ITT9132 Concrete Mathematics
Lecture 5 – 28 February 2023

Original slides 2010–2014 Jaan Penjam; modified 2016–2023 Silvio Capobianco

Last update: 28 February 2023

Contents

Next section

How to sum infinite number sequences?

Setting $\sum_{k\in\mathbb{N}} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$ seems meaningful ...

Example 1

Let

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots .$$

Then

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2 + S,$$

and

$$S = 2$$

How to sum infinite number sequences?

Setting $\sum_{k\in\mathbb{N}} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$ seems meaningful ...

Example 1

Let

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots .$$

Then

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2 + S,$$

and

$$S = 2$$

But can we manipulate such sums like we do with finite sums?

How to sum infinite number sequences?

Example 2

Let

$$T = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots$$

Then

$$2T = 2 + 4 + 8 + 16 + 32 + 64 + 128 \dots = T - 1$$

and

$$T = -1$$

How to sum infinite number sequences?

Example 2

Let

$$T = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \ldots$$

Then

$$2T = 2 + 4 + 8 + 16 + 32 + 64 + 128 \ldots = T - 1$$

and

$$T = -1$$

Problem:

- The sum T is infinite . . .
- and we cannot subtract an infinite quantity from another infinite quantity.

How to sum infinite number sequences?

Example 3

Let

$$\sum_{k \geqslant 0} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

Different ways to sum

$$(1-1)+(1-1)+(1-1)+(1-1)+\ldots = 0+0+0+0+\ldots = 0$$

and

$$1-(1-1)-(1-1)-(1-1)-(1-1)-\ldots = 1-0-0-0-0-0-\ldots = 1$$

How to sum infinite number sequences?

Example 3

Let

$$\sum_{k \geqslant 0} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

Different ways to sum

$$(1-1) + (1-1) + (1-1) + (1-1) + \dots = 0 + 0 + 0 + 0 + \dots = 0$$

and

$$1 - (1-1) - (1-1) - (1-1) - (1-1) - \dots = 1 - 0 - 0 - 0 - 0 - 0 - \dots = 1$$

Problem:

- The sequence of the partial sums does not converge ...
- and we cannot manipulate something that does not exist.

Defining Infinite Sums: Nonnegative Summands

Definition 1

If $a_k \geqslant 0$ for every $k \geqslant 0$, then:

$$\sum_{k \geqslant 0} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k = \sup_{K \subseteq \mathbb{N}, |K| < \infty} \sum_{k \in K} a_k$$

Note that:

- The definition as a limit is (sort of) a *Riemann integral*.
- The definition as a least upper bound is a *Lebesgue integral* with respect to the counting measure
$$\mu(X) = \text{if } |K| = n \in \mathbb{N} \text{ then } n \text{ else } +\infty$$
- The limit / least upper bound above can be finite or infinite, but are always equal.
Exercise: Prove this fact.

Defining Infinite Sums: Riemann Summation

Definition 2 (Riemann sum of a series)

A series $\sum_{k \geqslant 0} a_k$ with complex coefficients converges to a complex number S, called the sum of the series, if:

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k = S\,.$$

In this case, we write: $\sum_{k \geqslant 0} a_k = S$.
The values $S_n = \sum_{k=0}^{n} a_k$ are called the partial sums of the series.
The series $\sum_{k \geqslant 0} a_k$ converges absolutely if $\sum_{k \geqslant 0} |a_k|$ converges.

Note that the series $\sum_{k \geqslant 0} a_k = \sum_{k \geqslant 0} (b_k + ic_k)$ converges to $S = T + iU$ if and only if $\sum_{k \geqslant 0} b_k$ converges to T and $\sum_{k \geqslant 0} c_k$ converges to U.

A series that converges, but not absolutely

Let $a_k = \dfrac{(-1)^{k-1}}{k} [k > 0]$. Then $\displaystyle\sum_{k \geqslant 0} a_k = \ln 2$.

However, it is easy to prove by induction that $\displaystyle\sum_{k=0}^{2^n} |a_k| = H_{2^n} > \frac{n}{2}$ for every $n \geqslant 1$.

Infinite Sums: Associativity *within* a series

Associativity

A series $\sum_{k \geqslant 0} a_k$ has the associative property if for every two strictly increasing sequences

$$i_0 = 0 < i_1 < i_2 < \ldots < i_k < i_{k+1} < \ldots$$
$$j_0 = 0 < j_1 < j_2 < \ldots < j_k < j_{k+1} < \ldots$$

we have:

$$\sum_{k \geqslant 0} \left(\sum_{i=i_k}^{i_{k+1}-1} a_i \right) = \sum_{k \geqslant 0} \left(\sum_{j=j_k}^{j_{k+1}-1} a_j \right)$$

We have seen that the series $\sum_{k \geqslant 0} (-1)^k$ *does not* have the associative property.

Theorem

A series has the associative property if and only if it has a sum (finite or infinite).

Proof: Regrouping as in the definition means taking a subsequence of the sequence of partial sums, which can converge to any of the latter's limit points.

Defining Infinite Sums: Lebesgue Summation

Every real number can be written as $x = x^+ - x^-$, where:

$$x^+ = x \cdot [x > 0] = \max(x, 0) \text{ and } x^- = -x \cdot [x < 0] = \max(-x, 0)$$

Note that: $x^+ \geqslant 0$, $x^- \geqslant 0$, and $x^+ + x^- = |x|$.

Definition 3 (Lebesgue sum of a series)

Let $\{a_k\}_k$ be an absolutely convergent sequence of real numbers. Then:

$$\sum_k a_k = \sum_k a_k^+ - \sum_k a_k^-$$

The series $\sum_k a_k$:

- converges absolutely if $\sum_k a_k^+ < +\infty$ and $\sum_k a_k^- < +\infty$;
- diverges positively if $\sum_k a_k^+ = +\infty$ and $\sum_k a_k^- < +\infty$;
- diverges negatively if $\sum_k a_k^+ < +\infty$ and $\sum_k a_k^- = +\infty$.

If both $\sum_k a_k^+ = +\infty$ and $\sum_k a_k^- = +\infty$ then “Bad Stuff happens”.

Infinite Sums: Bad Stuff

Riemann series theorem

Let $\sum_k a_k$ be a series with real coefficients which converges, but not absolutely.
For every real number L there exists a permutation p of $\mathbb{N}$ such that:

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{p(k)} = L$$

Example: The harmonic series

If we rearrange the terms of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ as follows:

$$\begin{aligned} 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots &= \ldots + \frac{1}{2k-1} - \frac{1}{2(2k-1)} - \frac{1}{4k} + \ldots \\ &= \ldots + \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k}\right) + \ldots \end{aligned}$$

we obtain:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2 \quad \text{but} \quad 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots = \ln\sqrt{2}$$

Infinite Sums: Commutativity

Commutativity

A series $\sum_{k\geqslant 0} a_k$ has the commutative property if for every permutation p of $\mathbb{N}$,

$$\sum_{k\geqslant 0} a_{p(k)} = \sum_{k} a_k$$

The Riemann series theorem says that any series which is convergent, but not absolutely convergent, does not have the commutative property.

Theorem

A convergent series has the commutative property if and only if it is absolutely convergent.

Proof: (Sketch) Think of Lebesgue summation.

Infinite Sums: Commutativity

Commutativity

A series $\sum_{k\geqslant 0} a_k$ has the commutative property if for every permutation p of $\mathbb{N}$,

$$\sum_{k\geqslant 0} a_{p(k)} = \sum_k a_k$$

The Riemann series theorem says that any series which is convergent, but not absolutely convergent, does not have the commutative property.

Theorem

A convergent series has the commutative property if and only if it is absolutely convergent.

If we want to manipulate infinite sums like finite ones,
we must require absolute convergence.

Infinite sums: Associativity *between* two series

Definition

Two series $\sum_k a_k, \sum_k b_k$ satisfy the associative property if:

$$\sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$$

Can we say that any two series have the associative property?

Infinite sums: Associativity *between* two series

Definition

Two series $\sum_k a_k, \sum_k b_k$ satisfy the associative property if:

$$\sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$$

Can we say that any two series have the associative property? In general, no:

- Let $a_k = [k \geqslant m]$ and $b_k = -[k \geqslant n]$ with $m, n \in \mathbb{Z}$.
- Then $\sum_k a_k = +\infty$ and $\sum_k b_k = -\infty$, but $\sum_k (a_k + b_k) = n - m$.
- However, we have again the $+\infty - \infty$ issue...

Infinite sums: Associativity *between* two series

Definition

Two series $\sum_k a_k, \sum_k b_k$ satisfy the associative property if:

$$\sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$$

Can we say that any two series have the associative property?

Theorem

- If the a_k and the b_k are all nonnegative, then $\sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$.
- If $\sum_k a_k$ and $\sum_k b_k$ both have a limit and at most one of those limits is infinite, then $\sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$.
- If $\sum_k a_k$ and $\sum_k b_k$ both converge absolutely, then $\sum_k (a_k + b_k)$ also converges absolutely.

Next subsection

Limits of sums, sums of limits

Consider the double indexed sequence:

$$a_{j,k} = \frac{1}{j} [1 \leqslant k \leqslant j]$$

Then on the one hand:

$$\sum_k a_{j,k} = 1 \text{ for every } j \text{, hence } \lim_{j \to \infty} \sum_k a_{j,k} = 1$$

But on the other hand:

$$\lim_{j \to \infty} a_{j,k} = 0 \text{ for every } k \text{, hence } \sum_k \lim_{j \to \infty} a_{j,k} = 0$$

A positive result

Monotone Convergence Theorem

If the $a_{j,k}$ are all nonnegative and for every k the sequence $\langle a_{j,k} \rangle_{j \geqslant 0}$ is monotone nondecreasing, then:

$$\lim_{j \to \infty} \sum_k a_{j,k} = \sum_k \lim_{j \to \infty} a_{j,k} \,,$$

regardless of the two sides being finite or infinite.

Proof: Let $a_k = \lim_{j \to \infty} a_{j,k} = \sup_j a_{j,k}$ and $S_j = \sum_k a_{j,k}$.

- Then $\langle S_j \rangle$ is nondecreasing and $\lim_{j \to \infty} S_j = \sup_j S_j \leqslant \sum_k a_k = \sum_k \lim_{j \to \infty} a_{j,k}$.
- If the l.h.s. is $+\infty$ or the r.h.s. is 0, we have nothing else to do.
- Otherwise, suppose $\sum_k a_k > \alpha > 0$: We will prove that $\sup_j S_j > \alpha$ too.
- Fix $\delta > 0$ such that $\sum_k a_k > \alpha + 2\delta$ too.
- Choose $k_1, \ldots, k_n$ such that $\sum_{i=1}^n a_{k_i} > \alpha + \delta$.
- Choose j such that $a_{j,k_i} > a_{k_i} - \delta \cdot 2^{-i}$ for every $1 \leqslant i \leqslant n$. Then:

$$S_j \geqslant \sum_{i=1}^n a_{j,k_i} > \sum_k a_{k_i} - \delta \cdot \sum_{i=1}^n \frac{1}{2^i} > \alpha + \delta - \delta = \alpha$$

What can we be sure of, in general?

Fatou's Lemma

If the $a_{j,k}$ are all nonnegative, then:

$$\sum_k \liminf_j a_{j,k} \leqslant \liminf_j \sum_k a_{j,k}$$

Proof: (Sketch) Apply the monotone convergence theorem to $b_{j,k} = \inf_{i \geqslant j} a_{i,k}$.

What can we be sure of, in general?

Fatou's Lemma

If the $a_{j,k}$ are all nonnegative, then:

$$\sum_k \liminf_j a_{j,k} \leqslant \liminf_j \sum_k a_{j,k}$$

Dominated Convergence Theorem

If $a_k = \lim_{j\to\infty} a_{j,k}$ exists for every k and in addition there exists a sequence $\langle b_k \rangle$ such that:

1. $|a_{j,k}| \leqslant b_k$ for every $j \geqslant 0$, and
2. $\sum_k b_k < +\infty$,

then:

$$\lim_{j\to\infty} \sum_k |a_{j,k} - a_k| = 0\,;$$

consequently,

$$\lim_{j\to\infty} \sum_k a_{j,k} = \sum_k a_k = \sum_k \lim_{j\to\infty} a_{j,k}\,.$$

Proof: (Sketch) Apply Fatou's lemma to $c_{j,k} = 2b_k - |a_{j,k} - a_k|$.

A proof of the divergence of the harmonic series[1]

By contradiction, assume $\sum_{k \geqslant 1} \frac{1}{k} = S < +\infty$.

- For, $j, k \geqslant 1$ put $a_{j,k} = \frac{1}{j} [1 \leqslant k \leqslant j]$ and $b_k = \frac{1}{k}$.
- Then for every j and k, $\left| a_{j,k} \right| \leqslant b_k$, and $\sum_{k \geqslant 1} b_k$ converges.
- Now, $\lim_{j \to \infty} a_{j,k} = 0$ for every k, so $\sum_{k \geqslant 1} \lim_{j \to \infty} a_{j,k} = 0$.
- But $\sum_{k \geqslant 1} a_{j,k} = 1$ for every j, so $\lim_{j \to \infty} \sum_{k \geqslant 1} a_{j,k} = 1$.
- This contradicts the Dominated Convergence Theorem.

[1] From the MathExchange thread "Awfully sophisticated proofs of simple facts".

Next subsection

Multiple infinite sums

Definition: Double infinite sums

For every $j, k \geqslant 0$ let $a_{j,k} \geqslant 0$.

1. If $a_{j,k} \geqslant 0$ for every j and k, then:

$$\sum_{j,k} a_{j,k} = \sup_{K \subseteq \mathbb{N} \times \mathbb{N}, |K| < \infty} \sum_{K} a_{j,k} = \lim_{n \to \infty} \sum_{0 \leqslant j,k \leqslant n} a_{j,k} \,.$$

   (Recall that $\sum_{0 \leqslant j,k \leqslant n} a_{j,k} = \sum_{j,k} a_{j,k} \, [0 \leqslant j \leqslant n] \, [0 \leqslant k \leqslant n]$.)

2. If $\sum_{j,k} \left| a_{j,k} \right| < +\infty$, then:

$$\sum_{j,k} a_{j,k} = \sum_{j,k} a^{+}_{j,k} - \sum_{j,k} a^{-}_{j,k} \,.$$

Multiple infinite sums

Definition: Double infinite sums

For every $j,k \geqslant 0$ let $a_{j,k} \geqslant 0$.

1. If $a_{j,k} \geqslant 0$ for every j and k, then:

$$\sum_{j,k} a_{j,k} = \sup_{K \subseteq \mathbb{N}\times\mathbb{N}, |K|<\infty} \sum_{K} a_{j,k} = \lim_{n\to\infty} \sum_{0\leqslant j,k\leqslant n} a_{j,k}\,.$$

(Recall that $\sum_{0\leqslant j,k\leqslant n} a_{j,k} = \sum_{j,k} a_{j,k} [0 \leqslant j \leqslant n] [0 \leqslant k \leqslant n]$.)

2. If $\sum_{j,k} \left| a_{j,k} \right| < +\infty$, then:

$$\sum_{j,k} a_{j,k} = \sum_{j,k} a^{+}_{j,k} - \sum_{j,k} a^{-}_{j,k}\,.$$

Can we use $\displaystyle\sum_{j\geqslant 0}\sum_{k\geqslant 0} a_{j,k}$ or $\displaystyle\sum_{k\geqslant 0}\sum_{j\geqslant 0} a_{j,k}$ instead?

TAL TECH

Multiple infinite sums

Definition: Double infinite sums

For every $j, k \geqslant 0$ let $a_{j,k} \geqslant 0$.

1. If $a_{j,k} \geqslant 0$ for every j and k, then:

$$\sum_{j,k} a_{j,k} = \sup_{K \subseteq \mathbb{N}\times\mathbb{N}, |K|<\infty} \sum_{K} a_{j,k} = \lim_{n\to\infty} \sum_{0 \leqslant j,k \leqslant n} a_{j,k} \,.$$

   (Recall that $\sum_{0\leqslant j,k\leqslant n} a_{j,k} = \sum_{j,k} a_{j,k} [0 \leqslant j \leqslant n] [0 \leqslant k \leqslant n]$.)

2. If $\sum_{j,k} \left| a_{j,k} \right| < +\infty$, then:

$$\sum_{j,k} a_{j,k} = \sum_{j,k} a^{+}_{j,k} - \sum_{j,k} a^{-}_{j,k} \,.$$

Can we use $\displaystyle\sum_{j\geqslant 0} \sum_{k\geqslant 0} a_{j,k}$ or $\displaystyle\sum_{k\geqslant 0} \sum_{j\geqslant 0} a_{j,k}$ instead?

In general, no:

- One writing is the limit on j of a limit on k which is a function of j;
- The other writing is the limit on k of a limit on j which is a function of k.
- There are no guarantees that the double limits be equal!

Multiple sums: An example of noncommutativity

From Joel Feldman's notes[2]

Let $a_{j,k} = [j = k = 0] + [k = j+1] - [k = j-1]$:

	0	1	2	3	4	...
0	1	1	0	0	0	...
1	−1	0	1	0	0	...
2	0	−1	0	1	0	...
3	0	0	−1	0	1	...
⋮	⋮	⋮	⋮	⋮	⋮	

Then:

- for every $j \geqslant 0$, $\sum_{k \geqslant 0} a_{j,k} = 2 \cdot [j = 0]$;
- for every $k \geqslant 0$, $\sum_{j \geqslant 0} a_{j,k} = 0$; and
- for every $n \geqslant 0$, $\sum_{0 \leqslant j,k \leqslant n} a_{j,k} = 1$.

Hence:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = 2 \; ; \quad \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = 0 \; ; \quad \text{and} \quad \lim_{n \to \infty} \sum_{0 \leqslant j,k \leqslant n} a_{j,k} = 1 \, .$$

[2] http://www.math.ubc.ca/~feldman/m321/twosum.pdf retrieved 21.02.2019.

Multiple infinite sums: Swapping indices

Theorem

For $j, k \geqslant 0$ let $a_{j,k}$ be real numbers.

Tonelli If $a_{j,k} \geqslant 0$ for every j and k, then:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = \sum_{j,k} a_{j,k} \,,$$

regardless of the quantities above being finite or infinite.

Fubini If $\sum_{j,k} \left| a_{j,k} \right| < +\infty$, then:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = \sum_{j,k} a_{j,k} \,.$$

Fubini's theorem is proved in the textbook.

Multiple infinite sums: Swapping indices

Theorem

For $j, k \geqslant 0$ let $a_{j,k}$ be real numbers.

Tonelli If $a_{j,k} \geqslant 0$ for every j and k, then:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = \sum_{j,k} a_{j,k} \,,$$

regardless of the quantities above being finite or infinite.

Fubini If $\sum_{j,k} \left| a_{j,k} \right| < +\infty$, then:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = \sum_{j,k} a_{j,k} \,.$$

Fubini's theorem is proved in the textbook. Again:

If we want to manipulate infinite sums like finite ones,
we must require absolute convergence.

Next subsection

Cesàro summation

Given a series $\sum_k a_k$, consider the sequence $S_n = \sum_{k=0}^{n} a_k$ of the partial sums.

- Put $u(x) = \sum_{k=0}^{x-1} S_k$ and $v(x) = x$. Then $\Delta u(x) = S_x$ and $\Delta v(x) = 1$.
- Suppose $\sum_k a_k$ converges. Put $L = \sum_{k \geqslant 0} a_k = \lim_{n\to\infty} \frac{S_n}{1}$.
- We then have by the Stolz-Cesàro lemma: $\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_k}{n} = L$.

Given a (not necessarily convergent) series $\sum_k a_k$, the quantity:

$$C\sum_k a_k = \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_k}{n},$$

if it exists, is called the Cesàro sum of the series $\sum_k a_k$.

Cesàro sum without convergence

The series $\sum_{k \geqslant 0} (-1)^k$ does not converge. However:

$$S_n = \sum_{k=0}^{n} (-1)^k = [n \text{ is even}]$$

so for every $n \geqslant 1$:

$$\begin{aligned}
\sum_{k=0}^{n-1} S_k &= \sum_{k=0}^{n-1} [k \text{ is even}] \\
&= \frac{n}{2} [n-1 \text{ is odd}] + \left(\frac{n-1}{2} + 1 \right) [n-1 \text{ is even}] \\
&= \frac{n}{2} [n \text{ is even}] + \frac{n+1}{2} [n \text{ is odd}] = \frac{n + [n \text{ is odd}]}{2}
\end{aligned}$$

The Cesàro sum of $a_k = (-1)^k$ is thus:

$$C\sum_k (-1)^k = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n + [n \text{ is odd}]}{2} = \frac{1}{2}$$

n	0	1	2	3	4	5	6	7	8	9
a_n	1	-1	1	-1	1	-1	1	-1	1	-1
S_n	1	0	1	0	1	0	1	0	1	0
$\sum_{k=0}^{n-1} S_k$	0	1	1	2	2	3	3	4	4	5

Convergence of sequences of functions

Definition

Let $E \subseteq \mathbb{C}$ and, for every $n \in \mathbb{N}$, let $f_n : E \to \mathbb{C}$. Let also $f : E \to \mathbb{C}$.

- We say that f_n converges pointwise to f if for every $\varepsilon > 0$ and for every $x \in E$ there exists $n_\varepsilon \in \mathbb{N}$ such that:

$$|f_n(x) - f(x)| < \varepsilon \text{ for every } n > n_\varepsilon \,.$$

- We say that f_n converges uniformly to f if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for every $x \in E$:

$$|f_n(x) - f(x)| < \varepsilon \text{ for every } n > n_\varepsilon \,.$$

Difference:

- With pointwise convergence, n_ε depends on both ε and x.
- With uniform convergence, n_ε depends on ε, but not on x: The same n_ε works for every x.

Useful properties of uniform convergence

Uniform convergence has many desirable properties:

1. If f_n converges uniformly, then the order of limits can be swapped:

$$\lim_{n\to\infty}\lim_{x\to x_0} f_n(x) = \lim_{x\to x_0}\lim_{n\to\infty} f_n(x)$$

2. If f_n convergs uniformly to f and every f_n is continuous, then f is continuous.
   Not true for pointwise convergence: $(1-n|x|)\,[|x|\leqslant 1/n]$ converges to $[x=0]$.

3. If
   1. the functions f_n are all differentiable[3],
   2. f_n converges pointwise to f, and
   3. f_n' converges uniformly,

   then f is differentiable and $f'(x) = \lim_{n\to\infty} f_n'(x)$ for every $x \in E$.

[3] Complex derivative is defined similarly to real derivative: we will see more in Chapter 5.

A simple criterion for uniform convergence

Weierstrass M-test

Let $f_k : E \to \mathbb{C}$, $k \in \mathbb{N}$, be a sequence of functions.
Suppose that a sequence M_k of real numbers exists such that:

1. $|f_k(x)| \leqslant M_k$ for every $n \in \mathbb{N}$ and $x \in E$; and
2. $\sum_{k \geqslant 0} M_k = M \in \mathbb{R}$.

Then the series of functions:

$$S(x) = \sum_{k \geqslant 0} f_k(x) = \lim_{n \to \infty} \sum_{0 \leqslant k \leqslant n} f_k(x)$$

converges uniformly and absolutely in E.

If the sequence $f_k(x)$ satisfies the Weierstrass M-test, we also say that the series of functions $S(x)$ converges totally in E.
Total convergence plays an important role in the theory of generating functions.

Abel summation

Abel's summation theorem

Let the series $S(x) = \sum_{k \geqslant 0} a_k x^k$ converge for every $0 \leqslant x < 1$. If:

$$S(1) = \sum_{k \geqslant 0} a_k$$

converges, then:

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k x^k = S(x) \text{ uniformly in } [0..1]$$

In particular:

$$L = \lim_{x \to 1^-} S(x) = S(1)$$

We can then define the Abel sum of a series as:

$$A \sum_k a_k = \lim_{x \to 1^-} \sum_k a_k x^k \text{ if the right-hand side exists}$$

Abel sum without convergence

The series $a_k = (-1)^k$ does not converge. However, for $0 \leqslant x < 1$ the series:

$$S(x) = \sum_{k \geqslant 0} (-1)^k x^k = \sum_{k \geqslant 0} (-x)^k$$

converges to $\dfrac{1}{1+x}$, and:

$$\lim_{x \to 1^-} S(x) = \lim_{x \to 1^-} \frac{1}{1+x} = \frac{1}{2}$$

The Abel sum of $a_k = (-1)^k$ is thus:

$$A\sum_k (-1)^k = \frac{1}{2}$$

Tauber's theorems

Tauber's first theorem (partial converse of Abel's summation theorem)

Let $S(x) = \sum_{k \geqslant 0} a_k x^k$ be such that $L = \lim_{x \to 1^-} S(x)$ exists. If:

$$\lim_{k \to \infty} k a_k = 0$$

then $S(1) = \sum_{k \geqslant 0} a_k = L$.

The condition here is that a_k is infinitesimal of order greater than first.

Tauber's second theorem (full converse of Abel's summation theorem)

Let $S(x) = \sum_{k \geqslant 0} a_k x^k$ be such that $L = \lim_{x \to 1^-} S(x)$ exists. Then $\sum_{k \geqslant 0} a_k$ converges if and only if:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k a_k = 0$$

In this case, $\sum_{k \geqslant 0} a_k = L$.

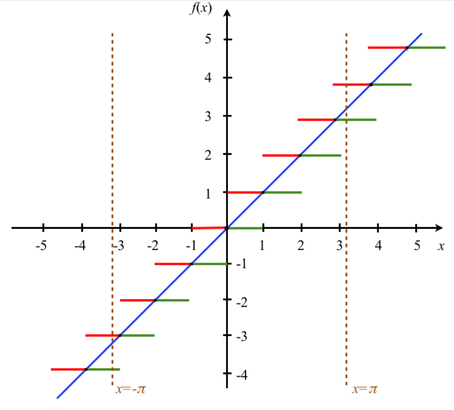
The condition here is that ka_k converges to zero in arithmetic mean.
This is more general than the previous one because of the Stolz-Cesàro lemma.

Next section

Floors and Ceilings

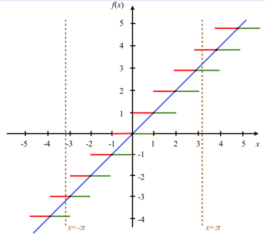
Definition

- The floor $\lfloor x \rfloor$ is the greatest integer not larger than x;
- The ceiling $\lceil x \rceil$ is the smallest integer not smaller than x.



$$\lfloor \pi \rfloor = 3 \qquad \lfloor -\pi \rfloor = -4$$
$$\lceil \pi \rceil = 4 \qquad \lceil -\pi \rceil = -3$$

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$



For every $x \in \mathbb{R}$:

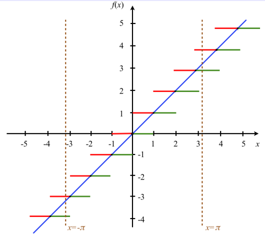
① $\lfloor x \rfloor = x = \lceil x \rceil$ iff $x \in \mathbb{Z}$

② $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$

③ $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$

④ $\lceil x \rceil - \lfloor x \rfloor = [x \notin \mathbb{Z}]$

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$



For every $x \in \mathbb{R}$:

① $\lfloor x \rfloor = x = \lceil x \rceil$ iff $x \in \mathbb{Z}$

② $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$

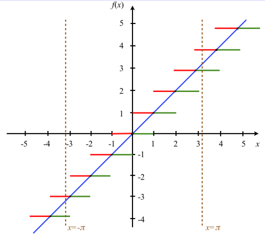
③ $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$

④ $\lceil x \rceil - \lfloor x \rfloor = [x \notin \mathbb{Z}]$

Why ②?

We could also call ③ the "flip the number, flip the room" rule.

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$



For every $x \in \mathbb{R}$:

① $\lfloor x \rfloor = x = \lceil x \rceil$ iff $x \in \mathbb{Z}$

② $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$

③ $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$

④ $\lceil x \rceil - \lfloor x \rfloor = [x \notin \mathbb{Z}]$

Why ②? Because the intervals $(x-1..x]$ and $[x..x+1)$ contain exactly one integer each.

We could also call ③ the "flip the number, flip the room" rule.

TAL TECH

Warmup: Representing numbers

Problem

Let $n = 2^m + \ell$. What are closed formulas for m and ℓ?

Warmup: Representing numbers

Problem

Let $n = 2^m + \ell$. What are closed formulas for m and ℓ?

Solution

First, $2^m \leqslant n < 2^{m+1}$.

- As lg, the logarithm in base 2, is an increasing function, $m \leqslant \lg n < m+1$.
- Then:

$$m = \lfloor \lg n \rfloor \ .$$

Next, $\ell = n - 2^m$. Then:

$$\ell = n - 2^{\lfloor \lg n \rfloor} \ .$$

From now on, the base-2 logarithm will be denoted by lg.

Warmup: the generalized Dirichlet box principle

Statement of the principle

Let m and n be positive integers. If n items are stored into m boxes, then:

- at least one box will contain at least $\lceil n/m \rceil$ objects, and
- at least one box will contain at most $\lfloor n/m \rfloor$ objects.

Warmup: the generalized Dirichlet box principle

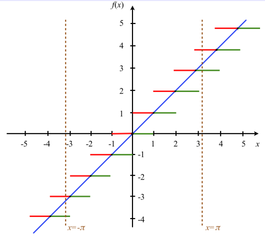
Statement of the principle

Let m and n be positive integers. If n items are stored into m boxes, then:

- at least one box will contain at least $\lceil n/m \rceil$ objects, and
- at least one box will contain at most $\lfloor n/m \rfloor$ objects.

- By contradiction, assume each of the m boxes contains fewer than $\lceil n/m \rceil$ objects.
- Then
$$n \leqslant m \cdot \left(\left\lceil \frac{n}{m} \right\rceil - 1 \right) \text{ or equivalently}, \ \frac{n}{m} + 1 \leqslant \left\lceil \frac{n}{m} \right\rceil :$$
which is impossible.
- Similarly, if each of the m boxes contained more than $\lfloor n/m \rfloor$ objects, we would have
$$n \geqslant m \cdot \left(\left\lfloor \frac{n}{m} \right\rfloor + 1 \right) \text{ or equivalently}, \ \frac{n}{m} - 1 \geqslant \left\lfloor \frac{n}{m} \right\rfloor :$$
which is also impossible.

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$ (cont.)



For every $x \in \mathbb{R}$ and $n \in \mathbb{Z}$:

5. $\lfloor x \rfloor = n$ iff $n \leqslant x < n+1$
6. $\lfloor x \rfloor = n$ iff $x-1 < n \leqslant x$
5. $\lceil x \rceil = n$ iff $n-1 < x \leqslant n$
8. $\lceil x \rceil = n$ iff $x \leqslant n < x+1$
9. $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ but, in general, $\lfloor nx \rfloor \neq n \lfloor x \rfloor$.
10. $\lceil x+n \rceil = \lceil x \rceil + n$ but, in general, $\lceil nx \rceil \neq n \lceil x \rceil$.
11. $x < n$ iff $\lfloor x \rfloor < n$
12. $n < x$ iff $n < \lceil x \rceil$
13. $x \leqslant n$ iff $\lceil x \rceil \leqslant n$
14. $n \leqslant x$ iff $n \leqslant \lfloor x \rfloor$

Generalization of property ⑨

Theorem

$$\lfloor x+y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor & \text{if } 0 \leqslant \{x\}+\{y\} < 1\,, \\ \lfloor x \rfloor + \lfloor y \rfloor + 1 & \text{if } 1 \leqslant \{x\}+\{y\} < 2\,. \end{cases}$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.

Proof. Let $x = \lfloor x \rfloor + \{x\}$ and $y = \lfloor y \rfloor + \{y\}$. Then:

$$\begin{aligned} \lfloor x+y \rfloor &= \lfloor \lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} \rfloor \\ &= \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor \end{aligned}$$

and clearly

$$\lfloor \{x\}+\{y\} \rfloor = \begin{cases} 0 & \text{if } 0 \leqslant \{x\}+\{y\} < 1\,, \\ 1 & \text{if } 1 \leqslant \{x\}+\{y\} < 2\,. \end{cases}$$

Q.E.D.

Warmup: When is $\lfloor nx \rfloor = n \lfloor x \rfloor$?

The problem

Give a necessary and sufficient condition on n and x so that

$$\lfloor nx \rfloor = n \lfloor x \rfloor$$

where n is a positive integer.

Warmup: When is $\lfloor nx \rfloor = n \lfloor x \rfloor$?

The problem

Give a necessary and sufficient condition on n and x so that

$$\lfloor nx \rfloor = n \lfloor x \rfloor$$

where n is a positive integer.

The solution

Write $x = \lfloor x \rfloor + \{x\}$. Then

$$\lfloor nx \rfloor = \lfloor n \lfloor x \rfloor + n\{x\} \rfloor = n \lfloor x \rfloor + \lfloor n\{x\} \rfloor$$

As $\{x\}$ is nonnegative, so is $\lfloor n\{x\} \rfloor$. Then

$$\lfloor nx \rfloor = n \lfloor x \rfloor \text{ if and only if } \{x\} < 1/n$$

Next section

Floor/Ceiling Applications

Theorem

The binary representation of a natural number $n > 0$ has $m = \lfloor \lg n \rfloor + 1$ bits.

Proof.

$$n = \underbrace{2^{m-1} + a_{m-2}2^{m-2} + \cdots + a_1 2 + a_0}_{m \text{ bits}}$$

Thus, $2^{m-1} \leqslant n < 2^m$, which gives $m - 1 \leqslant \lg n < m$. The last formula is valid if and only if $\lfloor \lg n \rfloor = m - 1$. Q.E.D.

As $\lceil x \rceil = \lfloor x \rfloor + [x \notin \mathbb{Z}]$, we cannot, in general replace $\lfloor \lg n \rfloor + 1$ with $\lceil \lg n \rceil$.

Example: $n = 35 = 100011_2$

$$m = \lfloor \lg 35 \rfloor + 1 = 5 + 1 = 6$$

Floor/Ceiling Applications

Theorem

The binary representation of a natural number $n > 0$ has $m = \lfloor \lg n \rfloor + 1$ bits.

Proof.

$$n = \underbrace{2^{m-1} + a_{m-2} 2^{m-2} + \cdots + a_1 2 + a_0}_{m \text{ bits}}$$

Thus, $2^{m-1} \leqslant n < 2^m$, which gives $m - 1 \leqslant \lg n < m$. The last formula is valid if and only if $\lfloor \lg n \rfloor = m - 1$. Q.E.D.

As $\lceil x \rceil = \lfloor x \rfloor + [x \notin \mathbb{Z}]$, we cannot, in general replace $\lfloor \lg n \rfloor + 1$ with $\lceil \lg n \rceil$.

Example: $n = 35 = 100011_2$

$$m = \lfloor \lg 35 \rfloor + 1 = 5 + 1 = 6$$

Floor/Ceiling Applications (2)

Theorem

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous, strictly increasing function with the property that, if $f(x) \in \mathbb{Z}$, then $x \in \mathbb{Z}$. Then:

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor \quad \text{and} \quad \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$$

whenever $f(x)$, $f(\lfloor x \rfloor)$, and $f(\lceil x \rceil)$ are all defined.

Proof. (for the ceiling function)

- If $x \in \mathbb{Z}$, then $x = \lceil x \rceil$, and there is nothing to prove.
- If $x \notin \mathbb{Z}$, then $x < \lceil x \rceil$, so $f(x) < f(\lceil x \rceil) \leqslant \lceil f(\lceil x \rceil) \rceil$ as f is strictly increasing.
- Also, by the special property, $f(x) \notin \mathbb{Z}$, so:

$$f(x) < \lceil f(x) \rceil \leqslant \lceil f(\lceil x \rceil) \rceil$$

- By contradiction, assume $\lceil f(x) \rceil < \lceil f(\lceil x \rceil) \rceil$.
- As f is continuous, by the intermediate value theorem there exists y such that $x \leqslant y < \lceil x \rceil$ and $f(y) = \lceil f(x) \rceil$.
- Such y is an integer, because of f's special property, so actually $x < y < \lceil x \rceil$.
- But there are no integers strictly between x and $\lceil x \rceil$: contradiction.

Q.E.D.

Floor/Ceiling Applications (2a)

Example

- $\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor$
- $\left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil$
- $\left\lceil \frac{\left\lceil \frac{\lceil x \rceil / 10}{10} \right\rceil}{10} \right\rceil = \left\lceil \frac{\lceil x \rceil / 10}{100} \right\rceil = \lceil x/1000 \rceil$
- $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$

In contrast:

$$\left\lceil \sqrt{\lfloor x \rfloor} \right\rceil \neq \lceil \sqrt{x} \rceil$$

For example, $\left\lceil \sqrt{\lfloor 1/4 \rfloor} \right\rceil = 0$ but $\left\lceil \sqrt{1/4} \right\rceil = 1$.

Floor/Ceiling Applications (3) : Intervals

For real numbers $\alpha \leqslant \beta$

Range	Nr. of integer values of t
$\alpha \leqslant t \leqslant \beta$	$\lfloor\beta\rfloor - \lceil\alpha\rceil + 1$
$\alpha \leqslant t < \beta$	$\lceil\beta\rceil - \lceil\alpha\rceil$
$\alpha < t \leqslant \beta$	$\lfloor\beta\rfloor - \lfloor\alpha\rfloor$
$\alpha < t < \beta$	$(\lceil\beta\rceil - \lfloor\alpha\rfloor - 1) \cdot [\alpha < \beta]$

Floor/Ceiling Applications (3) : Intervals

For real numbers $\alpha \leqslant \beta$

Range	Nr. of integer values of t
$\alpha \leqslant t \leqslant \beta$	$\lfloor\beta\rfloor - \lceil\alpha\rceil + 1$
$\alpha \leqslant t < \beta$	$\lceil\beta\rceil - \lceil\alpha\rceil$
$\alpha < t \leqslant \beta$	$\lfloor\beta\rfloor - \lfloor\alpha\rfloor$
$\alpha < t < \beta$	$(\lceil\beta\rceil - \lfloor\alpha\rfloor - 1) \cdot [\alpha < \beta]$

This is because, if $t \in \mathbb{Z}$, then:

$$\begin{array}{lll} \alpha \leqslant t & \text{if and only if} & \lceil\alpha\rceil \leqslant t \\ \alpha < t & \text{if and only if} & \lfloor\alpha\rfloor < t \;\; \text{if and only if } \lfloor\alpha\rfloor + 1 \leqslant t \\ t \leqslant \beta & \text{if and only if} & t \leqslant \lfloor\beta\rfloor \\ t < \beta & \text{if and only if} & t < \lceil\beta\rceil \;\; \text{if and only if } t \leqslant \lceil\beta\rceil - 1 \end{array}$$

and the slice $[m:n] = [m..n] \cap \mathbb{Z}$, $m \leqslant n$, has $n - m + 1$ elements.
(Note that, if $\alpha = \beta$ are both integers, then $\lceil\beta\rceil - \lfloor\alpha\rfloor - 1 = -1$.)