ITT9132 Concrete Mathematics
Lecture 4 – 20 February 2023

Chapter Two

Multiple Sums

Finite and Infinite Calculus

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Multiple sums

Definition

If H is a finite subset of $\mathbb{Z}^2$, we put:

$$\sum_{(j,k)\in H} a_{j,k} = \sum_{j,k} a_{j,k}\,[P(j,k)]$$

where $P(j,k) ::= (j,k) \in H$.

As only finitely many summands are nonzero, the usual properties of sums can be applied, and the following holds:

Law of interchange of order of summation

$$\sum_{j}\sum_{k} a_{j,k}\,[P(j,k)] = \sum_{j,k} a_{j,k}\,[P(j,k)] = \sum_{k}\sum_{j} a_{j,k}\,[P(j,k)]$$

Multiple sums with independent indices

If $P(j,k) = Q(j) \wedge R(k)$, then the indices j and k are independent and the double sum can be rewritten:

$$\begin{aligned}
\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k} \left([Q(j) \wedge R(k)] \right) \\
&= \sum_{j,k} a_{j,k} [Q(j)] [R(k)] \\
&= \sum_{j} [Q(j)] \sum_{k} a_{j,k} [R(k)] \quad \text{by commutativity, distributivity and associativity} \\
&= \sum_{j} \sum_{k} a_{j,k} \\
&= \sum_{k} [R(k)] \sum_{j} a_{j,k} [Q(j)] \\
&= \sum_{k} \sum_{j} a_{j,k}
\end{aligned}$$

Multiple sums with dependent indices

In general, the indices are not independent, but we can write:

$$P(j,k) = Q(j) \land R'(j,k) = R(k) \land Q'(j,k)$$

In this case, for $K'(j) = \{k \mid R'(j,k)\}$ and $J'(k) = \{j \mid Q'(j,k)\}$ we can proceed as follows:

$$\begin{aligned}
\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k}[Q(j)][R'(j,k)] \\
&= \sum_{j}[Q(j)]\sum_{k} a_{j,k}[R'(j,k)] = \sum_{j\in J}\sum_{k\in K'(j)} a_{j,k} \\
&= \sum_{k}[R(k)]\sum_{j} a_{j,k}[Q'(j,k)] = \sum_{k\in K}\sum_{j\in J'(k)} a_{j,k}
\end{aligned}$$

Warmup: what's wrong with this sum?

$$
\begin{aligned}
\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right) &= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k} \\
&= \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{a_k}{a_k} \\
&= \sum_{k=1}^{n} \sum_{k=1}^{n} 1 \\
&= n^2
\end{aligned}
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Solution

The second passage is seriously wrong:
It is not licit to turn two independent variables into two dependent ones.

Examples of multiple summing: Mutual upper bounds

Change of index order

However given a positive integer n and complex numbers $a_{j,k}$,

$$\sum_{k=1}^{n}\sum_{j=1}^{k} a_{j,k} = \sum_{1 \leqslant j \leqslant k \leqslant n} a_{j,k} = \sum_{j=1}^{n}\sum_{k=j}^{n} a_{j,k}$$

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Proof:

- For every positive integer n and integers j, k we have:

$$[1 \leqslant j \leqslant n]\,[j \leqslant k \leqslant n] = [1 \leqslant j \leqslant k \leqslant n] = [1 \leqslant k \leqslant n]\,[1 \leqslant j \leqslant k]$$

- Hence,

$$\begin{aligned}
\sum_{k=1}^{n} \sum_{j=1}^{k} a_{j,k} &= \sum_{k} \sum_{j} a_{j,k}\,[1 \leqslant k \leqslant n]\,[1 \leqslant j \leqslant k] \\
&= \sum_{j,k} a_{j,k}\,[1 \leqslant j \leqslant k \leqslant n] = \sum_{1 \leqslant j \leqslant k \leqslant n} a_{j,k} \\
&= \sum_{j} \sum_{k} a_{j,k}\,[1 \leqslant j \leqslant n]\,[j \leqslant k \leqslant n] = \sum_{j=1}^{n} \sum_{k=j}^{n} a_{j,k}
\end{aligned}$$

- Observe that the values of the $a_{j,k}$'s played no part.

Examples of multiple summing: Mutual upper bounds

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$$\sum_{k=1}^{n} \sum_{j=1}^{k} a_{j,k} = \sum_{1 \leqslant j \leqslant k \leqslant n} a_{j,k} = \sum_{j=1}^{n} \sum_{k=j}^{n} a_{j,k}$$

This can also be understood by considering the matrix:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

and observing that:

- The left-hand side is the sum S_U of the elements in the upper triangular part, counted from top to bottom and from left to right.
- The right-hand side is the sum of the same elements, counted from left to right and from top to bottom.

Upper and lower

What about the sum S_L of the elements of the lower triangular part?

- A quick application of the inclusion-exclusion formula gives us:

$$[1 \leqslant j \leqslant k \leqslant n] + [1 \leqslant k \leqslant j \leqslant n] = [1 \leqslant j \leqslant n, 1 \leqslant k \leqslant n] + [1 \leqslant k \leqslant n, j = k]$$

- Then:

$$\begin{aligned} S_U + S_L &= \sum_j \sum_k a_{j,k} [1 \leqslant j \leqslant k \leqslant n] + \sum_j \sum_k a_{j,k} [1 \leqslant k \leqslant j \leqslant n] \\ &= \sum_j \sum_k a_{j,k} ([1 \leqslant j \leqslant k \leqslant n] + [1 \leqslant k \leqslant j \leqslant n]) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} a_{j,k} ([1 \leqslant j \leqslant n, 1 \leqslant k \leqslant n] + [1 \leqslant k \leqslant n, j = k]) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} a_{j,k} + \sum_{k=1}^{n} a_{k,k} \end{aligned}$$

The case of symmetric summands

Now assume that the matrix is symmetric: that is, $a_{j,k} = a_{k,j}$ for every j and k.

- Then $S_L = S_U$ and the equality in the previous page can be rewritten:

$$\sum_{k=1}^{n} \sum_{j=1}^{k} a_{j,k} = \frac{1}{2} \left(\sum_{j=1}^{n} \sum_{k=1}^{n} a_{j,k} + \sum_{k=1}^{n} a_{k,k} \right)$$

- If, in particular, $a_{j,k} = a_j \cdot a_k$ for suitable $a_1, \ldots, a_n$, then:

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k = \sum_{j=1}^{n} a_j \sum_{k=1}^{n} a_k = \left(\sum_{k=1}^{n} a_k \right)^2$$

We thus get the following, remarkable rule:

Theorem

For every positive integer n and complex numbers $a_1, \ldots, a_n$,

$$\sum_{k=1}^{n} \sum_{j=1}^{k} a_j a_k = \frac{1}{2} \left(\left(\sum_{k=1}^{n} a_k \right)^2 + \sum_{k=1}^{n} a_k^2 \right)$$

Examples of multiple summation

Example 1: $S_{1,n} = \sum_{k=1}^{n} \sum_{j=1}^{k-1} \frac{1}{k-j}$

$$\begin{aligned}
S_{1,n} &= \sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant k-j < k} \frac{1}{j} \\
&\quad \text{swapping } j \text{ with } k-j \text{ as both vary between 1 and } k-1 \\
&= \sum_{1 \leqslant k \leqslant n} \sum_{0 < j \leqslant k-1} \frac{1}{j} \quad \text{rewriting} \\
&= \sum_{1 \leqslant k \leqslant n} H_{k-1} \\
&= \sum_{1 \leqslant k+1 \leqslant n} H_k \\
&= \sum_{0 \leqslant k < n} H_k
\end{aligned}$$

Examples of multiple summation

Example 2: $S_{2,n} = \sum_{j=1}^{n} \sum_{k=j+1}^{n} \frac{1}{k-j}$

$$\begin{aligned}
S_{2,n} &= \sum_{1 \leqslant j \leqslant n} \sum_{j < k+j \leqslant n} \frac{1}{k} \\
&\quad \text{swapping } k \text{ with } k-j \text{ as both vary between } j+1 \text{ and } n \\
&= \sum_{1 \leqslant j \leqslant n} \sum_{0 < k \leqslant n-j} \frac{1}{k} \ \text{ rewriting} \\
&= \sum_{1 \leqslant j \leqslant n} H_{n-j} \\
&= \sum_{1 \leqslant n-j \leqslant n} H_j \ \text{ reversing order of summation} \\
&= \sum_{0 \leqslant j < n} H_j
\end{aligned}$$

Examples of multiple summation

Example 3: $S_{3,n} = \sum_{1 \leqslant j < k \leqslant n} \frac{1}{k-j}$

Of course, $S_{1,n} = S_{2,n} = S_{3,n}$. But:

$$\begin{aligned}
S_{3,n} &= \sum_{1 \leqslant j < k+j \leqslant n} \frac{1}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant j \leqslant n-k} \frac{1}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \frac{n-k}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \frac{n}{k} - \sum_{1 \leqslant k \leqslant n} 1 \\
&= n \left(\sum_{1 \leqslant k \leqslant n} \frac{1}{k} \right) - n \\
&= nH_n - n
\end{aligned}$$

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Another way of “simplifying by complicating”

To compute a sum of the form $S_n = \sum_{1 \leqslant k \leqslant n} a_k$:

1. Expand the summand a_k by introducing a new variable j and new summands b_j, c_k such that:
$$a_k = \sum_{1 \leqslant j \leqslant k} b_j c_k$$
2. Rewrite the sum $\sum_{1 \leqslant k \leqslant n} a_k$ as the double sum $\sum_{1 \leqslant j \leqslant k \leqslant n} b_j c_k$.
3. Contract the summands into a sum over k parameterized by j:
$$S_n = \sum_{1 \leqslant k \leqslant n} \left(\sum_{1 \leqslant j \leqslant k} b_j \right) c_k = \sum_{1 \leqslant j \leqslant n} b_j \left(\sum_{j \leqslant k \leqslant n} c_k \right)$$
4. Sum over j to obtain a closed form for S_n.

Example: again, $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$

1. Expand: $k^2 = k \cdot k = \left(\sum_{j=1}^{k} 1 \right) \cdot k$.
2. Write the double sum: $\Box_n = \sum_{1 \leqslant j \leqslant k \leqslant n} k$.
3. Contract by summing over k:

$$\begin{aligned}
\Box_n = \sum_{j=1}^{n} \sum_{k=j}^{n} k &= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} k - \sum_{k=1}^{j-1} k \right) \\
&= \sum_{j=1}^{n} \left(\frac{n(n+1)}{2} - \frac{(j-1)j}{2} \right) \\
&= \frac{1}{2} \left(n^2(n+1) - \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} j \right) \\
&= \frac{n^2(n+1)}{2} - \frac{1}{2}\Box_n + \frac{n(n+1)}{4}
\end{aligned}$$

4. Derive a closed form for $\Box_n$:

$$\frac{3}{2}\Box_n = \frac{n+1}{4} \cdot (2n^2 + n), \text{ that is, } \Box_n = \frac{n(n+1)(2n+1)}{6}$$

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Derivative and Difference Operators

Infinite calculus: derivative

Euler's notation

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Lagrange's notation

$$f'(x) = \mathrm{D}f(x)$$

Leibniz's notation If $y = f(x)$, then

$$\frac{dy}{dx} = \frac{df}{dx}(x) = \frac{df(x)}{dx} = \mathrm{D}f(x)$$

Newton's notation

$$\dot{y} = f'(x)$$

Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$

In general, if $h \in \mathbb{R}$ (or $h \in \mathbb{C}$), then

Forward difference

$$\Delta_h[f](x) = f(x+h) - f(x)$$

Backward difference

$$\nabla_h[f](x) = f(x) - f(x-h)$$

Central difference

$$\delta_h[f](x) = f(x + \tfrac{1}{2}h) - f(x - \tfrac{1}{2}h)$$

Note that:

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{\Delta_h[f](x)}{h}$$

The derivative of a power

Example: $f(x) = x^3$

In this case,

$$\begin{aligned} \Delta_h[f](x) &= (x+h)^3 - x^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - x^3 \\ &= h \cdot (3x^2 + 3xh + h^2) \end{aligned}$$

Hence,

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{h \cdot (3x^2 + 3xh + h^2)}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$$

In general, for $m \geqslant 1$ integer:

$$\mathrm{D}(x^m) = mx^{m-1}$$

because of Newton's binomial theorem.

The (forward) difference of a power

Example: $f(x) = x^3$

In this case,

$$\Delta f(x) = \Delta_1 [f](x) = 3x^2 + 3x + 1$$

In general, for $m \geqslant 1$ integer:

$$\Delta(x^m) = \sum_{k=1}^{m} \binom{m}{k} x^{m-k} = \sum_{k=0}^{m-1} \binom{m}{k} x^k$$

again because of Newton's binomial theorem – but this time, we don't take a limit.

Falling and Rising Factorials

Definition

Let m be a positive integer.

- The falling factorial power, or simply falling factorial, is defined as:

$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$$

- The rising factorial power, or simply rising factorial, is defined as:

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$$

Read: "x to the mth falling" and "x to the mth rising", respectively.

From the definitions the following properties immediately follow:

1. $x^{\overline{m}} = (x+m-1)^{\underline{m}} = (-1)^m(-x)^{\underline{m}}$.
2. $m^{\underline{m}} = 1^{\overline{m}} = m!$.
3. $x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$.
4. $n^{\underline{m}} = m!\binom{n}{m}$ if n is a nonnegative integer.
5. $x^{\underline{m}} = \dfrac{x^{\underline{m+1}}}{x-m}$ if $x \neq m$.

Falling factorials with negative exponents

We want to define $x^{\underline{m}}$ with $m \leqslant 0$ integer so that the expansion rule:

$$x^{\underline{m+n}} = x^{\underline{m}} \cdot (x-m)^{\underline{n}}$$

is satisfied for every $m, n \in \mathbb{Z}$ and $x \in \mathbb{C}$.

- First of all, it must be $x^{\underline{0+n}} = x^{\underline{0}}(x-0)^{\underline{n}}$ for every $x \in \mathbb{C}$ and $n \in \mathbb{N}$. Then:

  $$x^{\underline{0}} = 1$$

  This is also consistent with defining an empty product as equal to 1.
- Next, it must be $x^{\underline{0}} = x^{\underline{-m}} \cdot (x+m)^{\underline{m}}$ for every $x \in \mathbb{C}$ and $m \in \mathbb{N}$ such that the right-hand side is nonzero. Then:

$$x^{\underline{-m}} = \frac{1}{(x+m)^{\underline{m}}} = \frac{1}{(x+1)^{\overline{m}}} \quad \text{for every } x \notin \{-1, \dots, -m\}$$

Dually,

$$x^{\overline{-m}} = \frac{1}{(x-1)^{\underline{m}}} \quad \text{for every } x \notin \{1, \dots, m\}$$

Difference of falling factorial with positive exponent

For $m \geqslant 1$ we have:

$$\begin{aligned}\Delta\left(x^{\underline{m}}\right) &= (x+1)^{\underline{m}} - x^{\underline{m}} \\ &= (x+1)\cdot x^{\underline{m-1}} - x^{\underline{m-1}}\cdot(x-m+1) \\ &= (x+1-(x-m+1))\cdot x^{\underline{m-1}} \\ &= m\cdot x^{\underline{m-1}}\end{aligned}$$

We thus obtain our first rule:

$$\Delta(x^{\underline{m}}) = m x^{\underline{m-1}} \ \forall m \geqslant 1$$

In fact, the family of functions $\{x^{\underline{m}} \mid m \in \mathbb{N}\}$ is the solution of the family of recurrences:

$$\Delta f_m(x) = m f_{m-1}(x) \text{ for } m \geqslant 1 \ ; \ \ f_m(0) = [m=0]$$

This is the same role of powers in Calculus, which satisfy:

$$D f_m(x) = m f_{m-1}(x) \text{ for } m \geqslant 1 \ ; \ \ f_m(0) = [m=0]$$

Differences of falling factorials with negative exponents

First, a simple example:

$$\begin{aligned}
\Delta x^{\underline{-2}} &= (x+1)^{\underline{-2}} - x^{\underline{-2}} \\
&= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)} \\
&= \frac{(x+1)-(x+3)}{(x+1)(x+2)(x+3)} \\
&= \frac{-2}{(x+1)(x+2)(x+3)} \\
&= -2 \cdot x^{\underline{-3}}
\end{aligned}$$

Differences of falling factorials with negative exponents

Now, for the general rule: let $m \in \mathbb{N}$. Then:

$$\begin{aligned}
\Delta x^{\underline{-m}} &= (x+1)^{\underline{-m}} - x^{\underline{-m}} \\
&= \frac{1}{(x+2)\cdots(x+m)(x+m+1)} - \frac{1}{(x+1)(x+2)\cdots(x+m)} \\
&= \frac{(x+1)-(x+m+1)}{(x+1)(x+2)\cdots(x+m)(x+m+1)} \\
&= \frac{-m}{(x+1)(x+2)\cdots(x+m)(x+m+1)} \\
&= -mx^{\underline{-(m+1)}} \\
&= -mx^{\underline{-m-1}}
\end{aligned}$$

Warmup: Initial conditions for all falling powers

Exercise 2.8

What is the value of $0^{\underline{m}}$, when m is a given integer?

1. For $m > 0$ a zero factor appears in $0^{\underline{m}}$, so $0^{\underline{m}} = 0$.
2. We observed that $x^{\underline{0}} = 1$ regardless of x, so $0^{\underline{0}} = 1$.
3. For $m < 0$ we have:

$$\begin{aligned} 0^{\underline{m}} &= \frac{1}{(x+1)^{\overline{-m}}} \text{ with } x = 0 \\ &= \frac{1}{1^{\overline{|m|}}} = \frac{1}{|m|!} \end{aligned}$$

A general expression is thus:

$$0^{\underline{m}} = \frac{1}{|m|!}\,[m \leqslant 0]$$

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A general expression is thus:

$$0^{\underline{m}} = \frac{1}{|m|!}[m \leqslant 0]$$

Next subsection

Indefinite Integrals and Sums

The Fundamental Theorem of Calculus

If g is continuous and f is differentiable, then:

$$\mathrm{D}f(x) = g(x) \quad \text{iff} \quad \int g(x)\mathrm{d}x = f(x) + \mathrm{C}$$

The symbol $\int g(x)\,dx$, without the additive constant, represents a family of functions whose derivative is $g(x)$.
We do something similar for sums:

Definition

The indefinite sum of the function $g(x)$ is the class of functions f such that $\Delta f(x) = g(x)$:

$$\Delta f(x) = g(x) \quad \text{iff} \quad \sum g(x)\delta x = f(x) + C(x)$$

where $C(x)$ is a function such that $C(x+1) = C(x)$ for any integer value of x.

For example, $C(x)$ could be periodic of period 1, and not necessarily constant.
Or, it could be zero on integers, and arbitrary everywhere else. Etc.

Definite Integrals and Sums

If $g(x) = \mathrm{D}f(x)$, then:

$$\int_a^b g(x)\mathrm{d}x = f(x)\Big|_a^b = f(b) - f(a)$$

We want definite sums to satisfy a similar property:

If $g(x) = \Delta f(x)$, then:

$$\sum_a^b g(x)\delta x = f(x)\Big|_a^b = f(b) - f(a)$$

Definite sums

Some observations

- $\sum_a^a g(x)\delta x = f(a) - f(a) = 0$
- $\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a)$
- $\sum_a^{b+1} g(x)\delta x - \sum_a^b g(x)\delta x = f(b+1) - f(b) = g(b)$

Hence, if $g(x) = \Delta f(x)$, then:

$$\begin{aligned}
\sum_a^b g(x)\delta x &= \sum_{k=a}^{b-1} g(k) = \sum_{a \leqslant k < b} g(k) \\
&= \sum_{a \leqslant k < b} (f(k+1) - f(k)) \\
&= (f(a+1) - f(a)) + (f(a+2) - f(a+1)) + \dots \\
&\quad + (f(b-1) - f(b-2)) + (f(b) - f(b-1)) \\
&= f(b) - f(a)
\end{aligned}$$

Integrals and Sums of Powers

Let's do a first "sanity check" for our definition:

If $m \neq -1$, then:

$$\int_0^n x^m \mathrm{d}x = \left.\frac{x^{m+1}}{m+1}\right|_0^n = \frac{n^{m+1}}{m+1}$$

Indeed:

If $m \neq -1$, then:

$$\sum_0^n x^{\underline{m}} \delta x = \sum_{0 \leqslant k < n} k^{\underline{m}} = \left.\frac{k^{\underline{m+1}}}{m+1}\right|_0^n = \frac{n^{\underline{m+1}}}{m+1}$$

Sums of Powers: applications

Case $m=1$

$$\sum_{0\leqslant k<n} k = \frac{n^{\underline{2}}}{2} = \frac{n(n-1)}{2}$$

Case $m=2$ Due to $k^2 = k^{\underline{2}} + k^{\underline{1}}$ we get:

$$\begin{aligned}\sum_{0\leqslant k<n} k^2 &= \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} \\ &= \frac{1}{3}n(n-1)(n-2) + \frac{1}{2}n(n-1) \\ &= \frac{1}{6}n(2(n-1)(n-2)+3(n-1)) \\ &= \frac{1}{6}n(n-1)(2n-4+3) \\ &= \frac{1}{6}n(n-1)(2n-1)\end{aligned}$$

Taking $n+1$ instead of n gives:

$$\square_n = \frac{(n+1)n(2n+1)}{6}$$

Sums of Powers (case $m = -1$)

As a first step, we observe that:

$$\begin{aligned}\Delta H_x &= H_{x+1} - H_x \\ &= \left(1 + \frac{1}{2} + \ldots + \frac{1}{x} + \frac{1}{x+1}\right) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{x}\right) \\ &= \frac{1}{x+1} = x^{\underline{-1}}\end{aligned}$$

We conclude:

$$\sum_a^b x^{\underline{-1}} \delta x = H_b - H_a \text{ for } 0 < a \leqslant b$$

This is yet another parallel between harmonic numbers and natural logarithms, as we know that:

$$\int_a^b \frac{dx}{x} = \ln b - \ln a \text{ for } 0 < a \leqslant b$$

Sums of Discrete Exponential Functions

- We have:
$$\mathrm{D}e^x = e^x$$
The finite analogue should have $\Delta f(x) = f(x)$. This means:
$$f(x+1) - f(x) = f(x), \text{ that is, } f(x+1) = 2f(x), \text{ only possible if } f(x) = 2^x$$
- For general base $c > 0$, the difference of c^x is:
$$\Delta(c^x) = c^{x+1} - c^x = (c-1)c^x$$
and the "anti-difference" for $c \neq 1$ is $\dfrac{c^x}{c-1}$.

As an application, we compute the sum of the geometric progression:
$$\sum_{a \leqslant k < b} c^k = \sum_a^b c^x \delta x = \frac{c^x}{c-1}\Big|_a^b = \frac{c^b - c^a}{c-1} = c^a \cdot \frac{c^{b-a} - 1}{c-1}.$$

Differential equations and difference equations

Differential equation	Solution	Difference equation	Solution
$Df_n(x) = nf_{n-1}(x)$ $f_n(0) = [n=0]\ ,\ n \geqslant 0$	$f_n(x) = x^n$	$\Delta u_m(x) = mu_{m-1}(x)$ $u_m(0) = [m=0]\ ,\ m \geqslant 0$	$u_m(x) = x^{\underline{m}}$
$Df_n(x) = nf_{n-1}(x)$ $f_n(1) = 1\ ,\ n < 0$	$f_n(x) = x^n$	$\Delta u_m(x) = mu_{m-1}(x)$ $u_m(0) = \dfrac{1}{\lvert m\rvert!}\ ,\ m < 0$	$u_m(x) = x^{\underline{m}}$
$Df(x) = x^{-1} \cdot [x > 0]$ $f(1) = 1$	$f(x) = \ln x$	$\Delta u(x) = x^{\underline{-1}} \cdot [x > 0]$ $u(1) = 1$	$u(x) = H_x$
$Df(x) = f(x)$ $f(0) = 1$	$f(x) = e^x$	$\Delta u(x) = u(x)$ $u(0) = 1$	$u(x) = 2^x$
$Df(x) = b \cdot f(x)$ $f(0) = 1$	$f(x) = a^x$ where $b = \ln a$	$\Delta u(x) = b \cdot u(x)$ $u(0) = 1$	$u(x) = c^x$ where $b = c - 1$

l'Hôpital's rule and Stolz-Cesàro lemma

l'Hôpital's rule: Hypotheses

1. $f(x)$ and $g(x)$ are both vanishing or both infinite at x_0.
2. $g'(x)$ is always positive in some neighborhood of x_0.

l'Hôpital's rule: Thesis

- If $\lim_{x\to x_0} \dfrac{f'(x)}{g'(x)} = L \in \mathbb{R}$,
- then $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = L$.

Stolz-Cesàro lemma: Hypotheses

1. $u(n)$ and $v(n)$ are defined for every value $n \in \mathbb{N}$.
2. $v(n)$ is positive, strictly increasing, and divergent.

Stolz-Cesàro lemma: Thesis

- If $\lim_{n\to\infty} \dfrac{\Delta u(n)}{\Delta v(n)} = L \in \mathbb{R}$,
- then $\lim_{n\to\infty} \dfrac{u(n)}{v(n)} = L$.

A useful corollary

Arithmetic mean theorem

If $\lim_{n\to\infty} a_n = L$, then $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k = L$ too.

That is:

If a sequence converges,
then the sequence of its arithmetic means converges to the same limit.

Proof:

- Let $u(x) = \displaystyle\sum_{k=0}^{x-1} a_k$ and $v(x) = x$.
- Then $\Delta u(x) = a_x$ and $\Delta v(x) = 1$.
- Apply the Stolz-Cesàro lemma.

Next subsection

Summation by Parts

Infinite analogue: integration by parts

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x$$

Now, when we take the difference of a product, we have a slightly more complex rule:

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x)\\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x)\\ &= \Delta u(x)v(x+1) + u(x)\Delta v(x)\\ &= u(x)\Delta v(x) + Ev(x)\Delta u(x)\end{aligned}$$

where E is the shift operator $Ef(x) = f(x+1)$. We then have the:

Rule for summation by parts

$$\sum u\Delta v\,\delta x = uv - \sum Ev\Delta u\,\delta x$$

Why the shift?

If we repeat our derivation with two continuous functions f and g of one real variable x, we find for any increment $h \neq 0$:

$$\begin{aligned} f(x+h)g(x+h) - f(x)g(x) &= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x) \\ &= f(x)(g(x+h) - g(x)) + g(x+h)(f(x+h) - f(x)) \end{aligned}$$

The incremental ratio is thus:

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = f(x) \cdot \frac{g(x+h) - g(x)}{h} + g(x+h) \cdot \frac{f(x+h) - f(x)}{h}$$

So there is a shift: but it is infinitesimal—and disappears by continuity of g.

Example: $S_n = \sum_{k=0}^{n} kH_k$

- We want to write $S_n = \sum_0^{n+1} u(x)\,\Delta v(x)\,\delta x$ for suitable $u(x)$ and $v(x)$.
- Let $u(x) = H_x$ and $v(x) = x^{\underline{2}}/2$.
- Then $\Delta u(x) = x^{\underline{-1}}$, $\Delta v(x) = x$, and $Ev(x) = (x+1)^{\underline{2}}/2$.
- Summing by parts:

$$
\begin{aligned}
\sum_0^{n+1} x\,H_x\,\delta x &= \frac{x^{\underline{2}}}{2} H_x \bigg|_0^{n+1} - \sum_0^{n+1} \frac{(x+1)^{\underline{2}}}{2}\, x^{\underline{-1}}\,\delta x \\
&= \frac{(n+1)n}{2} H_{n+1} - \frac{1}{2} \sum_0^{n+1} x^{\underline{-1}} (x-(-1))^{\underline{2}}\,\delta x \\
&= \frac{(n+1)n}{2} H_{n+1} - \frac{1}{2} \sum_0^{n+1} x^{\underline{1}}\,\delta x \\
&= \frac{(n+1)n}{2} H_{n+1} - \frac{(n+1)n}{4} \\
&= \frac{(n+1)n}{2} \left(H_{n+1} - \frac{1}{2} \right)
\end{aligned}
$$