ITT9132 Concrete Mathematics
Lecture 3 – 14 February 2023

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Next section

The repertoire method: Basic ideas

Let the recursion scheme

$$\begin{array}{rcl} g(0) & = & \alpha\,, \\ g(n+1) & = & \Phi(g(n)) + \Psi(n;\beta,\gamma,\ldots) \quad \text{for } n \geqslant 0\,. \end{array}$$

have the following properties:

1. Φ is linear in g:
   If $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$, then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.
   No hypotheses are made on the dependence of g on n.
2. Ψ is linear in each of the $m-1$ parameters $\beta, \gamma, \ldots$
   No hypotheses are made on the dependence of Ψ on n.

Then the whole system is linear in the parameters $\alpha, \beta, \gamma, \ldots$
We can then look for a general solution of the form

$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \ldots$$

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The repertoire method: Description

Suppose we have a *repertoire* of m pairs of the form $((\alpha_i, \beta_i, \gamma_i, \ldots), g_i(n))$ satisfying the following conditions:

1. For every $i = 1, 2, \ldots, m$, $g_i(n)$ is the solution of the system corresponding to the values $\alpha = \alpha_i, \beta = \beta_i, \gamma = \gamma_i, \ldots$
2. The m m-tuples $(\alpha_i, \beta_i, \gamma_i, \ldots)$ are linearly independent.

Then the functions $A(n), B(n), C(n), \ldots$ are uniquely determined.
The reason is that, for every fixed n,

$$\begin{array}{lllll} \alpha_1 A(n) & +\beta_1 B(n) & +\gamma_1 C(n) & +\ldots & = & g_1(n) \\ \vdots & & & & & \vdots \\ \alpha_m A(n) & +\beta_m B(n) & +\gamma_m C(n) & +\ldots & = & g_m(n) \end{array}$$

is a system of m linear equations in the m unknowns $A(n), B(n), C(n), \ldots$ whose coefficients matrix is invertible.

The repertoire method for recursive data types: Setting

Assume that a certain recursive data type S has:

- finitely many base cases $b_1, \ldots, b_k$;
- finitely many constructors, which we may assume to have all the same number m of arguments:
$$u_1, \ldots, u_r : S^m \to S\,.$$

Consider a recursion of the following form:

$$\begin{array}{rcll} g(b_i) & = & \alpha_i & \text{for } i = 1, \ldots, k\,; \\ g(u_j(x_1, \ldots, x_m)) & = & \Phi_j(g(x_1), \ldots, g(x_m)) + \Psi_j(n; \beta_{j,1}, \ldots, \beta_{j,s_j}) & \text{for every } j = 1, \ldots, r \end{array}$$

where $\alpha_1, \ldots, \alpha_k, \beta_{1,1}, \ldots, \beta_{1,s_1}, \ldots, \beta_{r,s_r} \in \mathbb{C}$ and $\Phi_j : S^m \to \mathbb{C}, \Psi_j : S^{r_j} \to \mathbb{C}$ for every $j = 1, \ldots, r$. Assume that:

1. each one of $\Phi_1, \ldots, \Phi_k$ is linear in all of its arguments; and
2. each one of $\Psi_1, \ldots, \Psi_r$ is linear in each one of its arguments except at most n.

Then we could look for a generic solution of the recurrence of the following form:

$$A_1, \ldots, A_k, B_{1,1}, \ldots, B_{1,s_1}, \ldots, B_{r,s_r} : S \to \mathbb{C}$$

such that the solution of the recurrence has the form:

$$g(x) = \sum_{i=1}^{k} \alpha_i A_i(x) + \sum_{j=1}^{r} \sum_{\ell=1}^{s_j} \beta_{j,\ell} B_{j,\ell}(x) \text{ for every } x \in S\,.$$

The repertoire method for recursive data types: Description

Theorem

Given the system in the previous slide, let $p = k + \sum_{j=1}^{r} s_j$ be the total number of parameters. Assume we have a repertoire of p pairs of the form:

$$((\alpha_{1,i}, \dots, \alpha_{k,i}, \beta_{1,1,i}, \dots, \beta_{r,s_r,i}), g_i(x))$$

with the following properties:

1. For every $i = 1, \dots, p$, g_i is the solution corresponding to the choice of parameters:

$$\alpha_1 = \alpha_{1,i}, \dots, \alpha_k = \alpha_{k,i}, \beta_{1,1} = \beta_{1,1,i}, \dots, \beta_{r,s_r} = \beta_{r,s_r,i}$$

2. The p p-tuples $(\alpha_{1,i}, \dots, \alpha_{k,i}, \beta_{1,1,i}, \dots, \beta_{r,s_r,i})$ are linearly independent.

Then the p functions $A_1, \dots, A_k, B_{1,1}, \dots, B_{r,s_r} : S \to \mathbb{C}$ are uniquely determined.

Reason why: For every $x \in S$, the p linear equations:

$$\alpha_{1,1} A_1(x) + \cdots + \alpha_{k,i} A_k(x) + \beta_{1,1,i} B_{1,1}(x) + \cdots + \beta_{r,s_r,i} B_{r,s_R}(x) = g_i(x), \;\; i = 1, \dots, p$$

in the p unknowns $A_1(x), \dots, A_k(x), B_{1,1}(x), \dots, B_{r,s_r}(x)$ form a system that has a nonsingular matrix of coefficients.

Next section

Binary representation of generalized Josephus function

Definition

The generalized Josephus function (GJ-function) is defined for α, β_0, β_1 as follows:

$$\begin{aligned} f(1) &= \alpha \\ f(2n+j) &= 2f(n) + \beta_j \ \text{ for } j = 0, 1 \text{ and } n > 0\,. \end{aligned}$$

We obtain the definition used before if to select $\beta_0 = \beta$ and $\beta_1 = \gamma$.

Binary representation of generalized Josephus function (2)

Case A: Argument is even

If $2n = 2^m + \ell$, then the binary notation is

$$2n = (b_m b_{m-1} \dots b_1 b_0)_2$$

or

$$2n = b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + b_0$$

where $b_i \in \{0, 1\}$, $b_0 = 0$ and $b_m = 1$.

Hence

$$n = b_m 2^{m-1} + b_{m-1} 2^{m-2} + \dots + b_2 2 + b_1$$

or

$$n = (b_m b_{m-1} \dots b_1)_2$$

Binary representation of generalized Josephus function (3)

Case B: Argument is odd

If $2n+1 = 2^m + \ell$, then the binary notation is

$$2n+1 = (b_m b_{m-1} \dots b_1 b_0)_2$$

or

$$2n+1 = b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + b_0$$

where $b_i \in \{0,1\}$, $b_0 = 1$ and $b_m = 1$.

We get

$$\begin{aligned} 2n+1 &= b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + 1 \\ 2n &= b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 \\ n &= b_m 2^{m-1} + b_{m-1} 2^{m-2} + \dots + b_2 2 + b_1 \end{aligned}$$

or

$$n = (b_m b_{m-1} \dots b_1)_2$$

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We get

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As the results for cases A and B are similar, we *don't* need to consider even and odd cases separately!

Binary representation of generalized Josephus function (4)

Let's evaluate:

$$
\begin{aligned}
f((b_m, b_{m-1}, \ldots, b_1, b_0)_2) &= 2f((b_m, b_{m-1}, \ldots, b_1)_2) + \beta_{b_0} \\
&= 2 \cdot (2f((b_m, b_{m-1}, \ldots, b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= 4f((b_m, b_{m-1}, \ldots, b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\
&= \vdots \\
&= f((b_m)_2)2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \\
&= f(1)2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \\
&= \alpha 2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \, ,
\end{aligned}
$$

where

$$
\beta_{b_j} = \begin{cases} \beta_1, & \text{if } b_j = 1 \\ \beta_0 & \text{if } b_j = 0 \end{cases}
$$

$$f((b_m b_{m-1} \ldots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_2$$

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&= f(1)2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \\
&= \alpha 2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \,,
\end{aligned}
$$

where

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\beta_{b_j} = \begin{cases} \beta_1, & \text{if } b_j = 1 \\ \beta_0 & \text{if } b_j = 0 \end{cases}
$$

$$
f((b_m b_{m-1} \ldots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_2
$$

Example

Original Josephus function: $\alpha = 1$, $\beta_0 = -1$, $\beta_1 = 1$ i.e.

$$\begin{aligned} f(1) &= 1 \\ f(2n) &= 2f(n) - 1 \\ f(2n+1) &= 2f(n) + 1 \end{aligned}$$

Compute

$$f((b_m b_{m-1} \dots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \dots \beta_{b_1} \beta_{b_0})_2$$

$$\begin{aligned} f(100) = f((1100100)_2) &= (1, 1, -1, -1, 1, -1, -1)_2 \\ &= 64 + 32 - 16 - 8 + 4 - 2 - 1 = 73 \end{aligned}$$

Generalized Josephus function: Multiple bases

Let $c, d \geqslant 2$ be integers.
Consider the following recurrence:

$$\begin{array}{rcll} f(j) & = & \alpha_j & \text{for } 1 \leqslant j < d\,; \\ f(dn+j) & = & cf(n)+\beta_j & \text{for } 0 \leqslant j < d \text{ and } n \geqslant 1\,. \end{array}$$

How can we compute $f(n)$ for an arbitrary positive integer n, without having to go through the entire iterative process?

Multiple bases representation

We can actually use the same technique!

Let $(b_m b_{m-1} \dots b_1 b_0)_d$ be the base-d writing of n. Then $b_m \neq 0$ and:

$$
\begin{aligned}
f((b_m, b_{m-1}, \dots, b_1, b_0)_d) &= cf((b_m, b_{m-1}, \dots, b_1)_d) + \beta_{b_0} \\
&= c \cdot (cf((b_m, b_{m-1}, \dots, b_2)_d) + \beta_{b_1}) + \beta_{b_0} \\
&= c^2 f((b_m, b_{m-1}, \dots, b_2)_d) + c\beta_{b_1} + \beta_{b_0} \\
&= \vdots \\
&= c^m \cdot f(b_m) + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0} \\
&= c^m \alpha_{b_m} + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0}
\end{aligned}
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&= c^m \alpha_{b_m} + c^{m-1}\beta_{b_{m-1}} + \ldots + c\beta_{b_1} + \beta_{b_0}
\end{aligned}
$$

With a slight abuse of notation: (the β_i's need not be base-c digits)

$$f((b_m b_{m-1} \ldots b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_c$$

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&= c^m \alpha_{b_m} + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0}
\end{aligned}
$$

Or, more precisely:

$$f((b_m b_{m-1} \dots b_1 b_0)_d) = p(c) \text{ where } p(x) = \alpha_{b_m} x^m + \beta_{b_{m-1}} x^{m-1} + \dots + \beta_{b_1} x + \beta_{b_0}$$

Next section

Sequences

Definition

A sequence of elements of a set A is a function $f : \mathbb{N} \to A$, where $\mathbb{N}$ is the set of natural numbers.

Notations used:

- $f = \langle a_n \rangle$, where we denote $a_n = f(n)$;
- $\{a_n\}_{n \in \mathbb{N}}$;
- $\langle a_0, a_1, a_2, a_3, \ldots \rangle$.

a_n is called the nth term of the sequence f

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Example

$$a_0 = 0,\ a_1 = \frac{1}{2 \cdot 3},\ a_2 = \frac{2}{3 \cdot 4},\ a_3 = \frac{3}{4 \cdot 5}, \cdots$$

or

$$\left\langle 0, \frac{1}{6}, \frac{1}{6}, \frac{3}{20}, \frac{2}{15}, \cdots, \frac{n}{(n+1)(n+2)}, \cdots \right\rangle$$

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Notation

$$f(n) = \frac{n}{(n+1)(n+2)}$$

or

$$a_n = \frac{n}{(n+1)(n+2)}$$

Sets of indices

- Default assumption: $\mathbb{N}$.
- Actually, any countably infinite set can be used as an index set. Examples:
  - $\mathbb{Z}^+ = \mathbb{N} - \{0\} \sim \mathbb{N}$.
  - $\mathbb{N} \setminus K$, where $K \subseteq \mathbb{N}$ is finite.
  - The set $\mathbb{Z}$ of relative integers.
  - $\{1, 3, 5, 7, \ldots\} = \mathrm{Odd}$.
  - $\{0, 2, 4, 6, \ldots\} = \mathrm{Even}$.

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The writing $A \sim B$ denotes that sets A and B are of the same cardinality.

- For finite sets, $|A|$ is the number of elements of A.
- In general, A and B are said to have the same cardinality if there exists a *bijection* between the two.
  We then write $A \sim B$, or $|A| = |B|$

(See http://www.mathsisfun.com/sets/injective-surjective-bijective.html for detailed explanation)

Finite sequences

- A finite sequence of elements of a set A is a function $f : K \to A$, where K is set a finite subset of natural numbers

  For example: $f : \{1, 2, 3, 4, \cdots, n\} \to A$, $n \in \mathbb{N}$

  Special case: $n = 0$, i.e. empty sequence: $f(\emptyset) = e$

Domain of a sequence

In general, we might be dealing with partial functions:

- Although a generic formula might be given, such formula might not be applicable in some cases.
- For example, the function $f : \mathbb{N} \to \mathbb{R}$ whose rule is:

$$a_n = \frac{n}{(n-2)(n-5)}$$

  is not defined for $n = 2$ and for $n = 5$.
- We define the domain of a function $f : A \to B$ as the subset D of A where f is defined.
- For example, the domain of our example function is $D = \mathbb{N} - \{2, 5\}$.

Next section

Notation

For a finite set $K = \{1, 2, \cdots, m\} = [1 : m]$ (a slice of $\mathbb{N}$) and a sequence $\langle a_n \rangle$ we write:

$$\sum_{k=1}^{m} a_k = a_1 + a_2 + \cdots + a_m$$

This specific writing takes into account the order of summation.
Other writings, in which the order is less or not important, are:

$$\sum_{1 \leqslant k \leqslant m} a_k\,;\ \sum_{k \in [1:m]} a_k\,;\ \sum_{k \in K} a_k\,;\ \sum_{K} a_k$$

Warmup: What does this notation mean?

$$\sum_{k=4}^{0} q_k$$

Options:

1. $$\sum_{k=4}^{0} q_k = q_4 + q_3 + q_2 + q_1 + q_0 = \sum_{k \in \{4,3,2,1,0\}} q_k = \sum_{k=0}^{4} q_k$$

   .

   This seems the sensible thing—but it forgets the ordering, against our convention from the previous slide...

2. $$\sum_{4 \leqslant k \leqslant 0} q_k = 0$$

   This also seems sensible—but it is counterintuitive...

3. But we might want to "sum from negative infinity" instead[1] in which case:

   $$\sum_{k=m}^{n} q_k = \sum_{k \leqslant n} q_k - \sum_{k<m} q_k \,,$$

   But then, $\sum_{k=4}^{0} q_k = \sum_{k \leqslant 0} q_k - \sum_{k<4} q_k = -q_1 - q_2 - q_3 \ldots$

[1] Not right now, though; we will do so in a future lecture.

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Warmup: What does this notation mean?

$$\sum_{k=4}^{0} q_k$$

Options:

1. $$\sum_{k=4}^{0} q_k = q_4 + q_3 + q_2 + q_1 + q_0 = \sum_{k \in \{4,3,2,1,0\}} q_k = \sum_{k=0}^{4} q_k$$
   .
   This seems the sensible thing—but it forgets the ordering, against our convention from the previous slide...
2. $$\sum_{4 \leqslant k \leqslant 0} q_k = 0$$
   This also seems sensible—but it is counterintuitive...
3. But we might want to "sum from negative infinity" instead[1] in which case:
   $$\sum_{k=m}^{n} q_k = \sum_{k \leqslant n} q_k - \sum_{k<m} q_k \,,$$
   But then, $\displaystyle\sum_{k=4}^{0} q_k = \sum_{k \leqslant 0} q_k - \sum_{k<4} q_k = -q_1 - q_2 - q_3 \ldots$

[1] Not right now, though; we will do so in a future lecture.

Warmup: Interpreting the Σ-notation

Compute $\sum_{\{0\leqslant k\leqslant 5\}} a_k$ and $\sum_{\{0\leqslant k^2\leqslant 5\}} a_{k^2}$.

First sum

$$\{0\leqslant k\leqslant 5\}=\{0,1,2,3,4,5\}:$$

thus, $\sum_{\{0\leqslant k\leqslant 5\}} a_k = a_0+a_1+a_2+a_3+a_4+a_5$.

Second sum

$$\{0\leqslant k^2\leqslant 5\}=\{0,1,2,-1,-2\}:$$

thus, $\sum_{\{0\leqslant k\leqslant 5\}} a_{k^2} = a_{0^2}+a_{1^2}+a_{2^2}+a_{(-1)^2}+a_{(-2)^2} = a_0+2a_1+2a_2$.

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$$\{0 \leqslant k^2 \leqslant 5\} = \{0, 1, 2, -1, -2\} :$$

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Warmup: Interpreting the Σ-notation

Compute $\sum_{\{0 \leqslant k \leqslant 5\}} a_k$ and $\sum_{\{0 \leqslant k^2 \leqslant 5\}} a_{k^2}$.

First sum

$$\{0 \leqslant k \leqslant 5\} = \{0, 1, 2, 3, 4, 5\} :$$

thus, $\sum_{\{0 \leqslant k \leqslant 5\}} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$.

Second sum

$$\{0 \leqslant k^2 \leqslant 5\} = \{0, 1, 2, -1, -2\} :$$

thus, $\sum_{\{0 \leqslant k \leqslant 5\}} a_{k^2} = a_{0^2} + a_{1^2} + a_{2^2} + a_{(-1)^2} + a_{(-2)^2} = a_0 + 2a_1 + 2a_2$.

A universal writing

We can decrease our worries about notation by using the *Iverson brackets:*

- $[\text{True}] = 1$ and $[\text{False}] = 0$;
- if a is infinite or undefined, then $a \cdot [\text{False}] = 0$.

Then we can write:

$$\sum_{k \in K} a_k = \sum_k a_k \, [k \in K]$$

or more generally:

$$\sum_{\{k \in \mathbb{Z} \mid P(k)\}} a_k = \sum_k a_k \, [P(k)]$$

where P is a property of (some) integers. For example:

$$\sum_{\{k \in \mathbb{Z} \mid k \text{ is prime}\}} \frac{1}{k} = \sum_p \frac{1}{p} \, [p \text{ is prime}]$$

Next section

Sums and Recurrences

A sum of the form $S_n = \sum_{k=0}^{n} a_k$ can be presented in recursive form:

$$\begin{array}{rcl} S_0 & = & a_0\,; \\ S_n & = & S_{n-1} + a_n \text{ for every } n \geqslant 1 \end{array}$$

that is, as the solution of a first-order recurrence.

Next subsection

Recalling the repertoire method

- Given

$$\begin{aligned} g(0) &= \alpha_1 \\ g(n) &= \Phi(g(n-1)) + \Psi_n(\alpha_2, \ldots, \alpha_k) \text{ for every } n > 0\,. \end{aligned}$$

  where Φ and Ψ_n are linear.
- Suppose we have k $(k+1)$-tuples $\left(g_i; \alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,k}\right)$ such that:
  1. $g_i(0) = \alpha_{i,1}$ and $g_i(n) = \Phi(g_i(n-1)) + \Psi_n(\alpha_{i,2}, \ldots, \alpha_{i,k})$ for every $i \in [1:k]$;
  2. the k k-tuples $\left(\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,k}\right)$ are linearly independent.
- Then the recurrence has a solution in closed form:

$$g(n) = \alpha_1 A_1(n) + \alpha_2 A_2(n) + \ldots + \alpha_k A_k(n)$$

  where the functions $A_1(n), A_2(n), \ldots, A_k(n)$ can be determined from the system of equations:

$$\begin{aligned} \alpha_{1,1} A_1(n) + \alpha_{1,2} A_2(n) + \ldots + \alpha_{1,k} A_k(n) &= g_1(n) \\ &\vdots \\ \alpha_{k,1} A_1(n) + \alpha_{k,2} A_2(n) + \ldots + \alpha_{k,k} A_k(n) &= g_k(n) \end{aligned}$$

Example 1: arithmetic sequence

The arithmetic sequence of initial term a and common difference b is the sequence $\langle a_n \rangle$ defined by:

$$a_n = a + b \cdot n \text{ for every } n \geqslant 0$$

Then the sum $S_n = \sum_{k=0}^{n} a_n$ is the solution of the recurrence:

$$\begin{aligned} S_0 &= a \\ S_n &= S_{n-1} + a + bn \text{ for every } n \geqslant 1 \end{aligned}$$

Everything is linear here, so we can safely apply the repertoire method to the family of recurrences:

$$\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + \beta + \gamma n \text{ for every } n \geqslant 1 \end{aligned}$$

Then S_n is the solution corresponding to $\alpha = a$, $\beta = a$, $\gamma = b$.

Evaluation of terms $R_n = R_{n-1} + (\beta + \gamma n)$

Evaluating the first terms gives:

$$\begin{aligned}
R_0 &= \alpha \\
R_1 &= \alpha + \beta + \gamma \\
R_2 &= \alpha + \beta + \gamma + (\beta + 2\gamma) = \alpha + 2\beta + 3\gamma \\
R_3 &= \alpha + 2\beta + 3\gamma + (\beta + 3\gamma) = \alpha + 3\beta + 6\gamma
\end{aligned}$$

There seem to be a constant term, a linear term, and something which resemble the triangular numbers.
So we apply the repertoire method with the following test functions:

1. $R_n = 1$ for all n
2. $R_n = n$ for all n
3. $R_n = n^2$ for all n

Repertoire method: case 1

Lemma 1

$A(n) = 1$ for every $n \in \mathbb{N}$.

Proof: With the choice $R_n = 1$ for every $n \geqslant 0$, the recurrence becomes:

$$\begin{aligned} 1 &= \alpha \\ 1 &= 1 + \beta + \gamma n \text{ for every } n \geqslant 1 \end{aligned}$$

This is only possible if $\alpha = 1$, $\beta = \gamma = 0$. We have proved that:

The particular solution $R_n = 1$ corresponds to the triple $(\alpha, \beta, \gamma) = (1, 0, 0)$

which is equivalent to Lemma 1.

Repertoire method: case 2

Lemma 1

$B(n) = n$ for every $n \in \mathbb{N}$.

Proof: With the choice $R_n = n$ for every $n \geqslant 0$, the recurrence becomes:

$$\begin{array}{rcl} 0 & = & \alpha \\ n & = & n - 1 + \beta + \gamma n \text{ for every } n \geqslant 1 \end{array}$$

This is only possible if $\alpha = 0$, $\beta = 1$, $\gamma = 0$. We have proved that:

The particular solution $R_n = n$ corresponds to the triple $(\alpha, \beta, \gamma) = (0, 1, 0)$

which is equivalent to Lemma 2.

Repertoire method: case 3

Lemma 3

$C(n) = \frac{n(n+1)}{2}$ for every $n \in \mathbb{N}$.

Proof: With the choice $R_n = n^2$ for every $n \geqslant 0$, the recurrence becomes:

$$\begin{array}{rcl} 0 & = & \alpha \\ n^2 & = & (n-1)^2 + \beta + \gamma n \text{ for every } n \geqslant 1 \end{array}$$

As $(n-1)^2 = n^2 - 2n + 1$, this is only possible if $\alpha = 0$, $\beta = -1$, $\gamma = 2$. We have proved that:

The particular solution $R_n = 1$ corresponds to the triple $(\alpha, \beta, \gamma) = (0, -1, 2)$

that is,

$$-B(n) + 2C(n) = n^2$$

As we know that $B(n) = n$, we can solve for $C(n)$ and obtain the thesis of Lemma 3.

Repertoire method: summing up

According to Lemma 1, 2, 3, we get:

1. $R_n = 1$ for all n $\implies$ $A(n) = 1$
2. $R_n = n$ for all n $\implies$ $B(n) = n$
3. $R_n = n^2$ for all n $\implies$ $C(n) = \frac{n^2+n}{2}$

Hence,

$$R_n = \alpha + n\beta + \left(\frac{n^2+n}{2}\right)\gamma$$

For $\alpha = \beta = a$ and $\gamma = b$ we get:

$$S_n = \sum_{k=0}^{n}(a+bk) = (n+1)a + \frac{n(n+1)}{2}b$$

Next subsection

Perturbation method

To find a closed form for $S_n = \sum_{0 \leqslant k \leqslant n} a_k$:

1. Rewrite S_{n+1} by isolating the first and the last term:

$$\begin{aligned} S_n + a_{n+1} &= a_0 + \sum_{1 \leqslant k \leqslant n+1} a_k \\ &= a_0 + \sum_{1 \leqslant k+1 \leqslant n+1} a_{k+1} \\ &= a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1} \end{aligned}$$

2. Work on the sum on the right and express it as a function of S_n.
3. Solve with respect to S_n.

Example 2: geometric sequence

Geometric sequence: $a_n = ax^n, x \neq 1$

Recurrence equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$
\begin{aligned}
S_0 &= a \\
S_n &= S_{n-1} + ax^n \,, \text{ for } n > 0 \,.
\end{aligned}
$$

Example 2: geometric sequence

Geometric sequence: $a_n = ax^n, x \neq 1$

Recurrence equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$\begin{aligned} S_0 &= a \\ S_n &= S_{n-1} + ax^n \text{, for } n > 0 \,. \end{aligned}$$

- Splitting off the first term gives

$$\begin{aligned} S_n + a_{n+1} &= a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1} \\ &= a + \sum_{0 \leqslant k \leqslant n} ax^{k+1} \\ &= a + x \sum_{0 \leqslant k \leqslant n} ax^k \\ &= a + xS_n \end{aligned}$$

Example 2: geometric sequence

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Recurrence equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$\begin{aligned} S_0 &= a \\ S_n &= S_{n-1} + ax^n \,, \text{ for } n > 0 \,. \end{aligned}$$

- From this we obtain the equality:

  $$S_n + ax^{n+1} = a + xS_n \,,$$

  that is: $(1-x)S_n = a - ax^{n+1}$.
- As $x \neq 1$ we can divide and obtain:

  $$S_n = a \cdot \frac{1 - x^{n+1}}{1 - x}$$

TAL TECH

Example 3: When perturbation doesn't work ...

Compute: $S_n = \sum_{k=0}^{n} k^2$.

1. Perturb the sum:

$$S_n + (n+1)^2 = 0 + \sum_{k=1}^{n+1} k^2$$

   Um ... that shifted k^2 sounds bad ...

Example 3: When perturbation doesn't work . . .

Compute: $S_n = \sum_{k=0}^{n} k^2$.

1. Perturb the sum:

$$S_n + (n+1)^2 = 0 + \sum_{k=1}^{n+1} k^2$$

   Um . . . that shifted k^2 sounds bad . . .

2. Rewrite the right-hand side so that it depends on S_n:

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=0}^{n} (k+1)^2 \\
&= \sum_{k=0}^{n} (k^2+2k+1) \\
&= S_n + \sum_{k=0}^{n} (2k+1) \\
&= S_n + 2\frac{n(n+1)}{2} + n + 1
\end{aligned}$$

TAL TECH

Example 3: When perturbation doesn't work ...

Compute: $S_n = \sum_{k=0}^{n} k^2$.

1. Perturb the sum:

$$S_n + (n+1)^2 = 0 + \sum_{k=1}^{n+1} k^2$$

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2. Rewrite the right-hand side so that it depends on S_n:

$$\sum_{k=1}^{n+1} k^2 = S_n + 2\frac{n(n+1)}{2} + n + 1$$

3. Solve with respect to S_n:

$$\begin{aligned} S_n + (n+1)^2 &= S_n + (n+1) + 2\frac{n(n+1)}{2} \\ (n+1)^2 &= (n+1) + n(n+1) \end{aligned}$$

... which is true, but where is S_n?

. . . try perturbing *another* sum!

In addition to S_n, consider the sum: $T_n = \sum_{k=0}^{n} k^3$.

1. Perturb T_n:

$$T_n + (n+1)^3 = 0 + \sum_{k=1}^{n+1} k^3$$

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1. Perturb T_n:

$$T_n + (n+1)^3 = 0 + \sum_{k=1}^{n+1} k^3$$

2. Rewrite the right-hand side so that it depends on T_n and on S_n:

$$\begin{aligned} \sum_{k=1}^{n+1} k^3 &= \sum_{k=0}^{n} (k+1)^3 \\ &= \sum_{k=0}^{n} (k^3 + 3k^2 + 3k + 1) \\ &= T_n + 3S_n + \sum_{k=0}^{n} (3k+1) \end{aligned}$$

. . . try perturbing *another* sum!

In addition to S_n, consider the sum: $T_n = \sum_{k=0}^{n} k^3$.

1. Perturb T_n:

$$T_n + (n+1)^3 = 0 + \sum_{k=1}^{n+1} k^3$$

2. Rewrite the right-hand side so that it depends on T_n and on S_n:

$$\sum_{k=1}^{n+1} k^3 = T_n + 3S_n + \sum_{k=0}^{n} (3k+1)$$

3. Solve with respect to S_n:

$$\begin{aligned}
(n+1)^3 &= 3S_n + 3\frac{n(n+1)}{2} + n + 1 \\
&= 3S_n + (n+1)\left(\frac{3}{2}n + 1\right) \\
3S_n &= (n+1)\left(n^2 + 2n + 1 - \frac{3}{2}n - 1\right) \\
S_n &= \frac{1}{3}(n+1)\left(n^2 + \frac{n}{2}\right) = \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Next subsection

Example 3: Hanoi sequence

Consider again the Tower of Hanoi recurrence:

$$
\begin{aligned}
T_0 &= 0 \\
T_n &= 2T_{n-1} + 1
\end{aligned}
$$

Example 3: Hanoi sequence

Consider again the Tower of Hanoi recurrence:

$$T_0 = 0$$
$$T_n = 2T_{n-1} + 1$$

This sequence can be transformed into a geometric sum using the following manipulations:

- Divide both equalities by 2^n:

$$T_0/2^0 = 0$$
$$T_n/2^n = T_{n-1}/2^{n-1} + 1/2^n$$

- Set $S_n = T_n/2^n$ to have:

$$S_0 = 0$$
$$S_n = S_{n-1} + 2^{-n}$$

This is almost the geometric sum with the parameters $a = 1$ and $x = 1/2$: Only the initial summand 1 is missing.

Example 3: Hanoi sequence

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\end{aligned}
$$

This is almost the geometric sum with the parameters $a = 1$ and $x = 1/2$: Only the initial summand 1 is missing.

Example 3: Hanoi sequence

Consider again the Tower of Hanoi recurrence:

$$\begin{aligned} T_0 &= 0 \\ T_n &= 2T_{n-1} + 1 \end{aligned}$$

Then $S_n = T_n / 2^n$ satisfies:

$$\begin{aligned} S_n &= \left(\sum_{k=0}^{n} \left(\frac{1}{2} \right)^n \right) - 1 \\ &= \frac{1 - (1/2)^{n+1}}{1 - 1/2} - 1 \\ &= 2 - 2^{-n} - 1 = 1 - 2^{-n} \end{aligned}$$

We conclude:

$$T_n = 2^n S_n = 2^n - 1$$

Just the same result we have proven by means of induction!

Next subsection

Summation factor: Idea

We want to solve a linear recurrence of the form:

$$a_n T_n = b_n T_{n-1} + c_n \quad \text{for every } n > 0$$

where:

1. $\langle a_n \rangle$, $\langle b_n \rangle$ and $\langle c_n \rangle$ are arbitrary sequences; and
2. for every $n > 0$, $a_n \neq 0$ and $b_n \neq 0$.

We also assume that the *initial value* T_0 is given.

The idea

Find a summation factor s_n satisfying the following property:

$$s_n b_n = s_{n-1} a_{n-1} \quad \text{for every } n \geqslant 1$$

Summation factor: Realization

If a sequence $\langle s_n \rangle$ as in the previous slide exists, then:

1. $s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n = s_{n-1} a_{n-1} T_{n-1} + s_n c_n$.
2. Set $S_n = s_n a_n T_n$ and rewrite the equation as:

$$\begin{aligned} S_0 &= s_0 a_0 T_0 \\ S_n &= S_{n-1} + s_n c_n \end{aligned}$$

3. This yields a *closed formula* for the solution:

$$T_n = \frac{1}{s_n a_n} \left(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k \right) = \frac{1}{s_n a_n} \left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k \right) \text{ for every } n > 0$$

Finding a summation factor

Assuming that $b_n \neq 0$ for every n:

1. Set $s_0 = 1$ and also $a_0 = 1$.
2. Compute the next elements using the property $s_n b_n = s_{n-1} a_{n-1}$:

$$\begin{aligned} s_1 &= \frac{1}{b_1} = \frac{a_0}{b_1} \\ s_2 &= \frac{s_1 a_1}{b_2} = \frac{a_0 a_1}{b_1 b_2} \\ s_3 &= \frac{s_2 a_2}{b_3} = \frac{a_0 a_1 a_2}{b_1 b_2 b_3} \\ &= \dots \\ s_n &= \frac{s_{n-1} a_{n-1}}{b_n} = \frac{a_0 a_1 \cdots a_{n-1}}{b_1 b_2 \cdots b_n} \end{aligned}$$

(To be proved by induction!)

Example: Hanoi tower again

The choice $a_n = c_n = 1$ and $b_n = 2$ gives the Hanoi Tower sequence.

- Evaluate the summation factor:

$$s_n = \frac{s_{n-1}\, a_{n-1}}{b_n} = \frac{a_0\, a_1 \cdots a_{n-1}}{b_1\, b_2 \cdots b_n} = \frac{1}{2^n}$$

- The solution is:

$$T_n = \frac{1}{s_n a_n}\left(s_1\, b_1\, T_0 + \sum_{k=1}^{n} s_k\, c_k\right) = 2^n \sum_{k=1}^{n} \frac{1}{2^k} = 2^n\left(1 - \frac{1}{2^n}\right) = 2^n - 1$$

Yet Another Example: Constant coefficients

Consider now the recurrence:

$$Z_n = aZ_{n-1} + b \text{ for every } n \geqslant 1\,, \quad a \neq 1$$

Taking $a_n = 1$, $b_n = a$ and $c_n = b$:

- Evaluate summation factor:

$$s_n = \frac{s_{n-1}\,a_{n-1}}{b_n} = \frac{a_0\,a_1 \ldots a_{n-1}}{b_1\,b_2 \ldots b_n} = \frac{1}{a^n}$$

- Apply the resolutive formula:

$$\begin{aligned} Z_n &= \frac{1}{s_n a_n}\left(s_1\,b_1\,Z_0 + \sum_{k=1}^{n} s_k\,c_k\right) = a^n\left(Z_0 + b\sum_{k=1}^{n}\frac{1}{a^k}\right) \\ &= a^n Z_0 + b\sum_{k=1}^{n} a^{n-k} \\ &= a^n Z_0 + b\sum_{k=0}^{n-1} a^k \text{ where the new } k \text{ is the old } n-k \\ &= a^n Z_0 + \frac{a^n - 1}{a-1}\,b \end{aligned}$$

Yet Another Example: Check up on results

We could also have solved the recurrence by iteration:

$$
\begin{aligned}
Z_n &= aZ_{n-1} + b \\
&= a^2 Z_{n-2} + ab + b \\
&= a^3 Z_{n-3} + a^2 b + ab + b \\
&= \cdots \\
&= a^k Z_{n-k} + (a^{k-1} + a^{k-2} + \ldots + 1)b \\
&= a^k Z_{n-k} + \frac{a^k - 1}{a - 1} b \qquad \text{(assuming } a \neq 1\text{)}
\end{aligned}
$$

We can do at most n iterations, so for $k = n$ we get:

$$Z_n = a^n Z_{n-n} + \frac{a^n - 1}{a - 1} b = a^n Z_0 + \frac{a^n - 1}{a - 1} b$$

TAL TECH

Next subsection

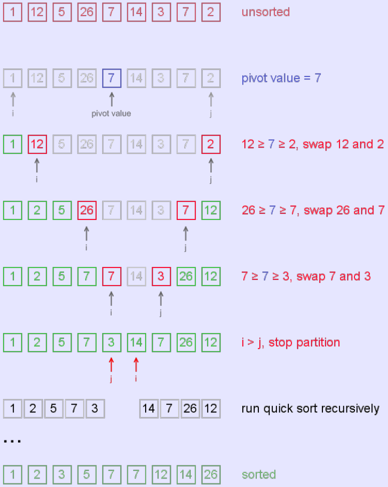
The Quicksort algorithm (C.A.R. Hoare, 1959-1961)

Input: An array A with n elements, indexed from 1 to n.

1. If $n = 0$ then return.
2. Choose a pivot $p = A[k]$ for a suitable $k \in [1 : n]$.
3. Initialize $i \leftarrow 0$, $j \leftarrow n + 1$.
4. Do forever:
    1. Do $i \leftarrow i + 1$ while $i \leqslant n$ and $A[i] \leqslant p$.
    2. Do $j \leftarrow j - 1$ while $j \geqslant 1$ and $A[j] > p$.
    3. If $i \geqslant j$: break.
    4. Swap $A[i]$ with $A[j]$.
5. Call Quicksort recursively on the subarrays $A[1 : k - 1]$ and $A[k + 1 : n]$.

Output: the array A with elements sorted.

Example: A run of Quicksort



How Quicksort earned its name

Quicksort uses the pivot to subdivide the array into "small" and "large" elements.

- This subdivision may be rough, but after it has been done, no "small" object will be compared with any "large" object ever again.
- This suggests very good performance in the average case.

Let C_n be the average number of comparisons made by Quicksort to sort an array of $n \geqslant 1$ elements.

- Each element is compared with the pivot except $A[k]$, which is the pivot.
- Each one of the n elements could be the pivor.
- The recursive call will work on an array of size $k-1$ and one of size $n-k$, for a total of $n-1$ objects.

We conclude:

$$\begin{aligned} C_0 &= 0 \\ C_n &= n+1+\frac{2}{n}\sum_{k=0}^{n-1} C_k \text{ for every } n \geqslant 1 \end{aligned}$$

Efficiency of Quicksort: Rewriting the recurrence

- Multiplying by n gives:

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-1} C_k$$

  We still cannot apply the summation factor method.

- However, if we write the recurrence for $n-1$:

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

  and subtract from the original, we obtain:

$$\begin{aligned}
nC_n - (n-1)C_{n-1} &= n^2 + n + 2C_{n-1} - (n-1)^2 - (n-1) \\
nC_n - nC_{n-1} + C_{n-1} &= n^2 + n + 2C_{n-1} - n^2 + 2n - 1 - n + 1 \\
nC_n - nC_{n-1} &= C_{n-1} + 2n \\
nC_n &= (n+1)C_{n-1} + 2n
\end{aligned}$$

- The last recurrence can be solved with a summation factor.

Efficiency of Quicksort: Summation factor in action

Let's solve the recurrence $nC_n = (n+1)C_{n-1} + 2n$ with a summation factor:

- We have $a_n = n$, $b_n = n+1$, and $c_n = 2n$, so:

$$s_n = \frac{a_1 a_2 \cdots a_{n-1}}{b_2 b_3 \cdots b_n} = \frac{1 \cdot 2 \cdots (n-1)}{3 \cdot 4 \cdots (n+1)} = \frac{2}{n(n+1)}$$

- We plug into the formula and obtain:

$$\begin{aligned} C_n &= \frac{1}{s_n a_n}\left(s_1 b_1 C_0 + \sum_{k=1}^{n} s_k c_k\right) \\ &= \frac{n+1}{2}\sum_{k=1}^{n}\frac{4k}{k(k+1)} \\ &= 2(n+1)\sum_{k=1}^{n}\frac{1}{k+1} = 2(n+1)\left(\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{n+1} - 1\right) \\ &= 2(n+1)H_n - 2n \end{aligned}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \ln n$ is the nth harmonic number.

Next subsection

A basic **con**tinuous method for dis**crete** mathematics

To compute a sum of the form $S_n = \sum_{k=1}^{n} a_k$:

1. Choose a continuous function $f(x)$ such that $f(k) = a_k$ for every $k > 0$ integer.
2. Identify the sequence $\langle a_k \rangle$ with the staircase function

$$a(x) = \sum_{k \geqslant 1} a_k \, [k-1 < x \leqslant k]$$

3. Determine an error term E_n such that:

$$S_n = \int_0^n f(x)\,dx + E_n \ \text{ for every } n \geqslant 1$$

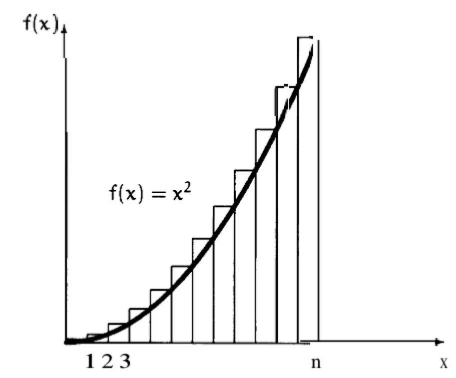
4. Express E_n itself as a sum:

$$E_n = \sum_{k=1}^{n} \left(a_k - \int_{k-1}^{k} f(x)dx \right)$$

5. Use a closed form for E_n to determine a closed form for S_n.

Example: Sum of perfect squares

Example: $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$



$$\int_0^n x^2\,dx = \frac{n^3}{3} \tag{1}$$

$$\Box_n = \int_0^n x^2\,dx + E_n \tag{2}$$

$$E_n = \sum_{k=1}^{n} \left(k^2 - \int_{k-1}^{k} x^2\,dx \right) \tag{3}$$

Example: Sum of perfect squares

Example: $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Evaluate (3):

$$\begin{aligned}
E_n &= \sum_{k=1}^{n} \left(k^2 - \int_{k-1}^{k} x^2 \, dx \right) \\
&= \sum_{k=1}^{n} \left(k^2 - \frac{k^3 - (k-1)^3}{3} \right) \\
&= \sum_{k=1}^{n} \left(k - \frac{1}{3} \right) \\
&= \frac{(n+1)n}{2} - \frac{n}{3} = \frac{3n^2 + n}{6} .
\end{aligned}$$

Finally, from (2) and (1) we get :

$$\Box_n = \frac{n^3}{3} + \frac{3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

Next section

Manipulation of Sums

For every finite set K and permutation $p(k)$ of K:

- Distributive law:
$$\sum_{k \in K} ca_k = c \sum_{k \in K} a_k$$
- Associative law:
$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k$$
- Commutative law:
$$\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}$$

All of the above work because the summands are nonzero at most finitely many times. (More on this later.)

Example: Arithmetic progressions

Let's compute again:

$$S = \sum_{0 \leqslant k \leqslant n} (a + bk)$$

$$\begin{aligned}
S &= \sum_{0 \leqslant n-k \leqslant n} (a + b(n-k)) \text{ by commutativity} \\
&= \sum_{0 \leqslant k \leqslant n} (a + bn - bk) \text{ because } [0 \leqslant k \leqslant n] = [0 \leqslant n-k \leqslant n] \\
2S &= \sum_{0 \leqslant k \leqslant n} ((a + bk) + (a + bn - bk)) \text{ by associativity} \\
&= \sum_{0 \leqslant k \leqslant n} (2a + bn) \\
2S &= (2a + bn) \sum_{0 \leqslant k \leqslant n} 1 \text{ by distributivity} \\
&= (2a + bn)(n+1)
\end{aligned}$$

Again, but only using basic properties:

$$S = (n+1)a + \frac{n(n+1)}{2} b$$

Yet Another Useful Equality

The Inclusion-Exclusion Principle

For any two finite sets K and K':

$$\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \sum_{k \in K \cup K'} a_k + \sum_{k \in K \cap K'} a_k$$

Examples:

1. For $1 \leqslant m \leqslant n$:

$$\sum_{k=1}^{m} a_k + \sum_{k=m}^{n} a_k = a_m + \sum_{k=1}^{n} a_k$$

2. For $n \geqslant 0$:

$$\sum_{0 \leqslant k \leqslant n} a_k = a_0 + \sum_{1 \leqslant k \leqslant n} a_k$$

3. For $n \geqslant 0$:

$$S_n + a_{n+1} = a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1}$$

that is, we recover the perturbation method!