ITT9132 Concrete Mathematics
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Exercise 1.8

Solve the recurrence:

$$Q_0 = \alpha \;\; ; \;\; Q_1 = \beta;$$
$$Q_n = (1 + Q_{n-1})/Q_{n-2} \;\; , \;\; \text{for } n > 1 \; .$$

Assume that $Q_n \neq 0$ for all $n \geqslant 0$. *Hint:* $Q_4 = (1+\alpha)/\beta$.

Solution. Let us just start computing. We get $Q_2 = (1+\beta)/\alpha$ and $Q_3 = (1 + ((1+\beta)/\alpha))/\beta = (1+\alpha+\beta)/\alpha\beta$. Then:

$$\begin{aligned}
Q_4 &= \left(1 + \frac{1+\alpha+\beta}{\alpha\beta}\right) \cdot \frac{\alpha}{1+\beta} \\
&= \frac{1+\alpha+\beta+\alpha\beta}{\alpha\beta} \cdot \frac{\alpha}{1+\beta} \\
&= \frac{1+\alpha+\beta+\alpha\beta}{\beta} \cdot \frac{1}{1+\beta} \\
&= \frac{(1+\alpha)(1+\beta)}{\beta} \cdot \frac{1}{1+\beta} \\
&= \frac{1+\alpha}{\beta}
\end{aligned}$$

and

$$\begin{aligned}
Q_5 &= \left(1 + \frac{1+\alpha}{\beta}\right) \cdot \frac{\alpha\beta}{1+\alpha+\beta} \\
&= \frac{1+\alpha+\beta}{\beta} \cdot \frac{\alpha\beta}{1+\alpha+\beta} \\
&= \alpha \\
&= Q_0 \, .
\end{aligned}$$

Thus, $Q_6 = (1+\alpha)/((1+\alpha)/\beta) = \beta = Q_1$, and the sequence is periodic.

Important note: Exercise 1.8 asks us to solve a *second order* recurrence with *two* initial conditions, corresponding to two consecutive indices. To be sure that the solution is a periodic sequence, we must then make sure that *two consecutive values* are repeated.

Exercise A.2

In Exercise session 2 we have used the repertoire method to solve the recurrence:

$$\begin{array}{ccc} g(0) & = & \alpha \ , \\ g(n+1) & = g(n) + \beta n + \gamma & \text{for every } n \geqslant 0 \ . \end{array} \tag{1}$$

What if the recurrence (1) had been

$$\begin{array}{rcl} g(0) & = & \alpha \ , \\ g(n+1) & = & \delta g(n) + \beta n + \gamma \ \text{ for every } n \geqslant 0 \end{array} \tag{2}$$

instead?

Solution. The recurrence (2), considered as a family of recurrence equations parameterized by $(\alpha, \beta, \gamma, \delta)$, does *not* have the form required by the repertoire method, because the function Φ depends on one of the parameters, not only on the function g: consequently, in general, $g_1(n) + g_2(n)$ is not the solution for $(\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2, \delta_1 + \delta_2)$, because $\delta_1 g_1(n) + \delta_2 g_2(n)$ is not, in general, equal to

$(\delta_1 + \delta_2)(g_1(n) + g_2(n))$. We cannot therefore use the repertoire method to express $g(n)$ as $g(n) = \alpha \cdot A(n) + \beta \cdot B(n) + \gamma \cdot C(n) + \delta \cdot D(n)$.

However, for every *fixed* δ, (2) *does* have the required form, with $\Phi(g) = \delta g$ and $\Psi(n; \beta, \gamma) = \beta n + \gamma$: thus, for every fixed δ, we can use the repertoire method to find three functions $A_\delta(n), B_\delta(n), C_\delta(n)$ such that

$$g_\delta(n) = \alpha \cdot A_\delta(n) + \beta \cdot B_\delta(n) + \gamma \cdot C_\delta(n)$$

for every $n \geqslant 0$. By reasoning as before, the choice $g_\delta(n) = 1$ corresponds to $(\alpha, \beta, \gamma) = (1, 0, 1-\delta)$, thus

$$A_\delta(n) + (1-\delta)C(n) = 1 \ : \tag{3}$$

the factor $1-\delta$ in front of $C_\delta(n)$ rings a bell, and suggests we might have to be careful about the cases $\delta = 1$ and $\delta \neq 1$. Choosing $g_\delta(n) = n$ corresponds to $(\alpha, \beta, \gamma) = (0, 1-\delta, 1)$, thus

$$(1-\delta)B_\delta(n) + C_\delta(n) = n \ . \tag{4}$$

We are left with one triple of values to choose. As we had put $g(n) = 2^n$ when $\delta = 2$, we are tempted to just put $g(n) = \delta^n$: but if $\delta = 1$ this would be the same as $g(n) = 1$, which we have already considered. We will then deal separately with the cases $\delta = 1$ and $\delta \neq 1$.

Let us start with the latter. For $\delta \neq 1$ the choice $g_\delta(n) = \delta^n$ corresponds to $(\alpha, \beta, \gamma) = (1, 0, 0)$, thus

$$A_\delta(n) = \delta^n \ : \tag{5}$$

by combining this with (3) and (4) we find

$$C_\delta(n) = \frac{1 - A_\delta(n)}{1 - \delta} = \frac{1 - \delta^n}{1 - \delta} = 1 + \delta + \ldots + \delta^{n-1}$$

and

$$B_\delta(n) = \frac{n - C_\delta(n)}{1 - \delta} = \frac{n - 1 - \delta - \ldots - \delta^{n-1}}{1 - \delta} \ .$$

Let us now consider the case $\delta = 1$. Then (3) becomes $A_1(n) = 1$ and (4) becomes $C_1(n) = n$: for the last case, we set $g_1(n) = n^2$, which corresponds to $(\alpha, \beta, \gamma) = (0, 2, 1)$, and find

$$2B_1(n) + C_1(n) = n^2 \ , \tag{6}$$

which yields $B_1(n) = (n^2 - n)/2$.

Exercise 2.2

Simplify the expression $x \cdot ([x > 0] - [x < 0])$.

Solution. If $x > 0$ then the expression has value $x \cdot (1 - 0) = x$. If $x = 0$ then the expression has value $0 \cdot (0 - 0) = 0$. If $x < 0$ then the expression has value $x \cdot (0 - 1) = -x$. Thus, $x \cdot ([x > 0] - [x < 0]) = |x|$

.

Exercise 2.12

Show that the function $p(k) = k + (-1)^k c$ is a permutation of the set of all integers, whenever c is an integer.

Solution. A way to solve the exercise is to prove that $p(k)$ has an *inverse function* $q(n)$, defined for every integer n, such that $p(k) = n$ if and only if $q(n) = k$.

So let $p(k) = k + (-1)^k c = n$. Then $n + c = k + (1 + (-1)^k)c$. But $1 + (-1)^k$ is 2 if k is even and 0 if k is odd, which means that k and $n + c$ are either both even or both odd: hence, $(-1)^k = (-1)^{n+c}$. We can thus rewrite $k = n + c - (1 + (-1)^k)c = n - (-1)^{n+c}c$: this is the inverse function $q(n)$ we were looking for.

Exercise 2.21

Evaluate the sums $S_n = \sum_{k=0}^{n}(-1)^{n-k}$, $T_n = \sum_{k=0}^{n}(-1)^{n-k}k$, and $U_n = \sum_{k=0}^{n}(-1)^{n-k}k^2$ by the perturbation method, assuming that $n \geqslant 0$.

Solution. By applying the permutation $p(k) = n - k$ we see that $S_n = [n \text{ is even}]$. Let's try to reach the same result via the perturbation method. First,

$$\begin{aligned} S_{n+1} &= \sum_{0 \leqslant k \leqslant n+1} (-1)^{n+1-k} \\ &= \sum_{0 \leqslant k \leqslant n} (-1)^{n+1-k} + 1 \\ &= -S_n + 1 \ ; \end{aligned}$$

next,

$$\begin{aligned} S_{n+1} &= (-1)^{n+1} + \sum_{1 \leqslant k \leqslant n+1} (-1)^{n+1-k} \\ &= (-1)^{n+1} + \sum_{0 \leqslant k \leqslant n} (-1)^{n-k} \\ &= (-1)^{n+1} + S_n \ . \end{aligned}$$

Together, the two equalities above yield $2S_n = 1 - (-1)^{n+1} = 1 + (-1)^n$, so that:

$$S_n = \frac{1 + (-1)^n}{2} = [n \text{ is even}] \ .$$

For T_n we use a similar trick. First,

$$\begin{aligned} T_{n+1} &= \sum_{0 \leqslant k \leqslant n} (-1)^{n+1-k}k + n + 1 \\ &= -T_n + n + 1 \ ; \end{aligned}$$

next,

$$\begin{aligned} T_{n+1} &= 0 + \sum_{1 \leqslant k \leqslant n+1} (-1)^{n+1-k}k \\ &= \sum_{0 \leqslant k \leqslant n} (-1)^{n-k}(k+1) \\ &= T_n + S_n \ ; \end{aligned}$$

together these yield $2T_n = n + 1 - S_n$. But as $S_n = [n \text{ is even}]$, $1 - S_n = [n \text{ is odd}]$: thus,

$$T_n = \frac{n + [n \text{ is odd}]}{2} \ .$$

With U_n the trick will be similar as with T_n, but we will have to be careful about the square:

$$\begin{aligned}
-U_n + (n+1)^2 &= \sum_{0 \leqslant k \leqslant n} (-1)^{n-k}(k+1)^2 \\
&= \sum_{0 \leqslant k \leqslant n} (-1)^{n-k}(k^2+2k+1) \\
&= U_n + 2T_n + S_n \ ,
\end{aligned}$$

which yields $2U_n = (n+1)^2 - 2T_n - S_n$. But

$$2T_n + S_n = n + [n \text{ is odd}] + [n \text{ is even}] = n+1 \ :$$

thus, $U_n = (n^2+n)/2$.

Exercise B.1 (from the classroom test of 23 November 2016)

Solve the recurrence:

$$\begin{aligned}
T_0 &= 1\,; \\
nT_n &= 2T_n + \frac{2^n}{n!}\left(1 + \frac{n}{3^n}\right) \quad \text{for every } n \geqslant 1\,.
\end{aligned} \tag{7}$$

Solution. The recurrence (7) has the form:

$$a_n T_n = b_n T_{n-1} + c_n \quad \text{for every } n \geqslant 1$$

with $a_n =$, $b_n = 2$, and $c_n = \dfrac{2^n}{n!}\left(1 + \dfrac{n}{3^n}\right)$ for every $n \geqslant 1$: this suggests using a summation factor. As usual, we put $s_0 = 1$ and, considering $a_0 = 1$, we have:

$$\begin{aligned}
s_n &= \frac{a_0 \cdot a_1 \cdots a_{n-1}}{b_1 \cdot b_2 \cdots b_n} \\
&= \frac{1 \cdot 1 \cdot \cdots \cdot (n-1)}{2 \cdot 2 \cdots 2} \\
&= \frac{(n-1)!}{2^n}
\end{aligned}$$

We then put $U_n = s_n a_n T_n = \dfrac{n!}{2^n} T_n$ and the recurrence becomes:

$$\begin{aligned}
U_0 &= 1\,; \\
U_1 &= U_{n-1} + \frac{1}{n} + \frac{1}{3^n} \quad \text{for every } n \geqslant 1
\end{aligned}$$

which has solution:

$$\begin{aligned} U_n &= 1 + \sum_{k=1}^{n} \left(\frac{1}{k} + \frac{1}{3^k} \right) \\ &= 1 + H_n + \frac{1}{3} \sum_{k=0}^{n-1} \frac{1}{3^k} \\ &= 1 + H_n + \frac{1}{3} \cdot \frac{1-(1/3)^n}{1-1/3} \\ &= 1 + H_n + \frac{1}{2} \cdot \left(1 - \frac{1}{3^n} \right) \end{aligned}$$

We conclude:

$$\begin{aligned} T_n = \frac{1}{s_n a_n} U_n &= \frac{2^n}{n!} \cdot \left(1 + H_n + \frac{1}{2} \cdot \left(1 - \frac{1}{3^n} \right) \right) \\ &= \frac{2^n}{n!} (1 + H_n) + \frac{2^{n-1}(3^n - 1)}{3^n n!} \, . \end{aligned}$$