ITT9132 Concrete Mathematics

Lecture 3: 9 February 2021

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Contents

Next section

The repertoire method: Basic ideas

Let the recursion scheme

$$\begin{aligned} g(0) &= \alpha\,, \\ g(n+1) &= \Phi(g(n)) + \Psi(n;\beta,\gamma,\dots) \quad \text{for } n \geq 0\,. \end{aligned}$$

have the following properties:

1. Φ is linear in g:
   If $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$, then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.
   No hypotheses are made on the dependence of g on n.
2. Ψ is linear in each of the $m-1$ parameters $\beta,\gamma,\dots$
   No hypotheses are made on the dependence of Ψ on n.

Then the whole system is linear in the parameters $\alpha,\beta,\gamma,\dots$
We can then look for a general solution of the form

$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \dots$$

The repertoire method: Description

Suppose we have a *repertoire* of m pairs of the form $((\alpha_i, \beta_i, \gamma_i, \ldots), g_i(n))$ satisfying the following conditions:

1. For every $i = 1, 2, \ldots, m$, $g_i(n)$ is the solution of the system corresponding to the values $\alpha = \alpha_i, \beta = \beta_i, \gamma = \gamma_i, \ldots$
2. The m m-tuples $(\alpha_i, \beta_i, \gamma_i, \ldots)$ are linearly independent.

Then the functions $A(n), B(n), C(n), \ldots$ are uniquely determined.
The reason is that, for every fixed n,

$$\begin{array}{lllllll} \alpha_1 A(n) & +\beta_1 B(n) & +\gamma_1 C(n) & +\ldots & = & g_1(n) \\ \vdots & & & & = & \vdots \\ \alpha_m A(n) & +\beta_m B(n) & +\gamma_m C(n) & +\ldots & = & g_m(n) \end{array}$$

is a system of m linear equations in the m unknowns $A(n), B(n), C(n), \ldots$ whose coefficients matrix is invertible.

Next section

Binary representation of generalized Josephus function

Definition

The generalized Josephus function (GJ-function) is defined for α, β_0, β_1 as follows:

$$
\begin{aligned}
f(1) &= \alpha \\
f(2n+j) &= 2f(n) + \beta_j \text{ for } j = 0, 1 \text{ and } n > 0\,.
\end{aligned}
$$

We obtain the definition used before if to select $\beta_0 = \beta$ and $\beta_1 = \gamma$.

Binary representation of generalized Josephus function (2)

Case A: Argument is even

If $2n = 2^m + \ell$, then the binary notation is

$$2n = (b_m b_{m-1} \dots b_1 b_0)_2$$

or

$$2n = b_m 2^m + b_{m-1} 2^{m-1} + \dots + b_1 2 + b_0$$

where $b_i \in \{0,1\}$, $b_0 = 0$ and $b_m = 1$.

Hence

$$n = b_m 2^{m-1} + b_{m-1} 2^{m-2} + \dots + b_2 2 + b_1$$

or

$$n = (b_m b_{m-1} \dots b_1)_2$$

Binary representation of generalized Josephus function (3)

Case B: Argument is odd

If $2n+1 = 2^m + \ell$, then the binary notation is

$$2n+1 = (b_m b_{m-1} \ldots b_1 b_0)_2$$

or

$$2n+1 = b_m 2^m + b_{m-1} 2^{m-1} + \ldots + b_1 2 + b_0$$

where $b_i \in \{0,1\}$, $b_0 = 1$ and $b_m = 1$.

We get

$$\begin{aligned} 2n+1 &= b_m 2^m + b_{m-1} 2^{m-1} + \ldots + b_1 2 + 1 \\ 2n &= b_m 2^m + b_{m-1} 2^{m-1} + \ldots + b_1 2 \\ n &= b_m 2^{m-1} + b_{m-1} 2^{m-2} + \ldots + b_2 2 + b_1 \end{aligned}$$

or

$$n = (b_m b_{m-1} \ldots b_1)_2$$

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We get

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Same results for cases A and B indicates that we don't need to consider even and odd cases separately.

Binary representation of generalized Josephus function (4)

Let's evaluate:

$$\begin{aligned}
f((b_m, b_{m-1}, \ldots, b_1, b_0)_2) &= 2f((b_m, b_{m-1}, \ldots, b_1)_2) + \beta_{b_0} \\
&= 2 \cdot (2f((b_m, b_{m-1}, \ldots, b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= 4f((b_m, b_{m-1}, \ldots, b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\
&= \vdots \\
&= f((b_m)_2)2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \\
&= f(1)2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \\
&= \alpha 2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} ,
\end{aligned}$$

where

$$\beta_{b_j} = \begin{cases} \beta_1, & \text{if } b_j = 1 \\ \beta_0 & \text{if } b_j = 0 \end{cases}$$

$$f((b_m b_{m-1} \ldots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \cdots \beta_{b_1} \beta_{b_0})_2$$

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&= \alpha 2^m + \beta_{b_{m-1}} 2^{m-1} + \ldots + \beta_{b_1} 2 + \beta_{b_0} \,,
\end{aligned}
$$

where

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$$

$$
f((b_m b_{m-1} \ldots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_2
$$

Example

Original Josephus function: $\alpha = 1$, $\beta_0 = -1$, $\beta_1 = 1$ i.e.

$$\begin{aligned} f(1) &= 1 \\ f(2n) &= 2f(n) - 1 \\ f(2n+1) &= 2f(n) + 1 \end{aligned}$$

Compute

$$f((b_m b_{m-1} \dots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \dots \beta_{b_1} \beta_{b_0})_2$$

$$\begin{aligned} f(100) = f((1100100)_2) &= (1, 1, -1, -1, 1, -1, -1)_2 \\ &= 64 + 32 - 16 - 8 + 4 - 2 - 1 = 73 \end{aligned}$$

TAL TECH

Generalized Josephus function: Multiple bases

Let $c, d \geqslant 2$ be integers.
Consider the following recurrent problem:

$$\begin{array}{rcll} f(j) & = & \alpha_j & \text{for } 1 \leqslant j < d\,; \\ f(dn+j) & = & cf(n) + \beta_j & \text{for } 0 \leqslant j < d \text{ and } n \geqslant 1\,. \end{array}$$

How can we compute $f(n)$ for an arbitrary positive integer n, without having to go through the entire iterative process?

Multiple bases representation

We can actually use the same technique!

Let $(b_m b_{m-1} \dots b_1 b_0)_d$ be the base-d writing of n. Then $b_m \neq 0$ and:

$$\begin{aligned}
f((b_m, b_{m-1}, \dots, b_1, b_0)_d) &= cf((b_m, b_{m-1}, \dots, b_1)_d) + \beta_{b_0} \\
&= c \cdot (cf((b_m, b_{m-1}, \dots, b_2)_d) + \beta_{b_1}) + \beta_{b_0} \\
&= c^2 f((b_m, b_{m-1}, \dots, b_2)_d) + c\beta_{b_1} + \beta_{b_0} \\
&= \vdots \\
&= c^m \cdot f(b_m) + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0} \\
&= c^m \alpha_{b_m} + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0}
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&= c^m \alpha_{b_m} + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0}
\end{aligned}$$

With a slight abuse of notation: (the β_i's need not be base-c digits)

$$f((b_m b_{m-1} \dots b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} \beta_{b_{m-2}} \dots \beta_{b_1} \beta_{b_0})_c$$

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&= c^m \alpha_{b_m} + c^{m-1}\beta_{b_{m-1}} + \dots + c\beta_{b_1} + \beta_{b_0}
\end{aligned}$$

Or, more precisely:

$$f((b_m b_{m-1} \dots b_1 b_0)_d) = p(c) \text{ where } p(x) = \alpha_{b_m} x^m + \beta_{b_{m-1}} x^{m-1} + \dots + \beta_{b_1} x + \beta_{b_0}$$

Next section

Sequences

Definition

A sequence of elements of a set A is a function $f : \mathbb{N} \to A$, where $\mathbb{N}$ is the set of natural numbers.

Notations used:

- $f = \langle a_n \rangle$, where we denote $a_n = f(n)$;
- $\{a_n\}_{n \in \mathbb{N}}$;
- $\langle a_0, a_1, a_2, a_3, \ldots \rangle$.

a_n is called the nth term of the sequence f

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Example

$$a_0 = 0,\ a_1 = \frac{1}{2 \cdot 3},\ a_2 = \frac{2}{3 \cdot 4},\ a_3 = \frac{3}{4 \cdot 5}, \cdots$$

or

$$\left\langle 0, \frac{1}{6}, \frac{1}{6}, \frac{3}{20}, \frac{2}{15}, \cdots, \frac{n}{(n+1)(n+2)}, \cdots \right\rangle$$

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Notation

$$f(n) = \frac{n}{(n+1)(n+2)}$$

or

$$a_n = \frac{n}{(n+1)(n+2)}$$

Sets of indices

- Default assumption: $\mathbb{N}$.
- Actually, any countably infinite set can be used as an index set. Examples:
  - $\mathbb{Z}^+ = \mathbb{N} - \{0\} \sim \mathbb{N}$.
  - $\mathbb{N} \setminus K$, where $K \subseteq \mathbb{N}$ is finite.
  - The set $\mathbb{Z}$ of relative integers.
  - $\{1, 3, 5, 7, \ldots\} = \mathrm{Odd}$.
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The writing $A \sim B$ denotes that sets A and B are of the same cardinality.

- For finite sets, $|A|$ is the number of elements of A.
- In general, A and B are said to have the same cardinality if there exists a *bijection* between the two.
  We then write $A \sim B$, or $|A| = |B|$

(See http://www.mathsisfun.com/sets/injective-surjective- bijective.html for detailed explanation)

Finite sequences

- A finite sequence of elements of a set A is a function
  $f : K \to A$,
  where K is set a finite subset of natural numbers

  For example: $f : \{1, 2, 3, 4, \cdots, n\} \to A,\ n \in \mathbb{N}$

  Special case: $n = 0$, i.e. empty sequence: $f(\emptyset) = e$

Domain of the sequence

$$f : T \to A$$

$$a_n = \frac{n}{(n-2)(n-5)}$$

The domain of f is $T = \mathbb{N} - \{2, 5\}$.

Next section

Notation

For a finite set $K = \{1, 2, \cdots, m\} = [1 : m]$ (a slice of $\mathbb{N}$) and a sequence $\langle a_n \rangle$ we write:

$$\sum_{k=1}^{m} a_k = a_1 + a_2 + \cdots + a_m$$

This specific writing takes into account the order of summation. Other writings, in which the order is less or not important, are:

$$\sum_{1 \leqslant k \leqslant m} a_k\,; \quad \sum_{k \in [1:m]} a_k\,; \quad \sum_{k \in K} a_k\,; \quad \sum_{K} a_k$$

Warmup: What does this notation mean?

$$\sum_{k=4}^{0} q_k$$

Options:

1. $\sum_{k=4}^{0} q_k = q_4 + q_3 + q_2 + q_1 + q_0 = \sum_{k \in \{4,3,2,1,0\}} q_k = \sum_{k=0}^{4} q_k$.
   This seems the sensible thing—**but:**
2. $\sum_{4 \leqslant k \leqslant 0} q_k = 0$ also looks like a feasible interpretation—**but:**
3. If

   $$\sum_{k=m}^{n} q_k = \sum_{k \leqslant n} q_k - \sum_{k<m} q_k \,,$$

   (provided the two sums on the right-hand side exist finite)
   then $\sum_{k=4}^{0} q_k = \sum_{k \leqslant 0} q_k - \sum_{k<4} q_k = -q_1 - q_2 - q_3$.

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Warmup: Interpreting the Σ-notation

Compute $\sum_{\{0\leqslant k\leqslant 5\}} a_k$ and $\sum_{\{0\leqslant k^2\leqslant 5\}} a_{k^2}$.

First sum

$$\{0\leqslant k\leqslant 5\} = \{0,1,2,3,4,5\}:$$

thus, $\sum_{\{0\leqslant k\leqslant 5\}} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$.

Second sum

$$\{0\leqslant k^2\leqslant 5\} = \{0,1,2,-1,-2\}:$$

thus,
$\sum_{\{0\leqslant k\leqslant 5\}} a_{k^2} = a_{0^2} + a_{1^2} + a_{2^2} + a_{(-1)^2} + a_{(-2)^2} = a_0 + 2a_1 + 2a_2$.

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A universal writing

We can decrease our worries about notation by using the *Iverson brackets:*

- $[\text{True}] = 1$ and $[\text{False}] = 0$;
- if a is infinite or undefined, then $a \cdot [\text{False}] = 0$.

Then we can write:

$$\sum_{k \in K} a_k = \sum_k a_k \, [k \in K]$$

or more generally:

$$\sum_{k \in \mathbb{Z} | P(k)} a_k = \sum_k a_k \, [P(k)]$$

where P is a property of (some) integers. For example:

$$\sum_{k \in \mathbb{Z} | k \text{ is prime}} \frac{1}{k} = \sum_p \frac{1}{p} \, [p \text{ is prime}]$$

TAL TECH

Next section

Sums and Recurrences

A sum of the form $S_n = \sum_{k=0}^{n} a_k$ can be presented in recursive form:

$$\begin{aligned} S_0 &= a_0\,; \\ S_n &= S_{n-1} + a_n \text{ for every } n \geq 1 \end{aligned}$$

that is, as the solution of a first-order recurrence.

Next subsection

Recalling the repertoire method

- Given

$$\begin{aligned} g(0) &= \alpha_1 \\ g(n) &= \Phi(g(n-1)) + \Psi_n(\alpha_2, \ldots, \alpha_k) \text{ for every } n > 0\,. \end{aligned}$$

  where Φ and Ψ_n are linear.
- Suppose we have k $(k+1)$-tuples $(g_i; \alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,k})$ such that:
  1. $g_i(0) = \alpha_{i,1}$ and $g_i(n) = \Phi(g_i(n-1)) + \Psi_n(\alpha_{i,2}, \ldots, \alpha_{i,k})$ for every $i \in [1:k]$;
  2. the k k-tuples $(\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,k})$ are linearly independent.
- Then the recurrence has a solution in closed form:

$$g(n) = \alpha_1 A_1(n) + \alpha_2 A_2(n) + \ldots + \alpha_k A_k(n)$$

  where the functions $A_1(n), A_2(n), \ldots, A_k(n)$ can be determined from the system of equations:

$$\begin{aligned} \alpha_{1,1} A_1(n) + \alpha_{1,2} A_2(n) + \ldots + \alpha_{1,k} A_k(n) &= g_1(n) \\ &\vdots \\ \alpha_{k,1} A_1(n) + \alpha_{k,2} A_2(n) + \ldots + \alpha_{k,k} A_k(n) &= g_k(n) \end{aligned}$$

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Example 1: arithmetic sequence

Arithmetic sequence: $a_n = a + bn$

Recurrence equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n$:

$$S_0 = a$$
$$S_n = S_{n-1} + (a + bn), \text{ for } n > 0.$$

Let's find a closed form for a bit more general recurrent equation:

$$R_0 = \alpha$$
$$R_n = R_{n-1} + (\beta + \gamma n), \text{ for } n > 0.$$

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Evaluation of terms $R_n = R_{n-1} + (\beta + \gamma n)$

$$R_0 = \alpha$$
$$R_1 = \alpha + \beta + \gamma$$
$$R_2 = \alpha + \beta + \gamma + (\beta + 2\gamma) = \alpha + 2\beta + 3\gamma$$
$$R_3 = \alpha + 2\beta + 3\gamma + (\beta + 3\gamma) = \alpha + 3\beta + 6\gamma$$

Observation

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$A(n), B(n), C(n)$ can be evaluated using repertoire method:
we will consider three cases

1. $R_n = 1$ for all n
2. $R_n = n$ for all n
3. $R_n = n^2$ for all n

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Repertoire method: case 1

Lemma 1: $A(n) = 1$ for all n

- $1 = R_0 = \alpha$
- From $R_n = R_{n-1} + (\beta + \gamma n)$ follows that $1 = 1 + (\beta + \gamma n)$. This must be true for every $n \in \mathbb{N}$, so $\beta = \gamma = 0$

Hence

$$1 = A(n) \cdot 1 + B(n) \cdot 0 + C(n) \cdot 0$$

□

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Repertoire method: case 2

Lemma 2: $B(n) = n$ for all n

- $\alpha = R_0 = 0$
- From $R_n = R_{n-1} + (\beta + \gamma n)$ follows that $n = (n-1) + (\beta + \gamma n)$.
  I.e. $1 = \beta + \gamma n$.
  This must be true for every $n \in \mathbb{N}$, so $\beta = 1$ and $\gamma = 0$

Hence

$$n = A(n) \cdot 0 + B(n) \cdot 1 + C(n) \cdot 0$$

□

Repertoire method: case 3

Lemma 3: $C(n) = \frac{n^2+n}{2}$ for all n

- $\alpha = R_0 = 0^2 = 0$.
- Equation $R_n = R_{n-1} + (\beta + \gamma n)$ can be rewritten as:
  - $n^2 = (n-1)^2 + \beta + \gamma n$.
  - $n^2 = n^2 - 2n + 1 + \beta + \gamma n$.
  - $0 = (1+\beta) + n(\gamma - 2)$.

  This must be true for every $n \in \mathbb{N}$, so $\beta = -1$ and $\gamma = 2$.

Hence:

$$\begin{aligned} n^2 &= A(n)\cdot 0 + B(n)\cdot(-1) + C(n)\cdot 2 \\ &= -n + 2C(n) \text{ by Lemma 2} \end{aligned}$$

□

Repertoire method: summing up

According to Lemma 1, 2, 3, we get:

1. $R_n = 1$ for all n $\Longrightarrow$ $A(n) = 1$
2. $R_n = n$ for all n $\Longrightarrow$ $B(n) = n$
3. $R_n = n^2$ for all n $\Longrightarrow$ $C(n) = \frac{n^2+n}{2}$

Hence,

$$R_n = \alpha + n\beta + \left(\frac{n^2+n}{2}\right)\gamma$$

For $\alpha = \beta = a$ and $\gamma = b$ we get:

$$S_n = \sum_{k=0}^{n}(a+bk) = (n+1)a + \frac{n(n+1)}{2}b$$

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Next subsection

Perturbation method

To find a closed form for $S_n = \sum_{0 \leqslant k \leqslant n} a_k$:

1. Rewrite S_{n+1} by splitting off first and last term:

$$\begin{aligned} S_n + a_{n+1} &= a_0 + \sum_{1 \leqslant k \leqslant n+1} a_k \\ &= a_0 + \sum_{1 \leqslant k+1 \leqslant n+1} a_{k+1} \\ &= a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1} \end{aligned}$$

2. Work on the sum on the right and express it as a function of S_n.
3. Solve with respect to S_n.

Example 2: geometric sequence

Geometric sequence: $a_n = ax^n, x \neq 1$

Recurrence equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$S_0 = a$$
$$S_n = S_{n-1} + ax^n \text{, for } n > 0 \text{.}$$

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Recurrence equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$S_0 = a$$
$$S_n = S_{n-1} + ax^n \text{, for } n > 0 \,.$$

- Splitting off the first term gives

$$\begin{aligned} S_n + a_{n+1} &= a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1} \\ &= a + \sum_{0 \leqslant k \leqslant n} ax^{k+1} \\ &= a + x \sum_{0 \leqslant k \leqslant n} ax^k \\ &= a + xS_n \end{aligned}$$

Example 2: geometric sequence

Geometric sequence: $a_n = ax^n, x \neq 1$

Recurrence equation for the sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{0 \leqslant k \leqslant n} ax^k$:

$$S_0 = a$$
$$S_n = S_{n-1} + ax^n \text{, for } n > 0 \,.$$

- From this we obtain the equality:

$$S_n + ax^{n+1} = a + xS_n \,,$$

  that is: $(1-x)S_n = a - ax^{n+1}$.
- As $x \neq 1$ we can divide and obtain:

$$S_n = a \cdot \frac{1 - x^{n+1}}{1 - x}$$

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Example 3: When perturbation doesn't work . . .

Compute: $S_n = \sum_{k=0}^{n} k^2$.

1. Perturb the sum:

$$S_n + (n+1)^2 = 0 + \sum_{k=1}^{n+1} k^2$$

   Um . . . that shifted k^2 sounds bad . . .

Example 3: When perturbation doesn't work ...

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   Um ... that shifted k^2 sounds bad ...

2. Rewrite the right-hand side so that it depends on S_n:

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=0}^{n} (k+1)^2 \\
&= \sum_{k=0}^{n} (k^2 + 2k + 1) \\
&= S_n + \sum_{k=0}^{n} (2k+1) \\
&= S_n + 2\frac{n(n+1)}{2} + n + 1
\end{aligned}$$

Example 3: When perturbation doesn't work ...

Compute: $S_n = \sum_{k=0}^{n} k^2$.

1. Perturb the sum:

$$S_n + (n+1)^2 = 0 + \sum_{k=1}^{n+1} k^2$$

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2. Rewrite the right-hand side so that it depends on S_n:

$$\sum_{k=1}^{n+1} k^2 = S_n + 2\frac{n(n+1)}{2} + n + 1$$

3. Solve with respect to S_n:

$$\begin{aligned} S_n + (n+1)^2 &= S_n + (n+1) + 2\frac{n(n+1)}{2} \\ (n+1)^2 &= (n+1) + n(n+1) \end{aligned}$$

   ... which is true, but where is S_n?

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. . . try perturbing *another* sum!

In addition to S_n, consider the sum: $T_n = \sum_{k=0}^{n} k^3$.

1. Perturb T_n:

$$T_n + (n+1)^3 = 0 + \sum_{k=1}^{n+1} k^3$$

…try perturbing *another* sum!

In addition to S_n, consider the sum: $T_n = \sum_{k=0}^{n} k^3$.

1. Perturb T_n:

$$T_n + (n+1)^3 = 0 + \sum_{k=1}^{n+1} k^3$$

2. Rewrite the right-hand side so that it depends on T_n and on S_n:

$$\begin{aligned} \sum_{k=1}^{n+1} k^3 &= \sum_{k=0}^{n} (k+1)^3 \\ &= \sum_{k=0}^{n} (k^3 + 3k^2 + 3k + 1) \\ &= T_n + 3S_n + \sum_{k=0}^{n} (3k+1) \end{aligned}$$

. . . try perturbing *another* sum!

In addition to S_n, consider the sum: $T_n = \sum_{k=0}^{n} k^3$.

1. Perturb T_n:

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$$\sum_{k=1}^{n+1} k^3 = T_n + 3S_n + \sum_{k=0}^{n} (3k+1)$$

3. Solve with respect to S_n:

$$\begin{aligned}
(n+1)^3 &= 3S_n + 3\frac{n(n+1)}{2} + n + 1 \\
&= 3S_n + (n+1)\left(\frac{3}{2}n + 1\right) \\
3S_n &= (n+1)\left(n^2 + 2n + 1 - \frac{3}{2}n - 1\right) \\
S_n &= \frac{1}{3}(n+1)\left(n^2 + \frac{n}{2}\right) = \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

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