ITT9132 Concrete Mathematics
Introductory exercises

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Inclusion-exclusion formula

How many integer numbers between 1 and 1000 are divisible by either 7 or 11, but not both?

Solution. This exercise takes a twist on the classical *inclusion-exclusion formula* for the numbers of elements of two finite sets:

$$|A| + |B| = |A \cup B| + |A \cap B| \; . \tag{1}$$

The twist is that we are considering a *disjoint* union, so we have to remove $|A \cap B|$ *twice* from $|A| + |B|$.[1] That is, while for union we have:

$$|A \cup B| = |A| + |B| - |A \cap B| \; ,$$

for symmetric difference we have instead:

$$|A \Delta B| = |A| + |B| - 2\,|A \cap B| \; .$$

As $1000 = 142 \cdot 7 + 6 = 90 \cdot 11 + 10 = 12 \cdot 77 + 76$, there are 142 numbers between 1 and 1000 that are divisible by 7, 90 that are divisible by 11, and 12 that are divisible by 77. By the observation above, there are $142 + 90 - 12 \cdot 2 = 208$ integer numbers between 1 and 1000 divisible by either 7 or 11, but not both.

[1] Thanks to Ahto Truu for this remark.

The recurrence equation for binomial coefficients

Recall that, for integers $n \geq 0$ and $0 \leq k \leq n$, the *binomial coefficient* $\binom{n}{k}$, read "n choose k", is the number of ways we can choose k objects of a set with n elements, without taking into account the order in which we take them. Then:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}, \tag{2}$$

where $k!$ is the number of ways to order a set of k elements. Note that $0! = \binom{n}{0} = \binom{n}{n} = 1$ for every $n \geq 0$, and $k! = k \cdot (k-1)!$ for every $k \geq 1$.

Prove that, for every $n \geq 0$ and every $1 \leq k \leq n$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \tag{3}$$

Solution (By calculation)**.** As $0 < k < n+1$, the following derivation is valid:

$$\begin{aligned}
\binom{n+1}{k} &= \frac{(n+1)\cdot n\cdots(n+1-k+1)}{k!} \\
&= \frac{n+1}{k}\cdot\frac{n\cdots(n-(k-1)+1)}{(k-1)!} \\
&= \left(\frac{n+1-k}{k}+1\right)\cdot\binom{n}{k-1} \\
&= \frac{n-k+1}{k}\cdot\frac{n\cdots(n-(k-1)+1)}{(k-1)!} + \binom{n}{k-1} \\
&= \binom{n}{k} + \binom{n}{k-1}.
\end{aligned}$$

Solution (By interpretation)**.** The left-hand side of (3) is the number of ways we can choose k objects from a set with $n+1$ elements, without taking the order into account. To do so, we have to decide whether we choose the first element, or not.

No. If we don't choose the first element, then all the k objects must be chosen between the other n elements: which can be done in $\binom{n}{k}$ ways.

Yes. If we choose the first element, then the remaining $k-1$ objects will have to be chosen between the remaining n objects: which can be done in $\binom{n}{k-1}$ ways.

Then the total of ways is precisely the right-hand side of (3).

Newton's Binomial Theorem

Let $n \geq 0$ be integer and let $x, y \in \mathbb{C}$. Prove *Newton's binomial formula:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{4}$$

in two different ways:

1. by induction on n;
2. by expanding the product.

Solution. 1. We start with the proof by induction.

- **Base case:** $n = 0$. Then the left-hand side is $(x+y)^0$, and the right-hand side is $\sum_{k=0}^{0} x^k y^{0-k} = x^0 y^0$: both are equal to 1.
- **Inductive step:** Suppose that, for a certain n, it is indeed the case that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. Then:

$$\begin{aligned}
(x+y)^{n+1} &= (x+y) \cdot (x+y)^n \\
&= (x+y) \cdot \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \text{by inductive hypothesis} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} x^{k+1} y^{n-k} + \binom{n}{n} x^{n+1} \\
&\quad + \binom{n}{0} \binom{n}{0} y^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k}
\end{aligned}$$

At this point, we notice that $\binom{n}{0}$ and $\binom{n+1}{0}$ are both equal to 1; the same holds for $\binom{n}{n}$ and $\binom{n+1}{n+1}$. Additionally, if in the first sum we replace k with $h = k+1$, then k will vary from 1 to n, the exponent of x becomes $k+1 = h-1+1 = h$, and the exponent of y becomes $n-k = n-(h-1) = n+1-h$. If we first do these substitutions, then rename h as k again, we obtain:

$$\begin{aligned}
(x+y)^{n+1} &= \sum_{k=1}^{n}\binom{n}{k}x^{k+1}y^{n-k} + \binom{n+1}{n+1}x^{n+1} \\
&\quad + \binom{n+1}{0}\binom{n}{0}y^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^k y^{n+1-k} \\
&= \binom{n+1}{0}y^{n+1} + \binom{n+1}{n+1}x^{n+1} \\
&\quad + \sum_{k=1}^{n}\left(\binom{n}{k-1}+\binom{n}{k}\right)x^k y^{n+1-k} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}x^{k+1}y^{n+1-k} \ :
\end{aligned}$$

that is, the formula also holds for $n+1$.

We have thus proved the following:

(a) the formula (4) is true for $n = 0$;

(b) if the formula (4) is true for a given n, then it is also true with $n+1$ in place of n.

By the *principle of mathematical induction*, the formula (4) is true for every integer $n \geq 0$.

2. We now give the second argument, which does not use induction. Having fixed x, y, and n, the power $(x+y)^n$ is a product of n factors, all equal to $x+y$; expanding

Arithmetic-geometric inequality

Let n be a positive integer. Prove that:

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \ldots + a_n}{n} \quad \text{for every } a_1, \ldots, a_n \in \mathbb{R}^+ \,, \tag{5}$$

that is, the *geometric mean* of any n positive real numbers $a_1, \ldots, a_n$ is no greater than their *arithmetic mean*. Do the proof as follows:

1. Prove that (5) is true for $n = 1$ and $n = 2$.
2. Prove that, if (5) is true for n, then it is also true for $n - 1$.
3. Prove that, if (5) is true for n *and for 2*, then it is also true for $2n$.
4. Explain why the three points above allow to conclude that (5) is true for every positive integer n.

Solution. We preliminarily observe that, as all the quantities involved are positive, (5) is equivalent to the following:

$$a_1 \cdots a_n \le \left(\frac{a_1 + \ldots + a_n}{n} \right)^n . \tag{6}$$

This is what we will use in the proof.

1. For $n = 1$, (6) becomes $a_1 \le a_1$, which is true. For $n = 2$, (6) becomes $a_1 a_2 \le \left(\frac{a_1 + a_2}{2} \right)^2$, which is true because it is equivalent to:

   $$a_1^2 + a_2^2 + 2a_1a_2 - 4a_1a_2 \ge 0 \,,$$

   which is true because the left-hand side is $(a_1 - a_2)^2$.

2. Suppose that (6) is true for n. Then it remains true if we choose the special value:

   $$a_n = \frac{a_1 + \ldots + a_{n-1}}{n-1} .$$

   But in this case:

   $$\begin{aligned} a_1 \cdots a_n &= a_1 \cdots a_{n-1} \cdot \frac{a_1 + \ldots + a_{n-1}}{n-1} \\ &\le \left(\frac{a_1 + \ldots + a_{n-1} + \frac{a_1 + \ldots + a_{n-1}}{n-1}}{n} \right)^n \\ &= \left(\frac{\frac{n \cdot (a_1 + \ldots + a_{n-1})}{n-1}}{n} \right)^n \\ &= \left(\frac{a_1 + \ldots + a_{n-1}}{n-1} \right)^n . \end{aligned}$$

   As $a_1, \ldots, a_n$ are all positive, we can simplify and obtain $a_1 \cdots a_{n-1} \le \left(\frac{a_1 + \ldots + a_{n-1}}{n-1} \right)^{n-1}$.

3. Suppose that (6) is true for n and for 2. Then:

$$\begin{aligned} a_1 \cdots a_{2n} &= (a_1 \cdots a_n) \cdot (a_{n+1} \cdots a_{2n}) \\ &\le \left(\frac{a_1 + \ldots + a_n}{n}\right)^n \cdot \left(\frac{a_{n+1} + \ldots + a_{2n}}{n}\right)^n \\ &= \left(\frac{a_1 + \ldots + a_n}{n} \cdot \frac{a_{n+1} + \ldots + a_{2n}}{n}\right)^n \\ &\le \left(\left(\frac{\frac{a_1+\ldots+a_n}{n} + \frac{a_{n+1}+\ldots+a_{2n}}{n}}{2}\right)^2\right)^n \\ &= \left(\frac{a_1 + \ldots + a_{2n}}{2n}\right)^{2n} . \end{aligned}$$

4. Every positive integer n larger than 2 can be reached by starting from 2, doubling $m-1$ times until we reach a value $2^m \ge n$, then decreasing by 1 for $2^m - n$ steps.

This proof is due to the Hungarian mathematician Paul Erdős.

Children and sweets

How many ways are there to distribute n identical sweets between k children (with $1 \le k \le n$) if each child must receive at least one sweet?

Solution (By experiment, intuition, and induction)**.** We make a first attempt with $n = 5$ and several values of k:

$k = 1$. Clearly, there is only one way of distributing the five sweets to the unique child!

$k = 2$. The first child will take either 1, 2, 3, or 4 sweets, and the other one will get those that are left: so there are 4 ways of distributing 5 sweets between 2 children, each child receiving at least one sweet.

$k = 3$. There are two possibilities: either one kid is *favored* in that s/he gets three sweets and the other two only get one each, or one kid is *disfavored* in the sense that s/he gets only one candy, and the other two get two each. In each case there are three ways to choose the (dis)favored child, so overall there are $3 + 3 = 6$ ways of distributing 5 sweets between 3 children, each child receiving at least one sweet.

$k = 4$. In this case, one of the children will receive two sweets, and the others will receive one each: there are thus 4 ways of distributing 5 sweets between 4 children, each child receiving at least one sweet.

$k = 5$. Clearly, there is only one way to distribute the five sweets between five children, who will have one each.

We then plot the values of the number $D(n, k)$ of the ways of distributing n sweets between k children (with $1 \leq k \leq n$) giving at least one sweet to each child, for $n = 5$ and $1 \leq k \leq 5$, and we observe a coincidence:

k	1	2	3	4	5
$D(5, k)$	1	4	6	4	1
$\binom{4}{k-1}$	1	4	6	4	1

Maybe it is true in general that $D(n, k) = \binom{n-1}{k-1}$? Let's try to prove it by induction *on* n: that is, we construct the proposition

$$P(n) \, : \, D(n, k) = \binom{n-1}{k-1} \; \forall k \in \{1, \ldots, n\}$$

and we prove, as the *induction base*, that $P(1)$ is true; and as the *inductive step*, that for every $n \geq 1$, if $P(n)$ is true, then so is $P(n + 1)$.

- *Induction base:* there is only $1 = \binom{0}{0}$ way of giving one sweet to one child, so that the child has at least one sweet. Therefore, $P(1)$ is true.

- *Inductive step:* Suppose $P(n)$ is true: that is, for every $k \in \{1, \ldots, n\}$, there exist exactly $\binom{n-1}{k-1}$ ways of distributing n sweets between k children, giving at least one sweet to each child. What if the sweets are $n + 1$? Again, if the children are either 1 or $n + 1$, there is only $1 = \binom{n}{0} = \binom{n}{n}$ way: all to one, or one each, respectively. So we only need to consider the case $k \in \{2, \ldots, n\}$.

  One idea to reduce the problem to a previous case, is to first distribute n sweets between the k children, then decide which child give the $(n+1)$st sweet. This idea, however, does not take into account that the sweets are identical, but the children are not!

  The issue is resolved if we observe that either the first child receives only one sweet, or he receives two or more. The last case corresponds to a division of n sweets between k children: as we are working under the hypothesis that $P(n)$ is true, we can assert that there are $\binom{n-1}{k-1}$ such distributions. The first case corresponds to a division of n sweets between $k-1$ children: again, as we are working under the hypothesis that $P(n)$ is true, we can assert that there are $\binom{n-1}{k-2}$ such distributions. Adding the two together, for $k \in \{2, \ldots, n\}$ there are

$$\binom{n-1}{k-1} + \binom{n-1}{k-2} = \binom{n}{k-1}$$

ways of distributing $n+1$ sweets between k children, so that each child receives at least one sweet. We have thus proved that, if $P(n)$ is true, then so is $P(n+1)$: and that such implication holds for every $n \geq 1$, because n always acted only as a parameter, and no special cases needed to be treated.

Solution (By change of point of view)**.** Since the sweets are identical, we can distribute them by putting them in a line, and give the leftmost ones to the current child. Distributing n sweets between k children so that each child receives at least one sweet, is then the same of choosing at which $k-1$ points we stop giving sweets to the current child, and go on to the next. There are $n-1$ points where we can switch from a child to the next one, so by definition there must be $\binom{n-1}{k-1}$ ways of distributing n identical sweets between k children (with $1 \leq k \leq n$) giving each child at least one sweet.

Estimating sums

Consider the two sums:

$$A = \sum_{i=1}^{3} i^2 \sum_{j=1}^{i} (j^2+1) \;\; ; \;\; B = \sum_{j=1}^{3} (j^2+1) \sum_{i=j}^{3} i^2 \, .$$

Which one is larger?

Solution. The two sums are equal. To see this more easily, we introduce the *Iverson brackets* as a function from the set $\{\text{True}, \text{False}\}$ to the set $\{0, 1\}$ defined as follows:

1. $[\text{True}] = 1$ and $[\text{False}] = 0$.

2. If a is either infinite or undefined, then $a \cdot [\text{False}] = 0$.

The Iverson brackets allow us to move the dependencies between the summation indices from the indices themselves to the summands. Then:

$$A = \sum_{i=1}^{3} \sum_{j=1}^{3} i^2(j^2+1)[j \leq i] \quad \text{and} \quad B = \sum_{j=1}^{3} \sum_{i=1}^{3} (j^2+1)i^2[i \geq j]\,,$$

which are easily seen to be equal.

A more “brutal” alternative is to check which pairs (i, j) will enter each summation. For the first sum, the pairs are:

$$(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\,;$$

for the second sum, they are:

$$(1,1), (2,1), (3,1), (2,2), (3,2), (3,3)\,.$$

So the pairs of indices are the same for both sums, and so are the summands corresponding to each pair: therefore, the sums must be equal. Now do it with 100 instead of 3 ...

Divisibility

100 switches, numbered from 1 to 100, are off. At time n, for n from 1 to 100, exactly those switches whose number is divisible by n are switched: on if they are off, of if they are on.

After time $n = 100$, which switches are on?

Solution. For k from 1 to 100, the switch k is flipped at time n if and only if k is divisible by n. As each switch is originally off, determining which ones are on after time 100 is equivalent to determining which integers k between 1 and 100 have an *odd* number of divisors.

To do this, we observe that, if n is a divisor of k, then so in k/n. Then, if we regroup the divisors of k in pairs of the form $(n, k/n)$, the number k has an odd number of divisors if and only if one of the pairs $(n, k/n)$ has both elements equal, that is, $n = k/n = a$. But this is the same as saying that $k = a^2$, that is, k is a perfect square.