ITT9132 Concrete Mathematics
Exercise session 5: 2 March 2023

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Exercise 2.28

At what point does the following derivation go astray?

$$
\begin{aligned}
1 &= \sum_{k\geqslant 1} \frac{1}{k\cdot(k+1)} && (1)\\
&= \sum_{k\geqslant 1} \left(\frac{k}{k+1} - \frac{k-1}{k}\right) && (2)\\
&= \sum_{k\geqslant 1}\sum_{j\geqslant 1} \left(\frac{k}{j}[j=k+1] - \frac{j}{k}[j=k-1]\right) && (3)\\
&= \sum_{j\geqslant 1}\sum_{k\geqslant 1} \left(\frac{k}{j}[j=k+1] - \frac{j}{k}[j=k-1]\right) && (4)\\
&= \sum_{j\geqslant 1}\sum_{k\geqslant 1} \left(\frac{k}{j}[k=j-1] - \frac{j}{k}[k=j+1]\right) && (5)\\
&= \sum_{j\geqslant 1} \left(\frac{j-1}{j} - \frac{j}{j+1}\right) && (6)\\
&= \sum_{j\geqslant 1} \frac{-1}{j\cdot(j+1)} && (7)\\
&= -1 && (8)
\end{aligned}
$$

Solution. Let us check each passage in detail.

- (1) is correct. The right hand side is the limit for $n \to \infty$ of the telescopic sum $\sum_{1 \leqslant k \leqslant n}(1/k - 1/(k+1)) = 1 - 1/(n+1)$.
- (2) is correct. We are rewriting the generic summand into a different form, whose value is the same as the previous.
- (3) is correct. We are replacing each summand with a new sum whose summands are all zero except finitely many.
- (4) is **WRONG!** We are re-arranging the terms of a sum which is not absolutely convergent: this operation is not guaranteed to preserve the value of the sum. In fact, the absolute values of the summands of the double series are either 0, or $\dfrac{k}{k+1}$, or $\dfrac{j-1}{j}$, and the summands of the last two types converge to 1 when the variable goes to infinity: this contradicts the fundamental fact that the generic term of a convergent series must be vanishing at infinity.
- (5) is correct. We are rewriting the conditions in Iverson brackets without altering them.
- (6) is correct. We are summing over k, which gives an immediate result since for every j only finitely many summands are nonzero.
- (7) is correct.
- (8) is correct.

Exercise B.3 (from the midterm test of 29.03.2019)

Find a closed form for $\sum_{1 \leqslant k \leqslant n} k(k-1)\, 2^{-k}$ as a function of n, and use it to compute $\sum_{k \geqslant 1} k^{\underline{2}}\, 2^{-k}$ *Hint:* use the expression for $\sum_{1 \leqslant k \leqslant n} k\, 2^{-k}$ from the solution of Exercise B.2.

Solution. We can find a closed form by either the perturbation method, or discrete calculus.

- Perturbation method:

Let $S_n = \sum_{1 \leqslant k \leqslant n} k(k-1)\, 2^{-k}$. Then the following chain of equalities holds:

$$
\begin{aligned}
S_n + (n+1)n\, 2^{-n-1} &= 0 + \sum_{k=2}^{n+1} k(k-1)\, 2^{-k} \\
&= \sum_{k=1}^{n} (k+1)k\, 2^{-k-1} \\
&= \frac{1}{2} \sum_{k=1}^{2} (k^2 + k)\, 2^{-k} \\
&= \frac{1}{2} \sum_{k=1}^{2} (k^2 - k + 2k)\, 2^{-k} \\
&= \frac{1}{2} \left(S_n + 2 \sum_{k=1}^{n} k\, 2^{-k} \right) \\
&= \frac{S_n}{2} + 2 \sum_{k=1}^{n} k\, 2^{-k}\,.
\end{aligned}
$$

Multiplying both sides by 2, we get:

$$
\begin{aligned}
2S_n + (n+1)n\, 2^{-n} &= S_n + 2 \sum_{k=1}^{n} k\, 2^{-k} \\
&= S_n + 2\,(2 - (n+2)\, 2^{-n})\,,
\end{aligned}
$$

according to Exercise B.2. From this we get:

$$
\begin{aligned}
S_n &= 4 - (2n+4)\, 2^{-n} - (n+1)n\, 2^{-n} \\
&= 4 - (2n + 4 + n^2 + n)\, 2^{-n} \\
&= 4 - (n^2 + 3n + 4)\, 2^{-n}\,.
\end{aligned}
$$

- Discrete calculus:

  We use summation by parts with $u(x) = x^{\underline{2}}$ and $\Delta v(x) = (1/2)^x$. Then

$\Delta u(x) = 2x$ and $v(x) = -2 \cdot (1/2)^x$. Then:

$$\begin{aligned}
\sum x^{\underline{2}} \left(\frac{1}{2}\right)^x \delta x &= -2x^{\underline{2}} \left(\frac{1}{2}\right)^x + 4 \sum x \left(\frac{1}{2}\right)^{x+1} \\
&= -2x^{\underline{2}} \left(\frac{1}{2}\right)^x + 4 \sum x \left(\frac{1}{2}\right)^{x+1} \\
&= -2x^{\underline{2}} \left(\frac{1}{2}\right)^x + 2 \sum x \left(\frac{1}{2}\right)^x .
\end{aligned}$$

Then

$$\begin{aligned}
\sum_1^{n+1} x^{\underline{2}} \left(\frac{1}{2}\right)^x \delta x &= -2\, x^{\underline{2}} \left(\frac{1}{2}\right)^x \Bigg|_1^{n+1} + 2 \sum_1^{n+1} x \left(\frac{1}{2}\right)^x \\
&= -2(n+1)n\, 2^{-n-1} + 0 + 2 \cdot \left(2 - (n+2)\, 2^{-n}\right),
\end{aligned}$$

according to Exercise B.2. Reorganizing, we conclude:

$$\sum_{k=1}^{n} k^{\underline{2}}\, 2^{-k} = 4 - 2^{-n} \cdot ((n+1)n + 2(n+2)) = 4 - (n^3 + 3n + 4)\, 2^{-n} .$$

Note that for $n = 1$ we have $n^2 + 3n + 4 = 8$, so the formula correctly returns $S_1 = 0$. By taking the limit for $n \to \infty$ we find:

$$\sum_{k \geqslant 1} k^{\underline{2}}\, 2^{-k} = 4 .$$

Exercise B.5

Use Abel's summation theorem to calculate $\displaystyle\sum_{k \geqslant 0} \frac{(-1)^k}{2k+1}$. *Hint:* for every $0 \leqslant x < 1$, $\displaystyle\sum_{k \geqslant 0} (-1)^k t^{2k} = \frac{1}{1+t^2}$ uniformly in $[0..x]$.

Solution. The series $\displaystyle\sum_{k \geqslant 0} \frac{(-1)^k}{2k+1}$ converges by Leibniz's criterion, so we might try to evaluate it as, for example, $\displaystyle\lim_{x \to 1^-} \sum_{k \geqslant 0} \frac{(-1)^k x^k}{2k+1}$. However, the hint tells us that it may be more convenient to consider the power series $\displaystyle\sum_{k\geqslant 0} \frac{(-1)^k x^{2k+1}}{2k+1}$.

Why so? Because $\displaystyle\frac{x^{2k+1}}{2k+1} = \int_0^x t^{2k}\,dt$, and uniform convergence in $[0..x]$ allows us to swap the integral with the sum of the series. We have:

$$
\begin{aligned}
\sum_{k\geqslant 0} \frac{(-1)^k x^{2k+1}}{2k+1} &= \sum_{k\geqslant 0} \int_0^x (-1)^k t^{2k}\,dt \\
&= \int_0^x \left(\sum_{k\geqslant 0} (-1)^k t^{2k} \right) dt \\
&= \int_0^x \frac{1}{1+t^2}\,dt = \arctan x\,.
\end{aligned}
$$

We conclude: $\displaystyle\sum_{k\geqslant 0} \frac{(-1)^k}{2k+1} = \lim_{x\to 1^-} \arctan x = \frac{\pi}{4}$.

Exercise 3.2

Give an explicit formula for the integer nearest to the real number x. Do this in the two cases when an integer plus $1/2$ is rounded up or down.

Solution. Put $x = n + t$ with n integer and $0 \leqslant t < 1$. Rounding x to the nearest integer must yield $n = \lfloor x \rfloor$ when $t < 1/2$, and $n + 1 = \lceil x \rceil$ when $t > 1/2$.

This can be done by rounding x to $\lfloor x + 1/2 \rfloor$. In fact, $\lfloor x + 1/2 \rfloor = \lfloor n + 1/2 + t \rfloor$ is n if $t < 1/2$, and $n + 1$ if $t > 1/2$. We also observe that $\lfloor x + 1/2 \rfloor = n + 1$ if $t = 1/2$, *i.e.*, this is the choice that corresponds to rounding *up*.

Another option is to reason as follows: Being $x = \lfloor x \rfloor + \{x\}$, rounding up means turning x into $\lfloor x \rfloor$ if $\{x\} < 1/2$, and into $\lceil x \rceil = \lfloor x \rfloor + 1$ if $\{x\} \geqslant 1/2$. Then we can just use Iverson's brackets and round x to $\lfloor x \rfloor + [\{x\} \geqslant 1/2]$.

Are there any options for rounding *down*? We may try reasoning "by symmetry" and swapping floor with ceiling, plus with minus: that is, round x to $\lceil x - 1/2 \rceil = \lceil n - 1/2 + t \rceil$. And in fact, we immediately check that this quantity is n for $t < 1/2$ and $n + 1$ for $t > 1/2$. What about $t = 1/2$? We quickly get $\lceil (n + 1/2) - 1/2 \rceil = \lceil n \rceil = n$. So this is the function that rounds down, as required.

Exercise 3.3

Let m and n be positive integers and let α be an irrational number greater than n. Evaluate $\lfloor\lfloor m\alpha\rfloor\, n/\alpha\rfloor$.

Solution. The floor inside the floor is threatening trouble, so we should try to make it disappear. Write $m\alpha = \lfloor m\alpha\rfloor + \{m\alpha\}$. Then:

$$\frac{\lfloor m\alpha\rfloor\, n}{\alpha} = \frac{(m\alpha - \{m\alpha\})n}{\alpha} = mn - \frac{\{m\alpha\}\, n}{\alpha}\,.$$

By hypothesis, $1 \leqslant n < \alpha$. Moreover, α is irrational, so $m\alpha$ is not an integer and $\{m\alpha\}$ is positive. Consequently, $0 < \{m\alpha\} \cdot (n/\alpha) < 1 \cdot 1$. We can thus conclude that:

$$\begin{aligned} \left\lfloor \frac{\lfloor m\alpha\rfloor\, n}{\alpha} \right\rfloor &= \left\lfloor mn - \frac{\{ma\}\, n}{\alpha} \right\rfloor \\ &= mn + \left\lfloor \frac{-\{ma\}\, n}{\alpha} \right\rfloor \\ &= mn - \left\lceil \frac{\{ma\}\, n}{\alpha} \right\rceil \\ &= mn - 1\,. \end{aligned}$$

Note that we used the rule $\lfloor n + x\rfloor = n + \lfloor x\rfloor$, which holds whatever integer n and real x are. To apply it correctly, we must keep the "plus" sign outside the floor and not change x. This means that $\lfloor n - x\rfloor$ is $n + \lfloor -x\rfloor = n - \lceil x\rceil$, and not (in general) $n - \lfloor x\rfloor$.

Exercise 3.9

Egyptian mathematicians in 1800 B.C. represented rational numbers between 0 and 1 as sums of unit fractions $1/x_1 + \ldots + 1/x_k$, where the x's were distinct positive integers. For example, they wrote $\frac{1}{3} + \frac{1}{15}$ instead of $\frac{2}{5}$. Prove that it is always possible to do this in a systematic way: if $0 < m/n < 1$, then

$$\frac{m}{n} = \frac{1}{q} + \left\{\text{representation of } \frac{m}{n} - \frac{1}{q}\right\}\ ,\quad q = \left\lceil \frac{n}{m} \right\rceil\,.$$

(This is *Fibonacci's algorithm*, due to Leonardo Fibonacci, A.D. 1202.)

Solution. Let ϕ be the function such that

$$\frac{m}{n} - \frac{1}{q} = \frac{\phi(n,m)}{qn}\,.$$

Then,

$$\phi(n,m) = mq - n = m\,\lceil n/m \rceil - n = m(\lceil n/m \rceil - n/m) < m\,,$$

because $\lceil n/m \rceil - n/m < 1$. This means that the recursive calls of Fibonacci's algorithm are made on fractions which have strictly decreasing numerators, and as such, one of them will end up being 0: at that point the algorithm will terminate. But as the numerators are *de*creasing, the denominators are *in*creasing, because by construction $q, n \geqslant 2$: in particular, the x's are strictly increasing, thus all different.

Exercise C.3 (from the midterm test of 29 March 2019)

Express

$$\sum_{k=1}^{n} \left\lceil \sqrt{\left\lfloor \sum_{j=0}^{k} \frac{1}{j!} \right\rfloor + \left[\sqrt[3]{k} \in \mathbb{Z}\right]} \right\rceil$$

as a function of n. *Hint:* this is a "don't panic" question and the answer is rather simple.

Solution. The scary part is the argument of the square root. The cubic root inside the Iverson brackets looks like a false alarm, because that summand will either be 0 or 1: let's focus on the floor instead. We know from Calculus that the series $\sum_{j\geqslant 0} 1/j!$ converges to $e = 2.71828\ldots < 3$: as for $k = 1$ it is $1/0! + 1/1! = 1 + 1 = 2$, the floor is always 2. Then the argument of the square root is always either 2 or 3, and its ceiling is always 2. In conclusion,

$$\sum_{k=1}^{n} \left\lceil \sqrt{\left\lfloor \left(1 + \frac{1}{k}\right)^{k} \right\rfloor + \left[\sqrt[3]{k} \in \mathbb{Z}\right]} \right\rceil = 2n\,.$$