Mathematics for Computer Science
Self-evaluation exercises for Week 4

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Exercise 4.1 (from the midterm test of 7 October 2020)

For $n \geq 3$ we define the union and intersection of n sets $B_1, \ldots, B_n$ recursively as follows:

$$\begin{aligned} B_1 \cup B_2 \cup \ldots \cup B_{n-1} \cup B_n &= (B_1 \cup B_2 \cup \ldots \cup B_{n-1}) \cup B_n \,; \\ B_1 \cap B_2 \cap \ldots \cap B_{n-1} \cap B_n &= (B_1 \cap B_2 \cap \ldots \cap B_{n-1}) \cap B_n \,. \end{aligned}$$

Use the Well Ordering Principle to prove the following: if n is an arbitrary positive integer and $A, B_1, B_2, \ldots, B_n$, are arbitrary sets, then

$$A \cap (B_1 \cup B_2 \cup \ldots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n)\,.$$

Hint: start with proving that, if m is the minimum counterexample, then $m \geq 3$. **Important:** Any solutions that do not use the Well Ordering Principle will receive zero points.

Exercise 4.2

Recall the rules of the Subset Take-Away game:

1. The initial position is a finite nonempty set.
2. Taking turns, the players take away subsets of the initial set.
3. It is not permitted to take away the entire initial set as the first move.
4. Once a subset has been taken away, no subset which contains it can be taken away anymore.

   In particular: no subset can be taken more than once.

5. A player who cannot take away a nonempty subset on his or her turn, loses the game.

We have seen in classroom that if the initial set has either 1, 2, or 3 elements, then the second player has a winning strategy.

Prove that if the initial set has 4 elements, then the second player still has a winning strategy. *Hint:* consider the cases where the first player chooses a subset with one, three, or two elements, the last case being the trickiest one.

Exercise 4.3 (cf. Problem 4.19)

Before you try this exercise, you might want to revise what you know from your Calculus course.

For each of the following real-valued total functions on the real numbers, indicate whether it is a bijection, a surjection but not a bijection, an injection but not a bijection, or neither an injection nor a surjection.

1. $f_1(x) = x + 2$.
2. $f_2(x) = 2x$.
3. $f_3(x) = x^2$.
4. $f_4(x) = x^3$.
5. $f_5(x) = \sin x$.
6. $f_6(x) = x \sin x$. *Hint:* intermediate value theorem.
7. $f_7(x) = e^x$.

Exercise 4.4

1. Let A and B be sets. Give an intuitive reason why, if A inj B, then there exists a total injective *function* from A to B.
2. Prove a similar, but not identical, fact for the "surject" relation: if A surj B *and in addition* B *is nonempty*, then there exists a *total* surjective function from A to B.
3. What happens to the previous point if B is empty?

Exercise 4.5 (cf. Problem 4.29(c),(e),(f),(h))

Recall the components of the search engine of Problem 4.29 which we discussed in Exercise session 4 and its addition:

- A set P of *pages* that the search engine knows about.
- A binary relation L (for *link*) over pages, defined such that $p_1 L p_2$ if and only if p_1 links to p_2.
- A set E of *endorsers*, people who have recorded their opinions about which pages are high-quality.
- A binary relation R (for *recommends*) between endorsers and pages, such that eRp iff person e has recommended page p.
- A set W of *words* that may appear on pages.
- A binary relation M (for *mentions*) between pages and words, where pMw iff word w appears on page p.

Use the sets P, E, and W, the relations L, R, and M, and the usual operations on sets (union, intersection, difference, complement) and on relations (composition, inversion, image, inverse image) to express the following:

(c) The set of endorsers who have recommended pages containing the word "algebra".

(e) The set of pages that have at least one incoming or outgoing link.

(f) The relation that relates word w and page p iff w appears on a page that links to p.

(h) The relation that relates pages p_1 and p_2 iff p_2 can be reached from p_1 by following a sequence of exactly 3 links.

Exercise 4.6

We have seen during Lecture 4 that if A and B are finite sets, then $|A| = |B|$ if and only if A bij B. In this exercise, we will prove a little more.

First, a definition. For any real number of x, the *ceiling* of x is the smallest integer k such that $k \geq x$: we denote such smallest k as $\lceil x \rceil$. For example, $\lceil 17 \rceil = 17$, $\lceil \pi \rceil = 4$, and $\lceil -\pi \rceil = -3$.

Now, a fact:

Lemma (The pigeonhole principle)**.** *Let m and b be positive integers. If m objects are placed into b boxes, then in the end at least one box will contain at least $\lceil m/b \rceil$ objects.*

You are *not* required to prove the pigeonhole principle (though it is a good exercise). However, you might want to use it to prove the following:

Theorem (Theorem E4.6)**.** *Let A and B be finite sets with $|A| = |B| = n$, and let $f : A \to B$ be a total function. Then f is injective if and only if f is surjective.*

Hint: Consider the arrows in the graph of f. Also, you may assume $n \geq 2$.

Exercise 4.7 (cf. Problem 4.37)

Let A and B be finite sets, both having *two or more* elements. From our discussion on Problem 4.39 in the addition to Exercise session 4 follows that $|A \times B|$ is larger than both $|A|$ and $|B|$, so there cannot be a bijection from either A or B to $A \times B$.

Let now A be the set of *infinite binary strings*, where we write $a \in A$ as the sequence of its bits, that is,

$$a = a_0 a_1 a_2 \ldots a_n \ldots$$

Define a bijection from A to $A \times A$, and prove that it is, indeed, a bijection. *Hint:* Regroup and split.

(This is a small taste of what we will discuss in Lecture 8.)

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Solutions

Exercise 4.1

Let C be the set of counterexamples:

$$C = \{c \geq 1 \mid \exists A, B_1, \ldots, B_c \,.\, A \cap (B_1 \cup \ldots \cup B_c) \neq (A \cap B_1) \cup \ldots \cup (A \cap B_c)\} \;.$$

By contradiction, assume that C is nonempty: by the Well Ordering Principle, C has a minimum m. Then it must be $m \geq 3$, because for $n = 1$ the equality is trivially satisfied, and for $n = 2$ we have:

$$\begin{array}{rcl} x \in A \cap (B_1 \cup B_2) & \textbf{iff} & x \in A \textbf{ and } (x \in B_1 \textbf{ or } x \in B_2) \\ & \textbf{iff} & (x \in A \textbf{ and } x \in B_1) \textbf{ or } (x \in A \textbf{ and } x \in B_2) \\ & \textbf{iff} & (x \in A \cap B_1) \textbf{ or } (x \in A \cap B_2) \\ & \textbf{iff} & x \in (A \cap B_1) \cup (A \cap B_2) \,. \end{array}$$

Let then the sets $A, B_1, \ldots, B_m$ be such that:

$$A \cap (B_1 \cup \ldots \cup B_m) \neq (A \cap B_1) \cup \ldots \cup (A \cap B_m) \,.$$

As $m \geq 3$, $m - 1$ is still a positive integer, and as it is smaller than m, for the sets $A, B_1, \ldots, B_{m-1}$ the equality holds:

$$A \cap (B_1 \cup \ldots \cup B_{m-1}) = (A \cap B_1) \cup \ldots \cup (A \cap B_{m-1}) \,.$$

But then,

$$\begin{array}{rcl} A \cap (B_1 \cup \ldots \cup B_m) & = & A \cap ((B_1 \cup \ldots \cup B_{m-1}) \cup B_m) \\ & = & (A \cap (B_1 \cup \ldots \cup B_{m-1})) \cup (A \cap B_m) \\ & = & (A \cap B_1) \cup \ldots \cup (A \cap B_{m-1}) \cup (A \cap B_m) \;: \end{array}$$

contradiction.

Exercise 4.2

If the initial set $\{a, b, c, d\}$ has four elements, then the first player can take away as the first move either a subset of cardinality 1, or a subset of cardinality 2, or a subset of cardinality 3.

1. Let's start by supposing that the first player takes away a subset of one element, say, $\{a\}$. If the second player chooses $\{b, c, d\}$, then any subset taken away in the next moves cannot contain a, so it will be a

subset of $\{b, c, d\}$. This means that the second player has turned the game on four objects into a new game on three objects, in which they are still the second player; and we know that the second player has a winning strategy if the initial set has three elements.

2. We now notice that the second player can reason similarly if the first player takes away as the first move a subset of cardinality 3, say, $\{a, b, c\}$. If the second player chooses $\{d\}$, then any subset taken away in the next moves cannot contain d, so it will be a subset of $\{a, b, c\}$. This means that the second player has once again turned the game on four objects into a new game on three objects, in which they are still the second player; and we know that the second player has a winning strategy if the initial set has three elements.

3. The last case, where the first move takes away a subset of cardinality 2, say, $\{a, b\}$, requires more care. For example, if the second player takes away $\{c, d\}$, then the moves $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, and $\{b, d\}$ are still allowed. However, if the first player take away a subset of cardinality 2 and the second player responds by taking away the complement, then *no subset with three elements can be chosen anymore:* for example, any subset of $\{a, b, c, d\}$ of cardinality 3 contains either both a and b, or both c and d. Also, there are six subsets of cardinality 2 of a set of cardinality 4, so while the first player keeps taking away subsets with two elements, the second player can always respond by taking the complement.

   Sooner or later, the first player will have to start taking singletons; let's say they take $\{a\}$. If the second player takes, to fix the ideas, $\{b\}$ (more in general, if they take a singleton $\{x\}$ such that $\{a, x\}$ was one of the previous moves) then the moves $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, and $\{b, d\}$ all become illegal. But now the game has become a new game on the two-elements set $\{c, d\}$, where the original second player is still the second player, and has a winning strategy.

Exercise 4.3

1. f_1 is a bijection: $y = x + 2$ if and only if $x = y - 2$.

2. f_2 is also a bijection: $y = 2x$ if and only if $x = y/2$.

3. f_3 is neither surjective, nor injective. It is not surjective, because if $y < 0$, then for no x it is $x^2 = y$. It is not injective, because for both $x = 1$ and $x = -1$ it is $x^2 = 1$.

4. f_4 is a bijection: $y = x^3$ if and only if $x = \sqrt[3]{y}$, and the cubic root of a real number is always defined, and has the same sign as the number.

5. f_5 is neither surjective, nor injective. It is not surjective, because for no x it is $\sin x = 2$. It is not injective, because $\sin 0 = \sin \pi$.

6. f_6 is not injective, because $0 \sin 0 = \pi \sin \pi = 0$; however, it is surjective. To see why, observe that f_6 is continuous on the entire real line as the product of two continuous functions. By the intermediate value theorem, given any two real numbers a and b with $a < b$, for every $y \in \mathbb{R}$ such that $\min(f_6(a), f_6(b)) < y < \max(f_6(a), f_6(b))$ there exists $x \in (a, b)$ such that $f_6(x) = y$. So let $y \in \mathbb{R}$ be arbitrary. If $y = 0$, take $x = 0$. If $y > 0$, take $k > 0$ so that $b = \dfrac{\pi}{2} + 2k\pi > y$: then, as $f_6(0) = 0$ and $f_6(b) = b$, there exists $x \in (0, b)$ such that $f_6(x) = y$. If $y < 0$, take $k < 0$ so that $a = \dfrac{\pi}{2} + 2k\pi < y$: then, as $f_6(0) = 0$ and $f_6(a) = a$, there exists $x \in (a, 0)$ such that $f_6(x) = y$.

7. f_7 is not surjective, because it only takes positive values; however, it is injective, because it is *strictly increasing*, that is, if $x < y$ then $f_7(x) < f_7(y)$.

Exercise 4.4

1. Let A be nonempty and let $R : A \to B$ be a total injective relation. Then R has the [≥ 1 **out**] and the [≤ 1 **in**] properties. We can then construct a relation which has the [$= 1$ **out**] and [≤ 1 **in**] properties—that is, a total injective function—by *choosing*, for every $a \in A$, exactly one $b \in B$ such that aRb, and defining $f(a)$ as *that* b. This relation has the [$= 1$ **out**] property by construction, and still has the [≤ 1 **in**] property, because we cannot add entering arrows by removing arrows.

   (The reason why this is an intuitive reason and not a proof, is that it isn't really clear *why we can make such choice at all.* We will see this in greater detail in Lecture 8.)

2. Let B be a nonempty set and let $f : A \to B$ be a surjective function. We can then define a total surjective function $g : A \to B$ by fixing an element $b_0 \in B$, and for every $a \in A$ putting $g(a) = f(a)$ if $f(a)$ is defined, and $g(a) = b_0$ if $f(a)$ is undefined.

3. If B is empty, then the empty relation is the only surjective function from A to B, but it is total if and only if A is also empty.

Exercise 4.5

(c) We want to express the set:

$$\{e \in E \mid \exists p \in P \,.\, eRp \wedge pM\text{“algebra”}\} \ .$$

This can be rewritten as:

$$\{e \in E \mid e(M \circ R)\text{“algebra”}\} \ ,$$

which is $(M \circ R)^{-1}$ (“algebra”); or as:

$$\left\{e \in E \,\middle|\, \exists p \in P \,.\, \text{“algebra”}\, M^{-1}p \wedge pR^{-1}e\right\} \ ,$$

which is $(R^{-1} \circ M^{-1})$ (“algebra”). That these two relations are equal, is not a case: it is true in general that, if $R : A \to B$ and $S : B \to C$ are relations, then $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. You might want to prove this fact as an additional exercise (or to look for a proof online).

(e) We want to express the set:

$$\{p \in P \mid (\exists q \in P \,.\, qLp) \vee (\exists r \in P \,.\, pLr)\} \ .$$

This is the union of the set:

$$\{p \in P \mid \exists q \in P \,.\, qLp\} \ ,$$

which is $L^{-1}(P)$, and of the set:

$$\{p \in P \mid \exists r \in P \,.\, pLr\} \ ,$$

which is $L(P)$.

(f) Let's provisionally denote by H the relation we are looking for. Then:

$$\begin{aligned} wHp &\longleftrightarrow \exists q \in P \,.\, (qMw \wedge qLp) \\ &\longleftrightarrow \exists q \in P \,.\, (wM^{-1}q \wedge qLp) \\ &\longleftrightarrow w(L \circ M^{-1})p \,. \end{aligned}$$

The relation H we are looking for is thus $L \circ M^{-1}$.

(h) Let's provisionally denote by Z the relation we are looking for. Then:

$$\begin{aligned} p_1Zp_2 &\longleftrightarrow \exists q \in P \,.\, (p_1Lq \wedge \exists r \in P \,.\, (qLr \wedge rLp_2)) \\ &\longleftrightarrow \exists q \in P \,.\, (p_1Lq \wedge q(L \circ L)p_2) \\ &\longleftrightarrow p_1(L \circ L \circ L)p_2 \,. \end{aligned}$$

We don't need additional parentheses, because we mentioned in the addition to Exercise session 4 that *composition is associative*. We can also denote the composition of three instances of L simply as L^3.

Exercise 4.6

As in the thesis of Theorem E4.6, let A and B be finite sets with $|A| = |B| = n$, and let $f : A \to B$ be a total function. If $n = 0$ then f is the empty relation with empty domain and codomain, which is a bijection; so let's assume $n \geq 1$. Then f has both the [≥ 1 **out**] (for being total) and the [≤ 1 **out**] (for being a function) property, so it has the [$= 1$ **out**] property: in particular, its relation graph has exactly n arrows. We will prove that the *negations* of the two statements in the "if and only if" are equivalent: that is, the total function f is *not* injective if and only if it is *not* surjective.

- First, assume that f is not injective. Then f does not have the [≤ 1 **in**] property, so there will be two arrows (maybe more, but we only need two) that will point to the same element b_0 of B. Then at most $n - 2$ *other* elements of B can have entering arrows, so at most $n - 1$ of the n elements of B will have entering arrows. Then f doesn't have the [≥ 1 **in**] property, so it is not surjective.

- Next, assume that f is not surjective. Then f does not have the [≥ 1 **in**] property, so there will be an element b_0 of B which has no arrows entering. Then we have $m = n$ objects, the arrows, to put into $b \leq n - 1$ boxes, the elements of B to which they point. By the pigeonhole principle, at least one element of B will have at least:

$$\left\lceil \frac{m}{b} \right\rceil \geq \left\lceil \frac{n}{n-1} \right\rceil = 2$$

  arrow pointing to it. (Note that the ceiling function is *weakly increasing:* if $x < y$, then $\lceil x \rceil \leq \lceil y \rceil$.) Then f does not have the [≤ 1 **in**] property, so it is not injective.

Exercise 4.7

Let $a \in A$. Let's write:

$$a = a_0 a_1 a_2 a_3 a_4 a_5 \ldots a_n \ldots$$

Let's follow the hint and, instead of reading every single bit, we read *blocks of two bits:*

$$[a_0 a_1][a_2 a_3][a_4 a_5] \ldots [a_{2n} a_{2n+1}] \ldots$$

For every $k \geq 0$, the kth block will have a_{2k} as its first element, and a_{2k+1} as its second element. We can then use the sequence of the first elements of the blocks as the first element of the pair of infinite sequences, and the sequence of the second elements of the blocks as the second element of the pair of infinite sequences. We then define $f : A \to A \times A$ as follows:

$$f\left(a_0 a_1 a_2 a_3 \dots a_{2n} a_{2n+1}\right) = \left(a_0 a_2 \dots a_{2n} \dots, a_1 a_3 \dots a_{2n+1} \dots\right) , \qquad (1)$$

that is, $f(a) = (b, c)$ where, for every $k \in \mathbb{N}$, $b_k = a_{2k}$ and $c_k = a_{2k+1}$). This f is, indeed, a bijection:

- f is a total function. As soon as $a \in A$ is given, there is a unique $(b, c) \in A \times A$ such that $f(a) = (b, c)$.
- f is injective. Assume $a, a' \in A$ differ at some index n, that is, $a_n \neq a'_n$. Let $f(a) = (b, c)$ and $f(a') = (b', c')$. If $n = 2k$ is even, then $b_k \neq b'_k$, so $b \neq b'$ and $(b, c) \neq (b', c')$ too; if $n = 2k + 1$ is odd, then $c_k \neq c'_k$, so $c \neq c'$ and $(b, c) \neq (b', c')$ too.
- f is surjective. Let $(b, c) \in A \times A$: we must determine an infinite binary string a such that $f(a) = (b, c)$. We do so by choosing, for every $n \in \mathbb{N}$, the nth bit of a as follows:

  $$a_n = \begin{cases} b_k & \text{if } n = 2k \text{ is even}\,, \\ c_k & \text{if } n = 2k+1 \text{ is odd}\,. \end{cases} \qquad (2)$$

  By construction, for every $k \in \mathbb{N}$ it is $b_k = a_{2k}$ and $c_k = a_{2k+1}$: thus, $f(a) = (b, c)$.

The sequence a defined by (2) is called the *interleaving* of b and c, in this order.