ITT9132 Concrete Mathematics

Konkreetne Matemaatika
Конкретная Математика

Silvio Capobianco

Department of Software Science, School of Information Technology

Tallinn University of Technology

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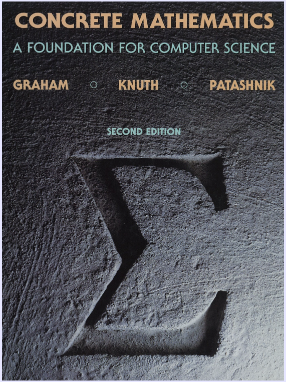
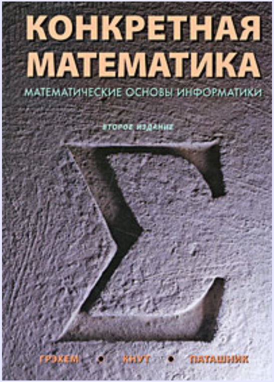
Original slides 2010–2014 Jaan Penjam; modified 2016–2023 Silvio Capobianco

CONtinuous + disCRETE
MATHEMATICS

The book





http://www-cs-faculty.stanford.edu/~uno/gkp.html

Concrete Mathematics is ...

- the controlled manipulation of mathematical formulas
- using a collection of techniques for solving problems

Goals of the book:

- to introduce the mathematics that supports advanced computer programming and the analysis of algorithms
- to provide a solid and relevant base of mathematical skills - the skills needed
  - to solve complex problems
  - to evaluate horrendous sums
  - to discover subtle patterns in data

Our additional goals

- to get acquainted with well-known and popular literature in Computer Science and in Mathematics;
- to develop mathematical skills and practice formulating complex problems mathematically;
- to practice presentation of results.

Contents of the Book

Chapters:

TAL TECH

Recurrent problems

Recurrences

- A sequence of complex numbers $\langle a_n \rangle = \langle a_0, a_1, a_2, \ldots \rangle$ is called recurrent, if for $n \geq 1$ its generic term a_n satisfies a recurrence (equation)

$$a_n = f_n(a_{n-1}, \ldots, a_0)\,,$$

  where $f_n : \mathbb{C}^n \to \mathbb{C}$ for every $n \geq 1$.
- If there exists $f : \mathbb{N} \times \mathbb{C}^k \to \mathbb{C}$ such that:

$$f_n = f(n; a_{n-1}, \ldots, a_{n-k}) \text{ for every } n \geq k\,,$$

  the number k is called the order of the recurrence.

recurrent – from the Latin recurrere to run back – in Estonian: taastuv

Two examples of recurrence equations

A recurrence of order 2

$$\begin{aligned} a_0 &= 0\,; \\ a_1 &= 1\,; \\ a_n &= a_{n-1} + a_{n-2} \text{ for every } n \geq 2 \end{aligned}$$

This recurrence defines the Fibonacci numbers.

A recurrence without a well-defined order

$$\begin{aligned} a_0 &= 1\,; \\ a_n &= a_0\,a_{n-1} + a_1\,a_{n-2} + \ldots + a_{n-1}\,a_0 \text{ for every } n \geq 1 \end{aligned}$$

This recurrence defines the Catalan numbers.

Why solve recurrences?

Reason 1: Efficiency.

- Consider the problem of adding together the first n positive integers.
- Naive algorithm:

```
sum  = 0
i    = 1
while i <= n:
    sum = sum + i
    i = i + 1
return sum
```

- Algorithm based on closed form:

```
return (n * (n+1)) / 2
```

- The second algorithm is clearly faster than the first one!

TAL TECH

Why solve recurrences?

Reason 2: Manageability.

- Stirling's approximation for the factorial:

$$\lim_{n\to\infty} \frac{e^n n!}{n^n \sqrt{2\pi n}} = 1$$

- Then, however given a_n, the two sequences:

$$n! a_n \,, \quad \left(\frac{n}{e}\right)^n a_n \sqrt{2\pi n}$$

  either both converge or both don't converge...
- ...and if they converge, the limit is the same!
- But the second sequence is more manageable than the first one: For example, we can use fast exponentiation.

Ad hoc techniques: Guess and Confirm

Example: $f(n) = (n^2 - 1 + f(n-1))/2$, $f(0) = 2$

- Let's compute some values:

n	0	1	2	3	4	5	6
$f(n)$	2	1	2	5	10	17	26

Guess: $f(n) = (n-1)^2 + 1$.

- Assuming that the guess holds for $n = k$, we prove that it holds for $n = k+1$:

$$\begin{aligned} f(k+1) &= ((k+1)^2 - 1 + f(k))/2 \\ &= (k^2 + 2k + (k-1)^2 + 1)/2 \\ &= (k^2 + 2k + k^2 - 2k + 1 + 1)/2 \\ &= (2k^2 + 2)/2 = k^2 + 1 \end{aligned}$$

QED.

TAL TECH

Ad hoc techniques: Guess and Confirm

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f(k+1) &= ((k+1)^2 - 1 + f(k))/2 \\
&= (k^2 + 2k + (k-1)^2 + 1)/2 \\
&= (k^2 + 2k + k^2 - 2k + 1 + 1)/2 \\
&= (2k^2 + 2)/2 = k^2 + 1
\end{aligned}
$$

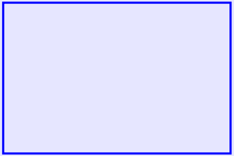
QED.

Next section

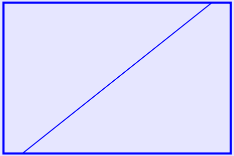
1. Recurrent Problems

1. The Tower of Hanoi
2. Lines in the Plane
3. The Josephus Problem

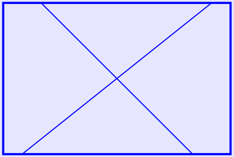
Regions of the plane defined by lines



$Q_0 = 1$



$Q_1 = 2$

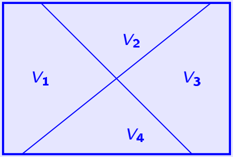


$Q_2 = 4$

In general: $Q_n = 2^n$?

Regions of the plane defined by lines

Actually ...



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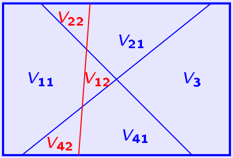

$$Q_3 = Q_2 + 3 = 7$$

Generally $Q_n = Q_{n-1} + n$.

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
Q_n	1	2	4	7	11	16	22	29	37	46	$\cdots$

Regions of the plane defined by lines

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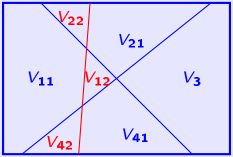

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TAL TECH

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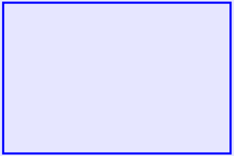
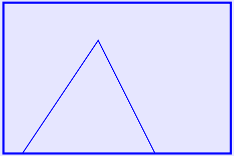
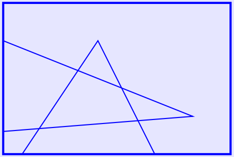


$T_0 = 1$



$T_1 = 2$

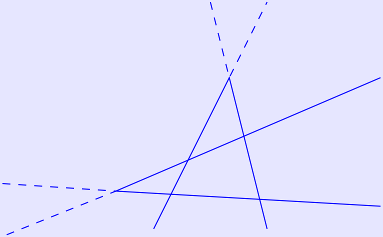


$T_2 = 7$

$T_3 = ?$

$T_n = ?$

Regions of the plane defined by lines

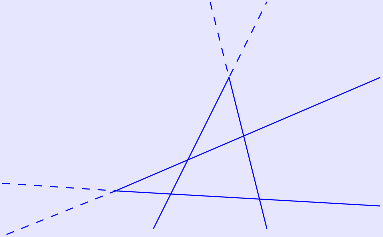


$$
\begin{aligned}
T_2 &= Q_4 - 2 \cdot 2 = 11 - 4 = 7 \\
T_3 &= Q_6 - 2 \cdot 3 = 22 - 6 = 16 \\
T_4 &= Q_8 - 2 \cdot 4 = 37 - 8 = 29 \\
T_5 &= Q_{10} - 2 \cdot 5 = 56 - 10 = 46
\end{aligned}
$$

$$T_n = Q_{2n} - 2n$$

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
Q_n	1	2	4	7	11	16	22	29	37	46	$\cdots$
T_n	1	2	7	16	29	46	67	92	121	156	$\cdots$

Regions of the plane defined by lines



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Next section

TAL TECH

2. Sums

1. Notation
2. Sums and Recurrences
3. Manipulation of Sums
4. Multiple Sums
5. General Methods
6. Finite and Infinite Calculus
7. Infinite Sums

Sums as solutions of recurrences

The simplest (nontrivial) recurrences have the form:

$$\begin{aligned} a_0 &= c_0 \; ; \\ a_n &= a_{n-1} + c_n \text{ for every } n \geq 1 \, . \end{aligned}$$

The solution to the above is clearly:

$$a_n = \sum_{k=0}^{n} c_k$$

Problem: find a closed form for the sum!

A simple case: Gauss' trick

Solve the recurrence:

$$S_0 = 0\,; \;\; S_n = S_{n-1} + n \;\text{ for every } n \geq 1$$

This is the same as calculating $\sum_{k=0}^{n} k$. Well:

- Addition is commutative, so:

$$0 + 1 + \cdots + (n-1) + n = n + (n-1) + \cdots + 1 + 0 = \sum_{k=0}^{n} (n-k)$$

- Addition is also associative, so:

$$2S_n = \sum_{k=0}^{n} k + \sum_{k=0}^{n} (n-k) = \sum_{k=0}^{n} n$$

- The right-hand side is a sum of $n+1$ summands, all equal to n. We conclude:

$$S_n = \frac{n(n+1)}{2}$$

The perturbation method

Consider a recurrence of the form:

$$S_0 = a_0\,;\;\; S_n = S_{n-1} + a_n \text{ for every } n \geq 1$$

Sometimes we can solve the recurrence by perturbing the sum:

1. Rewrite:

$$S_{n+1} = S_n + a_{n+1} = a_0 + \sum_{k=1}^{n+1} a_k$$

2. Manipulate $\sum_{k=1}^{n+1} a_k$ to express it as a function of S_n.
3. Solve for S_n.

Example: Sum of a geometric progression

Let $a \neq 1$. Consider the recurrence:

$$S_0 = 1\,;\;\; S_n = S_{n-1} + a^n \text{ for every } n \geq 1$$

1. Rewrite:

$$S_{n+1} = S_n + a^{n+1} = 1 + \sum_{k=1}^{n+1} a^k$$

2. Manipulate:

$$\sum_{k=1}^{n+1} a^k = a \cdot \sum_{k=0}^{n} a^k = a \cdot S_n$$

3. Solve for S_n:

$$\begin{aligned} S_n + a^{n+1} &= 1 + a \cdot S_n \\ (1-a)S_n &= 1 - a^{n+1} \\ S_n &= \frac{1-a^{n+1}}{1-a} = \frac{a^{n+1}-1}{a-1} \end{aligned}$$

Summation factors

Consider a recurrence of the following form:

$$a_n T_n = b_n T_{n-1} + c_n \text{ for every } n \geq 1$$

Assume that we can find a *summation factor* s_n such that:

$$s_n b_n = s_{n-1} a_{n-1} \text{ for every } n \geq 1$$

Then, putting $a_0 = s_0 = 1$ and $S_n = s_n a_n T_n$, we turn our recurrence into the much easier:

$$\begin{aligned} S_0 &= T_0 ; \\ S_n &= S_{n-1} + s_n c_n \text{ for every } n \geq 1 \end{aligned}$$

From this we recover:

$$T_n = \frac{1}{s_n a_n} \left(T_0 + \sum_{k=1}^{n} s_k c_k \right) \text{ for every } n \geq 1$$

and as soon as we have a closed formula for $\sum_{k=1}^{n} s_k c_k$, we have one for T_n too.

Example: A recurrence with a factor n

Consider the recurrence:

$$T_0 = 1\,;\;\; T_n = 2\,T_{n-1} + \left(\frac{3}{2}\right)^n \;\text{ for every } n \geq 1$$

Here $a_n = 1$, $b_n = 2$, and $c_n = (3/2)^n$, so we must solve:

$$s_n \cdot 2 = s_{n-1} \cdot 1 \;\text{ for every } n \geq 1$$

Then $s_n = 1/2^n$ and for $S_n = s_n a_n T_n = T_n/2^n$ we have:

$$S_0 = 1\,;\;\; S_n = S_{n-1} + \left(\frac{3}{4}\right)^n \;\text{ for every } n \geq 1$$

We know that this has the solution:

$$\begin{aligned} S_n &= 1 + \sum_{k=1}^{n} \left(\frac{3}{4}\right)^k = \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k \\ &= \frac{1-(3/4)^{n+1}}{1-(3/4)} = \frac{4^{n+1}-3^{n+1}}{4^n} \;\text{ for every } n \geq 0 \end{aligned}$$

and from this we conclude:

$$T_n = \frac{4^{n+1}-3^{n+1}}{2^n} \;\text{ for every } n \geq 0$$

Finite calculus

A way of "working on sums like they were integrals":

- Finite difference instead of derivative:

$$\Delta f(x) = f(x+1) - f(x) \text{ for every } x$$

- Idea: if $S_n = \sum_{k=0}^{n} a_k$, then $\Delta S_n = a_{n+1}$, and vice versa.
- A new family of *elementary functions* which solve specific difference equations (instead of "differential"):
  - Falling factorials in place of powers.
  - Harmonic numbers in place of logarithm.
- "Summation by parts".
- Stolz-Cesàro lemma in place of l'Hôpital's rule.

Infinite sums

On the one hand:

Example 1

Let

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots.$$

Then

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2 + S,$$

and

$$S = 2$$

Infinite sums

. . . but on the other hand:

Example 2

Let

$$T = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \ldots$$

Then

$$2T = 2 + 4 + 8 + 16 + 32 + 64 + 128 \ldots = T - 1$$

and

$$T = -1$$

TAL TECH

Infinite sums

Existence of the sum:

- Riemann summation as limit of partial sums.
- Lebesgue summation as a difference of least upper bounds.

Manipulation of sums:

- When are infinite sums commutative, associative, etc.?
- Riemann series theorem and absolute convergence.

Sums and limits:

- When does the limit of the sums coincide with the sum of the limits?
- Dominated convergence theorem and monotone convergence theorem.

Double sums:

- When does a simultaneous double sum coincide with an iterated double sum?
- Fubini's theorem on infinite double sums.

Other interpretations of the idea of convergence:

- Cesàro summation and Abel summation.

Next section

3. Integer Functions

1. Floors and Ceilings
2. Floor/Ceiling Applications
3. Floor/Ceiling Recurrences
4. 'mod': The Binary Operation
5. Floor/Ceiling Sums

Floor and ceiling

The ceiling of the real number x is the integer:

$$\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}$$

Dually, the floor of x is the integer:

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}$$

The following important chain of inequalities holds:

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

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$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

Generalized Pigeonhole Principle

If $m \geq 1$ pigeons are to be put in $n \geq 1$ pigeonholes, then:

- at least one pigeonhole will contain at least $\lceil m/n \rceil$ pigeons; and
- at least one pigeonhole will contain at most $\lfloor m/n \rfloor$ pigeons.

TAL
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Next section

4. Number Theory

1. Divisibility
2. Factorial Factors
3. Relative Primality
4. 'mod': The Congruence Relation
5. Independent Residues
6. Additional Applications
7. Phi and Mu

Divisibility and congruence

Integer divisibility:

- An integer a *divides*, or *is a factor of*, an integer b, written $a \backslash b$, if there exists an integer k such that $k \cdot a = b$.
- Note that, with this definition, every integer divides 0.

Modular congruence:

- If a, b, n are all integer, then *a is congruent to b modulo n*, written $a \equiv b \pmod{n}$, if n is a factor of $a - b$.
- Addition and multiplication "behave well" with respect to modular congruence. Not so exponentiation!

Special functions: ϕ and μ

Euler's function ϕ:

- For m positive integer, $\phi(m)$ is the number of integers a between 1 and m such that $\gcd(a,m)=1$.
- Euler's theorem: if $a,m>0$ and $\gcd(a,m)=1$, then $a^{\phi(m)}\equiv 1 \pmod{m}$.

Möbius' function μ:

- $\mu(m)$ is $(-1)^k$ if m is a product of k distinct primes, and 0 if m is divisible by the square of a prime.
- Möbius' inversion formula: for any two functions $f,g:\mathbb{Z}_+\to\mathbb{C}$ the following are equivalent:
  1. For every $m\geq 1$, $f(m)=\sum_{d\backslash m} g(d)$.
  2. For every $m\geq 1$, $g(m)=\sum_{d\backslash m}\mu(m/d)f(d)$.

Next section

5. Binomial Coefficients

Counting choices

Definition

The binomial coefficient "n choose k", denoted $\binom{n}{k}$, is the number of ways we can choose k objects from a set of n objects, regardless of the order in which we choose them.

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Then for every $n \geq 0$ and $0 \leq k \leq n$:

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!},$$

where $k!$, read k factorial, is the number of ways in which we can order k items, and can be defined by the recurrence:

$$\begin{array}{rcl} 0! & = & 1, \\ n! & = & n \cdot (n-1)! \text{ for every } n \geq 1. \end{array}$$

Counting choices

Definition

The binomial coefficient "n choose k", denoted $\binom{n}{k}$, is the number of ways we can choose k objects from a set of n objects, regardless of the order in which we choose them.

We also have the two-parameter recurrence:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \text{ for every } n \geq 0 \text{ and } 1 \leq k \leq n.$$

The Binomial Theorem

Theorem (Newton)

For every two real numbers x and y and nonnegative integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

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Proof: by expanding the product:

$$(x+y)\cdot(x+y)\cdots(x+y),\ n \text{ factors overall}$$

- This will be a sum of monomials of the form $x^k y^{n-k}$.
- Each such monomial is produced by choosing k factors $(x+y)$ from which to take the k factors x.

Next section

6. Special Numbers

1. Stirling Numbers
2. Eulerian Numbers
3. Harmonic Numbers
4. Harmonic Summation
5. Bernoulli Numbers
6. Fibonacci Numbers
7. Continuants

Stirling numbers

Definition

The Stirling number of the second kind "n subset k", denoted $\left\{ n \atop k \right\}$, is the number of ways we can partition a set of n objects into k nonempty subsets.

Computing Stirling numbers is harder than computing binomial coefficients, but the following two-parameter recurrence holds:

$$\left\{ n+1 \atop k \right\} = k \cdot \left\{ n \atop k \right\} + \left\{ n \atop k-1 \right\} \text{ for every } n \geq 0 \text{ and } 1 \leq k \leq n\,.$$

Definition

The Stirling number of the first kind "n cycle k", denoted $\left[n \atop k \right]$, is the number of ways we can partition a set of n objects into k nonempty cycles.

This time:

$$\left[n+1 \atop k \right] = n \cdot \left[n \atop k \right] + \left[n \atop k-1 \right] \text{ for every } n \geq 0 \text{ and } 1 \leq k \leq n\,.$$

Fibonacci numbers

Defined by the "simplest" second-order recurrence:

$$F_0 = 0\,;\ \ F_1 = 1\,;$$
$$F_n = F_{n-1} + F_{n-2} \quad \text{for every } n \geq 2$$

- Appear in several "natural" processes.
- Cassini's identity:

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n$$

- gcd law:

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}$$

- Played a crucial role in the solution of Hilbert's tenth problem.

Harmonic numbers

Definition

The harmonic numbers, denoted by H_n, are defined by the recurrence:

$$\begin{aligned} H_0 &= 0\,, \\ H_n &= H_{n-1} + \frac{1}{n} \text{ for every } n \geq 1\,. \end{aligned}$$

Note that $H_{2^n} > \frac{n}{2}$ for every $n \geq 0$:

$$H_{2^{n+1}} = H_{2^n} + \frac{1}{2^n+1} + \ldots + \frac{1}{2^{n+1}} \geq H_{2^n} + \frac{1}{2}\,.$$

In fact, the following estimate holds:

$$\ln n < H_n < 1 + \ln n$$

That is, harmonic numbers grow logarithmically.

TAL TECH

Harmonic numbers

Definition

The harmonic numbers, denoted by H_n, are defined by the recurrence:

$$\begin{aligned} H_0 &= 0\,, \\ H_n &= H_{n-1} + \frac{1}{n} \text{ for every } n \geq 1\,. \end{aligned}$$

More in general, the harmonic numbers of order s, denoted by $H_n^{(s)}$, are defined by the recurrence:

$$\begin{aligned} H_0^{(s)} &= 0\,, \\ H_n^{(s)} &= H_{n-1}^{(s)} + \frac{1}{n^s} \text{ for every } n \geq 1\,. \end{aligned}$$

For $s > 1$ the sequence $\left\langle H_n^{(s)} \right\rangle$ converges to a real number $\zeta(s)$: this defines the Riemann zeta function.

Bernoulli numbers

Jakob Bernoulli (1654-1705) studied the family of functions:

$$S_m(n) = \sum_{k=0}^{n-1} k^m$$

and discovered the following regularity:

Theorem

There exists a sequence $\langle B_n \rangle$ such that for every m, n nonnegative integers:

$$S_m(n) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}$$

The numbers B_n are called the Bernoulli numbers and have many remarkable properties.

Next section

7. Generating Functions

Solving recurrences with generating functions

Given a sequence $\langle g_n \rangle$ that satisfies a given recurrence, we seek a closed form for g_n which expresses it as a function of n, but not of $g_0, \ldots, g_{n-1}$.

The method of generating functions

1. Write a single equation that expresses g_n in terms of other elements of the sequence.
   This equation must hold for all integers n, assuming that $g_n = 0$ for every $n < 0$: this might need to add correction terms for the initial values.
2. Multiply both sides of the equation by z^n and sum over all n.
   This gives, on the left-hand side, the series $\sum_n g_n z^n$, which is the generating function $G(z)$ of the sequence $\langle g_n \rangle$.
   The right-hand side should be turned into some other expression involving $G(z)$.
3. Solve with respect to $G(z)$, obtaining an analytic form.
4. Expand the right-hand side into a power series and read off the coefficient of z^n: thanks to the properties of analytic functions in the complex plane, this is a closed form for g_n.

Example: Fibonacci numbers

1. Single equation holding for every $n \in \mathbb{Z}$:

$$g_n = g_{n-1} + g_{n-2} + [n = 1]$$

   where $[\text{True}] = 1$ and $[\text{False}] = 0$ are the *Iverson brackets.*

2. Multiply by z^n and obtain an equation for $G(z) = \sum_n g_n z^n$:

$$G(z) = zG(z) + z^2 G(z) + z$$

3. Solve with respect to $G(z)$:

$$G(z) = \frac{z}{1 - z - z^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \Phi z} - \frac{1}{1 - \widehat{\Phi} z} \right)$$

   where $\Phi = \dfrac{1 + \sqrt{5}}{2}$ is the *golden mean* and $\widehat{\Phi} = \dfrac{1 - \sqrt{5}}{2}$.

4. Derive an expression for g_n which only depends on n:

$$g_n = \frac{1}{\sqrt{5}} \left(\Phi^n - \widehat{\Phi}^n \right) \quad \text{for every } n \geq 0$$

Then for large n, F_n is the closest integer to $\Phi^n / \sqrt{5}$.

Next section

8. Discrete Probability

1. Definitions
2. Mean and Variance
3. Probability Generating Functions
4. Flipping Coins
5. Hashing

Next section

9. Asymptotics

Big-O notation

Definition

Let f and g be real-valued functions defined on the natural numbers.
We say that $f(n)$ is big-O of $g(n)$, and write $f(n) = O(g(n))$, if there exists $C > 0$ such that:

$$|f(n)| \leq C \cdot |g(n)| \text{ for every } n \text{ large enough}$$

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For example:

- $f(n) = O(1)$ if and only if f is *bounded*.
- $(\log n)^{\alpha} = O(n^{\beta})$ and $n^{\beta} = O(\gamma^{n})$ for every $\alpha, \beta > 0$ and $\gamma > 1$.
- $n^{\alpha} = O(n^{\beta})$ if and only if $\alpha \leq \beta$.

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Big-O notation can be tricky:

- Suppose $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$.
- Then we can conclude that $f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$ ...
- ...but only that $f_1(n) + f_2(n) = O(|g_1(n)| + |g_2(n)|)$.

It also loses the information about the *value* of C ...

Errors and the role of power series

We know that $e^x = \sum_{k \geq 0} \frac{x^k}{k!}$ and $\sin x = \sum_{k \geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ for every $x \in \mathbb{R}$. Then:

$$\sin \frac{1}{n} = \frac{1}{n} - \frac{1}{6n^3} + O\left(\frac{1}{n^5}\right)$$

so $\frac{1}{n} - \frac{1}{6n^3}$ approximates $\sin \frac{1}{n}$ with absolute error $O\left(\frac{1}{n^5}\right)$. Also:

$$\begin{aligned}
e^{\frac{1}{n}} &= 1 + \frac{1}{n} + \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right) \\
&= \left(1 + \frac{1}{n} + \frac{1}{2n^2}\right) \cdot \left(1 + \frac{1}{1 + \frac{1}{n} + \frac{1}{2n^2}} \cdot O\left(\frac{1}{n^3}\right)\right) \\
&= \left(1 + \frac{1}{n} + \frac{1}{2n^2}\right) \cdot \left(1 + O(1) \cdot O\left(\frac{1}{n^3}\right)\right) \\
&= \left(1 + \frac{1}{n} + \frac{1}{2n^2}\right) \cdot \left(1 + O\left(\frac{1}{n^3}\right)\right)
\end{aligned}$$

so $1 + \frac{1}{n} + \frac{1}{2n^2}$ approximates $e^{\frac{1}{n}}$ with relative error $O\left(\frac{1}{n^3}\right)$.

Pedagogical dilemma: what to teach?

Chapters:

1. Recurrent Problems
2. Sums
3. Integer Functions
4. Number Theory
5. Binomial Coefficients
6. Special Numbers
7. Generating Functions
8. Discrete Probability
9. Asymptotics

Course program (tentative)

- Week 1: Introduction
- Weeks 2 and 3: Recurrent Problems
- Weeks 4 and 5: Sums
- Week 6: Integer Functions
- Weeks 7 and 8: Number Theory
- Weeks 9 and 10: Binomial Coefficients
- Weeks 11 and 12: Special Numbers
- Weeks 13 and 14: Generating Functions
- Weeks 15 and 16: Asymptotics

Grading

Based on 100 points, distributed as follows:

- Two classroom presentations: 10 points each.
  One, two, or three each week, according to the number of participants.
- A midterm test: 30 points.
  On the ninth week.
- The final exam: 50 points.
  Three dates: one, two, and three weeks after the end of the course.

The final grade G is computed from the total score S as follows:

$$G = \max\left(0, \left\lceil \frac{S - 50}{10} \right\rceil\right)$$

- 91 or more: 5.
- 81 to 90: 4.
- 71 to 80: 3.
- 61 to 70: 2.
- 51 to 60: 1.
- 50 or less: 0.

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- A midterm test: 30 points.
  On the ninth week.
- The final exam: 50 points.
  Three dates: one, two, and three weeks after the end of the course.

The prerequisites to be admitted to the final exam are:

1. At least one classroom presentation.
2. At least 15 points at the midterm test.

Students who are not admitted to the final exam, or do not return their final assignment, will receive a "no show" mark.

Contact

Instructor:	**Silvio Capobianco**
Address:	Room B421, Tehnopol building, Akadeemia tee 21B
Office hours:	Thursdays from 15:00 to 17:00
Telephone:	620 4221
Email:	silvio.capobianco@taltech.ee silvio@cs.ioc.ee

Moodle page of the course

https://moodle.taltech.ee/course/view.php?id=31471

Enrolment key: **ConcMATH2023Spr** (case sensitive)