ITT9132 – Concrete Mathematics
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Exercise 1

Solve the recurrence:

$$\begin{aligned} g_0 &= 2\,, \\ g_1 &= \frac{5}{2}\,, \\ g_n &= \frac{3}{2}g_{n-1} - \frac{1}{2}g_{n-2} \text{ for every } n \geqslant 2\,. \end{aligned}$$

Solution. We use our usual four-step procedure:

1. We rewrite the recurrence so that it holds for every $n \in \mathbb{Z}$, with the convention that $g_n = 0$ if $n < 0$:

$$g_n = \frac{3}{2}g_{n-1}1\frac{1}{2}g_{n-2} + c_0\,[n = 0] + c_1\,[n = 1] \quad \text{for every } n \in \mathbb{Z}\,,$$

where a_0 and a_1 are correction terms to take into account the initial conditions:

(a) For $n = 0$ it is $g_0 = 2$ and $\frac{3}{2}g_{-1} - \frac{1}{2}g_{-2} = 0$: hence, $c_0 = 2$.

(b) For $n = 1$ it is $g_1 = \frac{5}{2}$ and $\frac{3}{2}g_0 - \frac{1}{2}g_{-1} = 3$. hence, $c_1 = -\frac{1}{2}$.

We thus obtain:

$$g_n = \frac{3}{2}g_{n-1} - \frac{1}{2}g_{n-2} + 2\,[n = 0] - \frac{1}{2}\,[n = 1] \quad \text{for every } n \in \mathbb{Z}\,.$$

2. Multiplying by z^n and summing over $n \in \mathbb{Z}$, we obtain:

$$\begin{aligned}
G(z) &= \sum_n g_n z^n \\
&= \frac{3}{2}\sum_n g_{n-1}z^n - \frac{1}{2}\sum_n g_{n-2}z^n + 2\sum_n [n=0]\, z^n - \frac{1}{2}\sum_n [n=1]\, z^n \\
&= \frac{3}{2}z\sum_n g_{n-1}z^{n-1} - \frac{1}{2}z^2\sum_n g_{n-2}z^{n-2} + 2 - \frac{1}{2}z \\
&= \frac{3}{2}zG(z) - \frac{1}{2}z^2G(z) + 2 - \frac{1}{2}z\,.
\end{aligned}$$

3. Solving with respect to $G(z)$, we obtain:

$$G(z) = \frac{2 - \frac{1}{2}z}{1 - \frac{3}{2}z + \frac{1}{2}z^2}\,.$$

4. The generating function is a rational function of the form $G(z) = P(z)/Q(z)$ with $P(z) = 2 - \frac{1}{2}z$ and $Q(z) = 1 - \frac{3}{2}z + \frac{1}{2}z^2$. The reverse of the denominator is:

$$Q^R(z) = z^2 - \frac{3}{2}z + \frac{1}{2} = (z-1)\left(z - \frac{1}{2}\right)\,,$$

so $Q(z) = (1-z)(1-z/2)$. We can now use either subdivision into partial fractions, or the Rational Expansion Theorem.

- **Subdivision into partial fractions.** We must find two constants A, B such that:

$$\frac{A}{1-z} + \frac{B}{1-z/2} = \frac{2 - \frac{1}{2}z}{1 - \frac{3}{2}z + \frac{1}{2}z^2}\,.$$

By multiplying and comparing the coefficient we get the system of two linear equations in two unknowns:

$$\begin{aligned}
A \quad &+B &= 2 \\
-\frac{1}{2}A \quad &-B &= -\frac{1}{2}
\end{aligned}$$

which has solution $A = 3, B = -1$. Then $G(z) = \frac{3}{1-z} - \frac{1}{1-z/2}$. By comparing the coefficients, we conclude: $g_n = 3 - 2^{-n}$.

- **Rational Expansion Theorem.** The denominator $Q(z)$ has inverse roots $\rho_1 = 1$ of multiplicity $d_1 = 1$ and $\rho_2 = 1/2$ with multiplicity $d_2 = 1$, and its derivative is $Q'(z) = z - \frac{3}{2}$. By the Rational Expansion Theorem, it is $g_n = a_1 \cdot 1^n + a_2 \cdot (1/2)^n$, where a_1 and a_2 are calculated as follows:

$$a_1 = \frac{-\rho_1 P(1/\rho_1)}{Q'(1/\rho_1)} = \frac{-\left(2 - \frac{1}{2}\right)}{1 - \frac{3}{2}} = \frac{3/2}{1/2} = 3\,;$$

$$a_2 = \frac{-\rho_2 P(1/\rho_2)}{Q'(1/\rho_2)} = \frac{-\frac{1}{2}\left(2 - \frac{1}{2} \cdot 2\right)}{2 - \frac{3}{2}} = \frac{-1/2}{1/2} = -1\,.$$

We conclude: $g_n = 3 - 2^{-n}$.

Exercise 2

For $n \in \mathbb{N}$ integer and $\alpha, \beta \in \mathbb{R}$ calculate:

$$S_n = \sum_{k=0}^{n} (-1)^k \binom{\alpha+1}{k} \binom{\beta+n-k}{n-k}$$

Solution. The sequence $\langle S_n \rangle$ is the convolution of the sequences $\left\langle (-1)^n \binom{\alpha+1}{n} \right\rangle$ and $\left\langle \binom{\beta+n}{n} \right\rangle$. The generating functions of the latter is $\frac{1}{(1-z)^{\beta+1}}$; for the other, we recall that $\sum_{k\geqslant 0}\binom{\alpha+1}{k}z^k = (1+z)^{\alpha+1}$, so $\sum_{k\geqslant 0}\binom{\alpha+1}{k}(-1)^k z^k$ is simply $(1-z)^{\alpha+1}$. Then the generating function of $\langle S_n\rangle$ is:

$$(1-z)^{\alpha+1}\cdot\frac{1}{(1-z)^{\beta+1}} = (1-z)^{\alpha-\beta}\,.$$

We conclude: $S_n = (-1)^n\binom{\alpha-\beta}{n}$.

Exercise 3

Give an asymptotic estimate of

$$\sqrt{1-\frac{1}{2n}+\frac{1}{3n^2}}$$

with absolute error $O(1/n^3)$.

Solution. Using the power series expansion $\sqrt{1+z} = \sum_{n\geqslant 0}\binom{1/2}{n}z^n$ we find:

$$\begin{aligned}\sqrt{1-\frac{1}{2n}+\frac{1}{3n^2}} &= 1+\binom{1/2}{1}\left(-\frac{1}{2n}+\frac{1}{3n^2}\right)+\binom{1/2}{2}\left(-\frac{1}{2n}+\frac{1}{3n^2}\right)^2\\ &\quad +O\left(\left(-\frac{1}{2n}+\frac{1}{3n^2}\right)^3\right)\end{aligned}$$

We have $\binom{1/2}{1}=\frac{1}{2}$ and $\binom{1/2}{2}=\frac{(1/2)\cdot(-1/2)}{2}=-\frac{1}{8}$. Substituting, we get:

$$\begin{aligned}\sqrt{1-\frac{1}{2n}+\frac{1}{3n^2}} &= 1-\frac{1}{4n}+\frac{1}{6n^2}-\frac{1}{8}\left(\frac{1}{4n^2}+O\left(\frac{1}{n^3}\right)\right)+O\left(\frac{1}{n^3}\right)\\ &= 1-\frac{1}{4n}+\left(\frac{1}{6}-\frac{1}{32}\right)\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)\\ &= 1-\frac{1}{4n}+\frac{13}{96n^2}+O\left(\frac{1}{n^3}\right).\end{aligned}$$

Exercise 4

For each of the following questions, mark the only correct answer:

1. How many steps are required to move a Hanoi tower with 5 disks from peg 1 to peg 3, if direct moves between pegs 1 and 3 are disallowed?

   (a) 31.
   No: it would be so if direct moves between pegs 1 and 3 were allowed.

   (b) 242.
   Yes: this case requires $3^6 - 1 = 729 - 1 = 728$ moves.

   (c) 243.
   No: see above.

2. What function satisfies the difference equation $\Delta f(x) = f(x)$ with the initial condition $f(0) = 17$?

   (a) $f(x) = 17^{x+1}$.
   No: this function satisfies $\Delta f(x) = 16 \cdot 17^{x+1}$.

   (b) $f(x) = 2^x$.
   No: this function satisfies $\Delta f(x) = f(x)$, but with $f(0) = 1$.

   (c) $f(x) = 17 \cdot 2^x$.
   Yes.

3. How is $x^{\overline{m}}$ defined for $m < 0$?

   (a) $x^{\overline{m}} = \dfrac{1}{(x+1)^{\overline{-m}}}$.
   No: this defines $x^{\underline{m}}$, not $x^{\overline{m}}$, for $m < 0$.

   (b) $x^{\overline{m}} = \dfrac{1}{(x-1)^{\underline{-m}}}$.
   Yes.

   (c) $x^{\overline{m}} = \dfrac{1}{x^{\underline{-m}}}$.
   No: see above.

4. True or false: if $\langle a_n \rangle_{n\geqslant 1}$ converges, then $\langle \left(\sum_{k=1}^{n} a_k\right)/n \rangle_{n\geqslant 1}$ converges to the same limit.

   True: this is the arithmetic mean theorem.

5. Which one of the following series has the commutative property?

   (a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n.$

   No: this series does not converge.

   (b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n}.$

   No: this series converges, but not absolutely, so it does not have the commutative property.

   (c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}.$

   Yes: this series converges absolutely, so it has the commutative property.

6. True or false: for every $x \in \mathbb{R}$, $\left\lceil \lceil x \rceil^3 \right\rceil = \lceil x^3 \rceil$.

   False: for example, $\left\lceil \lceil 3/2 \rceil^3 \right\rceil = 8$ but $\lceil (3/2)^3 \rceil = \lceil 27/8 \rceil = 7$. The function $f(x) = x^3$ is continuous and strictly increasing, but does not satisfy the requirement that x is integer if so is x^3.

7. True or false: $35^{64} - 1$ is divisible by 24.

   True: $\gcd(35, 24) = 1$ and $\phi(24) = 24 \cdot \dfrac{1}{2} \cdot \dfrac{2}{3} = 8$, so $35^{64} = (35^8)^8 \equiv 1^8 = 1 \pmod{24}$.

8. Which one of the following is the definition of a multiplicative function?

   (a) A function f such that $f(mn) = mf(n)$ for every n and for a fixed m.

   No.

   (b) A function f such that $f(mn) = f(m)f(n)$ for every m, n positive integers.

   No: a crucial hypothesis is lacking.

(c) A function f such that $f(mn) = f(m)f(n)$ for every m, n positive integers such that $m \perp n$.

Yes.

9. Let μ be the Möbius function. What is $\mu(65537!)$? *Hint:* this is a "don't panic" question.

(a) -1.

No: see below.

(b) 1.

No: see below.

(c) 0.

Yes: $n!$ is divisible by 4 for every $n \geqslant 4$.

10. True or false: for every $n \geqslant 0$, $\binom{-1/2}{n} = \left(\frac{1}{4}\right)^n \binom{2n}{n}$.

False: $\binom{-1/2}{n} = \left(-\frac{1}{4}\right)^n \binom{2n}{n}$.

11. True or false: $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k} 2^{-j} = 2$.

True: this is the Cesàro sum of the series $\sum_{n\geqslant 0} 2^{-n}$.

12. Which one of the following is the correct recurrence equation for the Stirling numbers of the first kind?

(a) $\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}$.

No: this is the recurrence for binomial coefficients.

(b) $\begin{bmatrix} n+1 \\ k \end{bmatrix} = k\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}$.

No: this is the recurrence for the Stirling numbers of the *second* kind.

(c) $\begin{bmatrix} n+1 \\ k \end{bmatrix} = n\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}$.

Yes.

13. Which one of the following equalities holds for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$?

(a) $\displaystyle\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k = x^{\underline{n}}$.
No: the roles of powers and falling powers are swapped.

(b) $\displaystyle\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k = x^{\overline{n}}$.
No: the formula uses the wrong Stirling numbers.

(c) $\displaystyle\sum_{k=0}^{n} \left[{n \atop k} \right] x^k = x^{\overline{n}}$.
Yes.

14. Which one of the following formulas returns, for large n, the Fibonacci number F_n?

(a) $\left\lfloor \dfrac{\Phi^n}{\sqrt{5}} + \dfrac{1}{2} \right\rfloor$
Yes: if n is large, then $\widehat{\Phi}^n$ is small, and F_n is simply the closest integer to $\dfrac{\Phi^n}{\sqrt{5}}$.

(b) $\Phi^n + \widehat{\Phi}^n$.
No: that formula returns the nth Lucas number.

(c) $\left\lfloor \dfrac{\Phi^n}{\sqrt{5}} - \dfrac{1}{2} \right\rfloor$
No: the sign inside the floor should be plus, not minus.

15. What is the value of the Bernoulli number B_{65537}? *Hint:* this is another "don't panic" question.

(a) 0.
Yes: Bernoulli numbers of odd index $n > 1$ are all zero.

(b) 1.
No: see above.

(c) $\dfrac{3489067234}{3489673}$.
No: that was just random key smashing.

16. What is the generating function of the sequence of the squares of the natural numbers?

(a) $\dfrac{z}{(1-z)^2}$.

No: this is the generating function of $\langle n \rangle$, not of $\langle n^2 \rangle$.

(b) $\dfrac{z(1+z)}{(1-z)^3}$.

Yes.

(c) $\ln \dfrac{1}{1-z}$.

No: this is the generating function of $\left\langle \frac{1}{n}\,[n>0] \right\rangle$.

17. True or false: $\displaystyle\sum_{n\geqslant 0} \binom{-1/2}{n} 3^n = \frac{1}{2}$.

False: the power series $\displaystyle\sum_{n\geqslant 0} \binom{-1/2}{n} z^n$ does not converge at $z=3$.

18. Let $G(z)$ be the generating function of the sequence $\langle g_n \rangle$. What is the generating function of the sequence $\left\langle \displaystyle\sum_{k=0}^{n} (-1)^{n-k} g_k g_{n-k} \right\rangle$?

(a) $(G(z))^2$.

No: this is the generating function of $\left\langle \displaystyle\sum_{k=0}^{n} g_k g_{n-k} \right\rangle$.

(b) $G(z)G(-z)$.

Yes: if $G(z) = \displaystyle\sum_{n\geqslant 0} g_n z^n$, then $\displaystyle\sum_{n\geqslant 0} (-1)^n g_n z^n = G(-z)$.

(c) $\dfrac{G(z)}{1-z}$.

No: this is the generating function of $\left\langle \displaystyle\sum_{k=0}^{n} g_k \right\rangle$.

19. Which one of the following is an approximation of $\cos\left(\frac{1}{n} - \frac{1}{n^2}\right)$ with absolute error $O(1/n^4)$?

(a) $1 - \frac{1}{2n^2} + \frac{1}{n^3} + O\left(\frac{1}{n^4}\right)$.

Yes.

(b) $1 - \frac{1}{2n^2} + \frac{1}{2n^3} + O\left(\frac{1}{n^4}\right)$.

No: the coefficient of $\frac{1}{n^3}$ in $\left(\frac{1}{n} - \frac{1}{n^2}\right)^2$ is -2.

(c) $1 - \frac{1}{2n^2} + O\left(\frac{1}{n^4}\right)$.

No: this is an approximation of $\cos\left(\frac{1}{n}\right)$, not of $\cos\left(\frac{1}{n} - \frac{1}{n^2}\right)$.

20. True or false: if $f(n) \prec 1$, then $\ln(1 + f(n)) = 1 + f(n) - \frac{1}{2}(f(n))^2 + O((f(n))^3)$.

False: the right-hand side has a summand 1 which does not appear in the expansion of $\ln(1+z)$ up to $O(z^3)$.