ITT9132 Concrete Mathematics
Exercise session 2: 4 February 2021

Silvio Capobianco

11 October 2022

Exercise 1.2

Find the shortest sequence of moves that transfers a tower of n disks from the left peg A to the right peg B, if direct moves between A and B are disallowed.

Exercise 1.3

Show that, in the previous exercise, each legal arrangement of n disks is encountered exactly once.

Exercise 1.4

Are there any starting and ending configurations of n disks on three pegs that are more than $2^n - 1$ moves apart, according to Lucas's original rules?

The technique of minimum counterexample

The proof technique we employed to solve the previous exercise is based on the following, intuitive[1] fact:

Every nonempty set of natural numbers has a minimum.

Suppose that we have a sequence $\{P(n)\}_{n \geq 0}$ of propositions depending on natural numbers, and that we want to prove that they are all true. To do

[1] Actually, it is equivalent to induction.

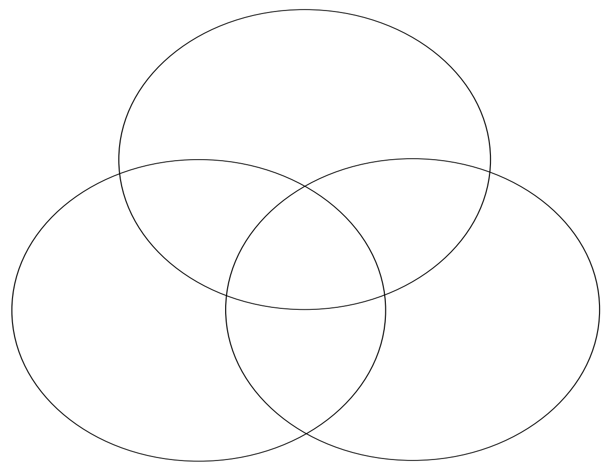

Figure 1: Venn diagram for three sets.

this, we may reason by contradiction and suppose that they are *not* all true: then the set $F = \{n \geq 0 \mid P(n) \,\text{is false}\}$ is nonempty, and has a minimum $\bar{n}$.

Now, either $\bar{n} = 0$, or $\bar{n} > 0$. To prove our original statement, we may then prove the following two other statements instead:

1. $P(0)$ is true.
2. If $n > 0$ and $P(n)$ is false, then so is $P(m)$ for some $m < n$.

Indeed, let us suppose to have proved points 1 and 2: consider then the set F of which $\bar{n}$ is the minumum. Then $\bar{n}$ cannot be zero: but it cannot be positive either, or there would be some $m < \bar{n}$ belonging to F, against $\bar{n}$ being a minimum. The only way to escape such contradiction, is to acknowledge that there is no such $\bar{n}$, and that F is empty.

This technique, or variants of it, also works with other kinds of induction such as *structural induction*.

Exercise 1.5

A *Venn diagram* with three overlapping circles is often used to illustrate the eight possible subsets associated with three given sets (see Figure 1). Can the sixteen possibilities that arise with four given sets be illustratd by four overlapping circles?

Exercise 1.8

Solve the recurrence:

$$Q_0 = \alpha \ ; \ \ Q_1 = \beta;$$
$$Q_n = (1 + Q_{n-1})/Q_{n-2} \ , \ \ \text{for } n > 1 \ .$$

Assume that $Q_n \neq 0$ for all $n \geq 0$. *Hint:* $Q_4 = (1 + \alpha)/\beta$.