** $\forall y \,.\, Q(y)$, which is true; but with this choice of the type of y, $Q(y)$ can only be false, so $P(x_{\mathbf{T}})$ **implies** $\exists y \,.\, Q(y)$ is interpreted as $\mathbf{T}$ **implies** $\exists y \,.\, \mathbf{F}$, which is false.

Exercise 3.7

1. We prove the contrapositive: if $\forall x \,.\, G(x)$ *does* have a counter-model, then F is *not* valid.

   Consider a domain D, a type X for the variable x, and an interpretation of the predicates $Q_1(x), \ldots, Q_n(x)$ that makes $\forall x \,.\, G(x)$ false. By definition, there exists $x_0 \in X$ such that $G(x_0)$ is false. Define the truth values of the variables P_i of F as being the same as those of the corresponding propositions $Q_i(x_0)$ in the counter-model we have defined: that is, if $Q_i(x_0)$ is true in the counter-model, then $P_i = \mathbf{T}$, and if $Q_i(x_0)$ is false in the counter-model, then $P_i = \mathbf{F}$. By construction, this assignment of truth values makes F false, so F is not valid.

2. We prove the contrapositive: if F is *not* valid, then $\forall x \,.\, G(x)$ *does* have a counter-model.

   Assume that for a certain assignment of truth values to the variables $P_1, P_2, \ldots, P_n$ the formula F is false. Choose one of these assignment: call it A, just for convenience. Now construct a counter-model for $\forall x \,.\, G(x)$ as follows:

   - The domain is the arithmetics of natural numbers.
   - The type of the variable x is the set of natural numbers.
   - The interpretation of $Q_i(x)$ is "the variable P_i is true in the assignment A".

   (This isn't the simplest possible counter-model, but works well enough as an example.) Then, whatever the formula F is, the truth value of $Q_i(0)$ is the same as the one that P_i has in the assignment A: this means that the two formulas F and $G(0)$ are either both true or both false. As F is false in the assignment A, taking $x = 0$ we conclude that our choices of domain, types of variables, and interpretations of predicates make the formula $\forall x \,.\, G(x)$ false.