Concrete Mathematics
Exercise session 1, 2 February 2023

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Exercise I.1: A divisibility issue

If A and B are finite sets with $|A|$ and $|B|$ elements, respectively, then:

$$|A| + |B| = |A \cup B| + |A \cap B| \ .$$

Use a variant of the inclusion-exclusion formula to determineow many integer numbers between 1 and 1000 are divisible by either 7 or 11, but not both.

Solution. Every 7th positive integer is divisible by 7, so the number of integers between 1 and 1000 which are divisible by 7 is simply the quotient of the division of 1000 by 7. We have $1000 = 142 \cdot 7 + 6$, so there are 142 multiples of 7 between 1 and 1000. Similarly, as $1000 = 90 \cdot 11 + 10$, there are 90 multiples of 11 between 1 and 1000.

Now, the common multiples of 7 and 11 are the multiples of the *least common multiple* of 7 and 11, which is 77: as $1000 = 12 \cdot 77 + 76$, there are 12 multiples of 77 between 1 and 1000.

The twist now is that we are not counting the multiples of *either* 7 and 11, but those of *one between 7 and 11 but not both*. This means that we are not looking for a union, but for a *symmetric difference*, which is defined as follows:

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B) \, .$$

Then the number we are looking for is $142 + 90 - 2 \cdot 12 = 208$

Exercise I.2: Factorials

The *factorial* of the nonnegative integer n, denoted by $n!$, is the number of possible orderings of n distinct objects.

1. Give a recursive definition of $n!$. That is:
   - establish the value of $0!$; and
   - give a rule to calculate $(n+1)!$, given $n!$.
2. How do you justify your choice for $0!$ according to the definition above?

Solution. Although many of you may have already seen the definition of factorial, the aim of this exercise is to illustrate the following basic fact. In combinatorics, one is usually given a *description* of a set, and from that description, one has to determine the *number of elements* of that set. We will do this in this exercise and, later on, in Exercise I.3.

1. We can first define the rule, then choose a sensible initial value.
   - Assume that we know that we can order n objects in $n!$ different ways. Now we must understand how to order $n+1$ objects. We can do so by first, choosing the *first* element; then, order the *remaining* objects. Of course, we have $n+1$ possible choices for the first element, so we have $(n+1)\cdot n!$ ways of ordering $n+1$ objects. This suggests the recursive formula:

     $$(n+1)! = (n+1)\cdot n!$$

   - We want the formula above to be meaningful for $n=0$ too. To do this, we observe that if $n>0$, then $n! = \dfrac{(n+1)!}{n+1}$; if we want this equality to hold by $n=0$ too, we must put $0! = \dfrac{1!}{1} = 1!$. There is clearly only one way of ordering one object, so $0! = 1! = 1$.
2. There is exactly one way to sort zero objects: doing nothing.

Intermezzo: The induction principle

Lemma (The principle of mathematical induction)**.** *Let $P(n)$ be a predicate whose truth or falsity depends on the value of a certain nonnegative integer n.*

Suppose that following two things happen:

1. *There exists $n_0 \in \mathbb{N}$ such that $P(n_0)$ is true.*

2. *For every $n \geq n_0$, if $P(n)$ is true, then $P(n+1)$ is true.*

Then $P(n)$ is true for every $n \geq n_0$.

There are several other forms of the induction principle; this is the main one. We will see some more in the next few lectures.

Exercise I.3: Binomial coefficients

Let n and $k \leq n$ be nonnegative integers. The *binomial coefficient* $\dbinom{n}{k}$, read "n choose k", is the number of ways of choosing k elements from a set of n elements, without considering the ordering in which they are taken.

If $n < 0$, $k < 0$, or $k > n$, we put $\binom{n}{k} = 0$.

1. What is the "reasonable" value for $\dbinom{n}{0}$? Why so?

2. Give an explicit formula for $\dbinom{n}{k}$ as a function of n and k.

3. Prove the recursive formula:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \text{ for every } n, k \geq 0, k \leq n$$

   in one (or both) of the following ways:

   (a) by explicit calculation;

   (b) by interpreting the two sides of the equality according to the definition given above.

Solution. 1. There is exactly one way to choose zero objects among n objects: doing nothing. So the reasonable choice is $\binom{n}{0} = 1$.

2. We can choose the first object in n ways. After we have done this, we can choose the second object in $n-1$ ways; for the third one, we will have $n-2$ options; and so on, until we will have $n-k+1$ choices for the kth objects. But since we don't care about the order in which we have taken the objects, we must divide this number by the number of possible orderings of k objects. We conclude:

$$\begin{aligned}\binom{n}{k} &= \frac{n(n-1)\cdots(n-k+1)}{k!} \\ &= \frac{n!}{k!(n-k)!}\end{aligned}$$

Note, however, that while the writing on the second rows requires both n and k to be integers, the one on the first row could be applied to any numerical value of n. This will be exploited in Chapters 2 and 5.

3. For explicit calculation:

$$\begin{aligned}\binom{n+1}{k} &= \frac{(n+1)\cdot n\cdots(n-k+2)}{k!} \\ &= \frac{n+1}{k}\cdot\frac{n\cdots(n-(k-1)+1)}{(k-1)!} \\ &= \frac{n+1}{k}\cdot\binom{n}{k-1} \\ &= \frac{(n-k+1)+k}{k}\cdot\binom{n}{k-1} \\ &= \left(\frac{n-k+1}{k}+1\right)\cdot\binom{n}{k-1} \\ &= \frac{n-k+1}{k}\cdot\binom{n}{k-1}+\binom{n}{k-1} \\ &= \binom{n}{k}+\binom{n}{k-1}\end{aligned}$$

For interpretation: Put the $n+1$ objects into a line. We have to decide what to do with the first one: either we take it, or we don't take it, and we can't do both things at once. If we take it, then we need to choose the remaining $k-1$ objects among the remaining n objects; if we don't, then we need to choose all the k objects among the other n objects.

Exercise I.4: Newton's binomial theorem

Prove the following:

Theorem (Newton's binomial theorem)**.** *Let x and y be complex numbers and let n be a nonnegative integer. Then:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{1}$$

in one of the following ways:

1. by induction on $n \geq 0$;
2. by expanding the product.

Solution. In classroom, we decided to use this exercise for an example of how to use the induction principle. So here it goes.

Let $P(n)$ be the following predicate:

$$\forall x, y \in \mathbb{C} \,.\, (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \,.$$

(Read: "for all x and y complex numbers, $(x+y)^n$ equals the sum, for k from 0 to n, of $\binom{n}{k}$ times x^k times y^{n-k}".) We want to prove that $P(n)$ is true for every $n \in \mathbb{N}$. Here, the symbol $\mathbb{N}$ represents the set of *nonnegative* integers.

- **Induction base:** Let $n = 0$. Then the left-hand side of (1) is $(x+y)^0 = 1$ and the right-hand side is $\binom{0}{0} x^0 y^{0-1} = 1$. We conclude that $P(0)$ is true.

  Incidentally: let's be careful not to confuse the *number* 0^0 with the *indeterminate form* 0^0. The number is an *empty product* of no factors, all equal to zero: empty products take value 1. The indeterminate form denotes a *family of functions* of the form $f(x) = (g(x))^{h(x)}$ where $g(x)$ and $h(x)$ both vanish in a neighborhood of a point x_0: in general, this information alone is not enough to determine the behavior of $f(x)$ in a neighborhood of x_0.

- **Inductive step:** assume that $P(n)$ is true for a certain n, on which we make no special hypotheses. We substitute n with $n+1$ in the right-hand side of (1) and expand:

$$
\begin{aligned}
\sum_{k=0}^{n+1} \binom{n}{k} x^k y^{n+1-k} &= \binom{n+1}{0} y^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + \binom{n+1}{n+1} x^{n+1} \\
&= y^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) x^k y^{n+1-k} + x^{n+1} \\
&\quad \text{because of Exercise I.3} \\
&= y^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} \\
&\quad + \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k} + x^{n+1} \\
&= y^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} \\
&\quad + \sum_{j=0}^{n-1} \binom{n}{j} x^{j+1} y^{n+1-(j-1)} + x^{n+1} \\
&\quad \text{with the substitution } j = k-1 \\
&= y \cdot \left(\binom{n}{0} y^n + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k} \right) \\
&\quad + \left(\sum_{j=0}^{n-1} \binom{n}{j} x^j y^{n-j} + \binom{n}{n} x^n \right) \cdot x \\
&= y \cdot (x+y)^n + (x+y)^n \cdot x \\
&\quad \text{by inductive hypothesis} \\
&= (x+y)^{n+1}.
\end{aligned}
$$

We conclude that if $P(n)$ is true, then so is $P(n+1)$. As our argument doesn't depend on the specific value of $n \in \mathbb{N}$, the previous implication is true for every n.

Now for the expansion: $(x+y)^n$ is a product of n factors, all equal to $x+y$. When we expand the product, we get a sum of monomials of the form $x^k y^{n-k}$ for some k from 0 to n, each one obtained by choosing an x from k of the factors $x+y$ and a y from the remaining $n-k$; for each k we have $\binom{n}{k}$ choices.

Exercise I.5: Distributing candies to children

Determine the number of ways of distributing $n \geq 1$ candies between $k \geq 1$ children ($k \leq n$) so that each child receives at least one candy.

Solve the problem above in one (or both) of the following ways:

1. by formulating a good hypothesis, and prove it by induction;
2. by reinterpreting the problem so that it matches one of the situations you have met in one of the previous slides.

Solution. Let $D(n,k)$ be the number of ways of distributing n candies between k children so that each child receives at least one candy. We can make some experiments to see if there is any sort of regularity:

- $D(n,n) = 1$. The only way of distributing n candies between n children so that each child receives at least one candy, is to give one candy to each child.
- $D(n,n-1) = n-1$. Distributing n candies between $n-1$ children so that each child receives at least one candy, is the same as choosing the one child who will receive two candies instead of one.
- Calculating $D(n,n-2)$ is already more complicated. In this case, either two children will receive two candies, or one child will receive three candies; all the others will receive one candy. Reasoning as before, there are $\binom{n-2}{2}$ ways to do the first thing, and $\binom{n-2}{1}$ to do the other; but then, there are

$$D(n,n-2) = \binom{n-2}{2} + \binom{n-2}{1} = \binom{n-1}{2}$$

ways of distributing n candies between $n-2$ children so that each child receives at least one candy.[1]

- $D(n, 1) = 1$, of course.

The experiments suggest some links with binomial coefficients, but with upper index $n-1$ when the candies are n. This suggests the following change of point of view:

Choosing how many candies to give to each child, is the same as choosing *when to stop giving candies to a child, and start with the next child:* these moments in time will be one less than the number of children. As each child must receive at least one candy, if we put the latter in a line, these changes will correspond to the *gaps* between one candy and the next: of course, there are $n-1$ such gaps.

Distributing n candies between k children so that each child receives at least one candy, is then the same as choosing $k-1$ gaps among $n-1$ gaps. We conclude that $D(n,k) = \binom{n-1}{k-1}$.

Exercise I.6: Flipping switches

Consider the following situation:

- A panel has 100 switches, numbered from 1 to 100.
- Initially, all the switches are OFF.
- A clock marks the time, at integer intervals.
- At each time $t \in \{1, 2, \ldots, 99, 100\}$, an operator flips *all and only* the switches with a number which is a *multiple* of t.

After time $t = 100$, which switches will be ON?

Solution. Let's start by doing some experiments:

- Switch number 1 will only be flipped at time $t = 1$, so it will be on.
- Switch number 2 will only be flipped at times $t = 1$ and $t = 2$, so it will be off.

[1] This discussion fixes some mistakes done during the lecture.

- Actually, any switch whose number is a prime p will be flipped exactly twice, so it will be OFF. Remember that 1 is not a prime number.
- Switch number 4 will be flipped at times $t = 1$, $t = 2$, $t = 4$, and never again; so it will be ON.
- Switch number 6 will be flipped at times $t = 1$, $t = 2$, $t = 3$, $t = 6$, and never again; so it will be OFF.
- Switch number 12 will be flipped at times $t = 1$, $t = 2$, $t = 3$, $t = 4$, $t = 6$, $t = 12$, and never again; so it will be OFF.

Until now, the only switches which we know will be ON are numbers 1 and 4: both of which have an *odd* number of positive divisors. But this must be it: since all switches start in the OFF position, they can only be ON after time $t = 100$ if they have an *odd* number of divisors. But how can we determine if the number of (positive) divisors of a positive integers is even or odd?

Well, for every positive integer n, consider the set:

$$S_n = \{(a, b) \mid a, b \in \mathbb{Z}_+, a \leq b, ab = n\}.$$

Then every divisor of n appears as either the left element or the right element of one of the pairs which are elements of S_n. Then these divisors can be oddly many if and only of there is one of these pairs (a, b) where $a = b$: but this is the same as saying that $n = a^2$ is a perfect square.

We conclude that the switches which will be ON after time $t = 100$ are numbers 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100.

Intermezzo: The Iverson brackets

The *Iverson brackets* are the function $[\cdot] : \{\text{False}, \text{True}\} \to \{0, 1\}$ satisfying the following two properties:

1. $[\text{True}] = 1$ and $[\text{False}] = 0$.
2. If a is infinite or undefined, then $a \cdot [\text{False}] = 0$.

This notation was originally introduced by Kenneth Iverson in the specification of the APL language; our textbook "tweaks" the original definition a little bit. Donald Knuth, one of the authors, who is also the creator of TeX and of Metafont, is a very strong proponent of this notation.

Iverson brackets behave well with respect to Boolean operations:

1. $[P \wedge Q] = [P] \cdot [Q]$.
2. $[\neg P] = 1 - [P]$.
3. $[P] + [Q] = [P \vee Q] + [P \wedge Q]$.
4. etc.

Exercise I.7: Estimating sums

Let n be a positive integer. Consider the two sums:

$$A_n = \sum_{i=1}^{n} i^2 \sum_{j=1}^{i} (j^2 + 1) \;\; ; \;\; B_n = \sum_{j=1}^{n} (j^2 + 1) \sum_{i=j}^{3} i^2 \, .$$

1. Let $n = 3$. Which one of the two sums A_3, B_3 is larger?
2. What if $n = 17$? Or even larger?

Hint: This is a "don't panic" exercise. The Iverson brackets can be useful.

Solution. The purpose of this exercise is to illustrate one convenient use of the Iverson brackets: if we have a sum with some constraints on the indices, we can move those constraints *from the sum to the summands.*

1. For $n = 3$ we have, on the one hand:

$$\begin{aligned} A_3 = \sum_{k=1}^{3} k^2 \sum_{j=1}^{k} (j^2 + 1) &= 1 \cdot 2 + 4 \cdot (2 + 5) + 9 \cdot (2 + 5 + 10) \\ &= 2 + 28 + 153 = 183\, ; \end{aligned}$$

   and on the other hand:

$$\begin{aligned} B_3 = \sum_{j=1}^{3} (j^2 + 1) \sum_{k=j}^{3} k^2 &= 2 \cdot (1 + 4 + 9) + 5 \cdot (4 + 9) + 10 \cdot 9 \\ &= 28 + 65 + 90 = 183\, . \end{aligned}$$

   We conclude that $A_3 = B_3$.

2. No, we *don't* want to redo all the calculations for $n = 17$. But why should we have, for example, the upper bound in the first double sum depend on the index? We could just add $n - k$ *summands all equal to zero*, and we could stop at $j = n$ instead of $j = k$. That is:

$$\begin{aligned} A_n = \sum_{k=1}^{n} k^2 \sum_{j=1}^{k} (j^2+1) &= \sum_{k=1}^{n} k^2 \sum_{j=1}^{n} (j^2+1)\,[j \le k] \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} k^2 (j^2+1)\,[j \le k] \;. \end{aligned}$$

But we could do the same with the lower bound in the second sum, adding $n - j - 1$ summands all equal to zero, and obtain:

$$\begin{aligned} B_n = \sum_{j=1}^{n} (j^2+1) \sum_{k=j}^{n} k^2 &= \sum_{j=1}^{n} (j^2+1) \sum_{k=1}^{n} k^2\,[k \ge j] \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} (j^2+1) k^2\,[k \ge j] \;. \end{aligned}$$

It is now easy to see that $A_n = B_n$ whatever n is.