ITT9132 Concrete Mathematics

Lecture 5: 23 February 2021

Chapter Two

Finite and infinite calculus

Infinite sums

Cesàro and Abel summation

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Derivative and Difference Operators

Infinite calculus: derivative

Euler's notation

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Lagrange's notation

$$f'(x) = \mathrm{D}f(x)$$

Leibniz's notation If $y = f(x)$, then

$$\frac{dy}{dx} = \frac{df}{dx}(x) = \frac{df(x)}{dx} = \mathrm{D}f(x)$$

Newton's notation

$$\dot{y} = f'(x)$$

Finite calculus: difference

$$\Delta f(x) = f(x+1) - f(x)$$

In general, if $h \in \mathbb{R}$ (or $h \in \mathbb{C}$), then

Forward difference

$$\Delta_h[f](x) = f(x+h) - f(x)$$

Backward difference

$$\nabla_h[f](x) = f(x) - f(x-h)$$

Central difference

$$\delta_h[f](x) = f(x + \tfrac{1}{2}h) - f(x - \tfrac{1}{2}h)$$

$$\mathrm{D}f(x) = \lim_{h \to 0} \frac{\Delta_h[f](x)}{h}$$

Derivative of Power function

Example: $f(x) = x^3$

In this case,

$$\begin{aligned}\Delta_h[f](x) &= f(x+h) - f(x) \\ &= (x+h)^3 - x^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - x^3 \\ &= h \cdot (3x^2 + 3xh + h^2)\end{aligned}$$

Hence,

$$\mathrm{D}f(x) = \lim_{h\to 0} \frac{h \cdot (3x^2 + 3xh + h^2)}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

In general, for $m \geqslant 1$ integer:

$$\mathrm{D}(x^m) = mx^{m-1}$$

(Forward) Difference of Power Function

Example: $f(x) = x^3$

In this case,

$$\Delta f(x) = \Delta_1 [f](x) = 3x^2 + 3x + 1$$

In general, for $m \geqslant 1$ integer:

$$\Delta(x^m) = \sum_{k=1}^{m} \binom{m}{k} x^{m-k}$$

because of Newton's binomial theorem.

Falling and Rising Factorials

Definition

The falling factorial (power) is defined for $m \geqslant 0$ by:

$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1)$$

The rising factorial (power) is defined for $m \geqslant 0$ by:

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$$

Properties

$$x^{\overline{m}} = (-1)^m(-x)^{\underline{m}}$$

$$n! = n^{\underline{n}} = 1^{\overline{n}}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$$

$$x^{\underline{m}} = \frac{x^{\underline{m+1}}}{x-m} \quad \text{if } x \neq m$$

$$x^{\underline{-m}} = \frac{1}{(x+1)^{\overline{m}}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}$$

Falling factorials with negative exponents

We want to define $x^{\underline{m}}$ with $m \leqslant 0$ integer so that the expansion rule:

$$x^{\underline{m+n}} = x^{\underline{m}} \cdot (x-m)^{\underline{n}}$$

is satisfied for every $m, n \in \mathbb{Z}$ and $x \in \mathbb{C}$.

- First of all, it must be $x^{\underline{0+n}} = x^{\underline{0}}(x-0)^{\underline{n}}$ for every $x \in \mathbb{C}$ and $n \in \mathbb{N}$.
  Then it must be:
  $$x^{\underline{0}} = 1$$
  This is also consistent with defining an empty product as equal to 1.
- Next it must be $x^{\underline{0}} = x^{\underline{-m}} \cdot (x+m)^{\underline{m}}$ for every $x \in \mathbb{C}$ and $m \in \mathbb{N}$ such that the right-hand side is nonzero.
  Then it must be:

$$x^{\underline{-m}} = \frac{1}{(x+m)^{\underline{m}}} = \frac{1}{(x+1)^{\overline{m}}} \text{ for every } x \notin \{1, \dots, m\}$$

Dually,

$$x^{\overline{-m}} = \frac{1}{(x-1)^{\underline{m}}} \text{ for every } x \notin \{-1, \dots, -m\}$$

Difference of falling factorial with positive exponent

$$\begin{aligned}\Delta\left(x^{\underline{m}}\right) &= (x+1)^{\underline{m}} - x^{\underline{m}} \\ &= (x+1)\cdot(x\cdots(x-m+2)) - (x\cdots(x-m+2))\cdot(x-m+1) \\ &= (x+1-(x-m+1))\cdot(x\cdots(x-m+2)) \\ &= m\cdot x^{\underline{m-1}}\end{aligned}$$

Hence:

$$\Delta(x^{\underline{m}}) = m x^{\underline{m-1}} \ \forall m \geqslant 1$$

Differences of falling factorials with negative exponents

First, a simple example:

$$\begin{aligned}
\Delta x^{\underline{-2}} &= (x+1)^{\underline{-2}} - x^{\underline{-2}} \\
&= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)} \\
&= \frac{(x+1)-(x+3)}{(x+1)(x+2)(x+3)} \\
&= \frac{-2}{(x+1)(x+2)(x+3)} \\
&= -2 \cdot x^{\underline{-3}}
\end{aligned}$$

Differences of falling factorials with negative exponents

Now, for the general rule: let $m \in \mathbb{N}$. Then:

$$\begin{aligned}
\Delta x^{\underline{-m}} &= (x+1)^{\underline{-m}} - x^{\underline{-m}} \\
&= \frac{1}{(x+2)\cdots(x+m)(x+m+1)} - \frac{1}{(x+1)(x+2)\cdots(x+m)} \\
&= \frac{(x+1)-(x+m+1)}{(x+1)(x+2)\cdots(x+m)(x+m+1)} \\
&= \frac{-m}{(x+1)(x+2)\cdots(x+m)(x+m+1)} \\
&= -mx^{\underline{-(m+1)}} \\
&= -mx^{\underline{-m-1}}
\end{aligned}$$

Next subsection

Indefinite Integrals and Sums

The Fundamental Theorem of Calculus

$$\mathrm{D}f(x) = g(x) \quad \text{iff} \quad \int g(x)\mathrm{d}x = f(x) + \mathrm{C}$$

Definition

The indefinite sum of the function $g(x)$ is the class of functions f such that $\Delta f(x) = g(x)$:

$$\Delta f(x) = g(x) \quad \text{iff} \quad \sum g(x)\delta x = f(x) + C(x)$$

where $C(x)$ is a function such that $C(x+1) = C(x)$ for any integer value of x.

Definite Integrals and Sums

If $g(x) = \mathrm{D}f(x)$, then:

$$\int_a^b g(x)\mathrm{d}x = f(x)\Big|_a^b = f(b) - f(a)$$

Similarly:

If $g(x) = \Delta f(x)$, then:

$$\sum_a^b g(x)\delta x = f(x)\Big|_a^b = f(b) - f(a)$$

Definite sums

Some observations

- $\sum_a^a g(x)\delta x = f(a) - f(a) = 0$
- $\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a)$
- $\sum_a^{b+1} g(x)\delta x - \sum_a^b g(x)\delta x = f(b+1) - f(b) = g(b)$

Hence, if $g(x) = \Delta f(x)$, then:

$$
\begin{aligned}
\sum_a^b g(x)\delta x &= \sum_{k=a}^{b-1} g(k) = \sum_{a \leqslant k < b} g(k) \\
&= \sum_{a \leqslant k < b} (f(k+1) - f(k)) \\
&= (f(a+1) - f(a)) + (f(a+2) - f(a+1)) + \ldots \\
&\quad + (f(b-1) - f(b-2)) + (f(b) - f(b-1)) \\
&= f(b) - f(a)
\end{aligned}
$$

Integrals and Sums of Powers

If $m \neq -1$, then:

$$\int_0^n x^m \mathrm{d}x = \frac{x^{m+1}}{m+1}\Big|_0^n = \frac{n^{m+1}}{m+1}$$

Analogous finite case:

If $m \neq -1$, then:

$$\sum_0^n x^{\underline{m}} \delta x = \sum_{0 \leqslant k < n} k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\Big|_0^n = \frac{n^{\underline{m+1}}}{m+1}$$

Sums of Powers: applications

Case $m=1$

$$\sum_{0\leqslant k<n} k = \frac{n^{\underline{2}}}{2} = \frac{n(n-1)}{2}$$

Case $m=2$ Due to $k^2 = k^{\underline{2}} + k^{\underline{1}}$ we get:

$$\begin{aligned}
\sum_{0\leqslant k<n} k^2 &= \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} \\
&= \frac{1}{3}n(n-1)(n-2) + \frac{1}{2}n(n-1) \\
&= \frac{1}{6}n(2(n-1)(n-2)+3(n-1)) \\
&= \frac{1}{6}n(n-1)(2n-4+3) \\
&= \frac{1}{6}n(n-1)(2n-1)
\end{aligned}$$

Taking $n+1$ instead of n gives:

$$\square_n = \frac{(n+1)n(2n+1)}{6}$$

Sums of Powers (case $m = -1$)

As a first step, we observe that:

$$\begin{aligned}\Delta H_x &= H_{x+\mathbf{1}} - H_x \\ &= \left(1 + \frac{1}{2} + \ldots + \frac{1}{x} + \frac{1}{x+1}\right) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{x}\right) \\ &= \frac{1}{x+1} = x^{\underline{-\mathbf{1}}}\end{aligned}$$

We conclude:

$$\sum_a^b x^{\underline{-\mathbf{1}}} \delta x = H_x \Big|_a^b$$

Sums of Discrete Exponential Functions

- We have:

$$\mathrm{D}e^x = e^x$$

The finite analogue should have $\Delta f(x) = f(x)$. This means:

$$f(x+1) - f(x) = f(x), \text{ that is, } f(x+1) = 2f(x), \text{ only possible if } f(x) = 2^x$$

- For general base $c > 0$, the difference of c^x is:

$$\Delta(c^x) = c^{x+1} - c^x = (c-1)c^x$$

and the "anti-difference" for $c \neq 1$ is $\dfrac{c^x}{c-1}$.

As an application, we compute the sum of the geometric progression:

$$\sum_{a \leqslant k < b} c^k = \sum_a^b c^x \delta x = \frac{c^x}{c-1}\bigg|_a^b = \frac{c^b - c^a}{c-1} = c^a \cdot \frac{c^{b-a} - 1}{c-1}\,.$$

Differential equations and difference equations

Differential equation	Solution	Difference equation	Solution
$Df_n(x) = nf_{n-1}(x)$ $f_n(0) = [n = 0]\ ,\ n \geqslant 0$	$f_n(x) = x^n$	$\Delta u_m(x) = mu_{m-1}(x)$ $u_m(0) = [m = 0]\ ,\ m \geqslant 0$	$u_m(x) = x^{\underline{m}}$
$Df_n(x) = nf_{n-1}(x)$ $f_n(1) = 1\ ,\ n < 0$	$f_n(x) = x^n$	$\Delta u_m(x) = mu_{m-1}(x)$ $u_m(0) = \dfrac{1}{\lvert m\rvert !}\ ,\ m < 0$	$u_m(x) = x^{\underline{m}}$
$Df(x) = \dfrac{1}{x} \cdot [x > 0]$ $f(1) = 1$	$f(x) = \ln x$	$\Delta u(x) = \dfrac{1}{x+1} \cdot [x \geqslant 1]$ $u(1) = 1$	$u(x) = H_x$
$Df(x) = f(x)$ $f(0) = 1$	$f(x) = e^x$	$\Delta u(x) = u(x)$ $u(0) = 1$	$u(x) = 2^x$
$Df(x) = b \cdot f(x)$ $f(0) = 1$	$f(x) = a^x$ where $b = \ln a$	$\Delta u(x) = b \cdot u(x)$ $u(0) = 1$	$u(x) = c^x$ where $b = c - 1$

l'Hôpital's rule and Stolz-Cesàro lemma

l'Hôpital's rule: Hypotheses

1. $f(x)$ and $g(x)$ are both vanishing or both infinite at x_0.
2. $g'(x)$ is always positive in some neighborhood of x_0.

l'Hôpital's rule: Thesis

- If $\lim_{x \to x_0} \dfrac{f'(x)}{g'(x)} = L \in \mathbb{R}$,
- then $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = L$.

Stolz-Cesàro lemma: Hypotheses

1. $u(n)$ and $v(n)$ are defined for every value $n \in \mathbb{N}$.
2. $v(n)$ is positive, strictly increasing, and divergent.

Stolz-Cesàro lemma: Thesis

- If $\lim_{n \to \infty} \dfrac{\Delta u(n)}{\Delta v(n)} = L \in \mathbb{R}$,
- then $\lim_{n \to \infty} \dfrac{u(n)}{v(n)} = L$.

Proof of Stolz-Cesàro lemma in the case of real limit

Suppose $\lim_{n\to\infty} \dfrac{\Delta u(n)}{\Delta v(n)} = L \in \mathbb{R}$. Fix $\varepsilon > 0$.

- As $\langle v(n) \rangle$ is strictly increasing, for n large enough we have:

$$(v(n+1)-v(n))\left(L-\frac{\varepsilon}{2}\right) < u(n+1)-u(n) < (v(n+1)-v(n))\left(L+\frac{\varepsilon}{2}\right)$$

- Summing p consecutive terms, we find:

$$(v(n+p)-v(n))\left(L-\frac{\varepsilon}{2}\right) < u(n+p)-u(n) < (v(n+p)-v(n))\left(L+\frac{\varepsilon}{2}\right)$$

- As $\langle v(n) \rangle$ is positive, we can divide by $v(n+p)$ and obtain:

$$\left(1-\frac{v(n)}{v(n+p)}\right)\left(L-\frac{\varepsilon}{2}\right) < \frac{u(n+p)}{v(n+p)} - \frac{u(n)}{v(n+p)} < \left(1-\frac{v(n)}{v(n+p)}\right)\left(L+\frac{\varepsilon}{2}\right)$$

- As $\lim_{n\to\infty} v(n) = +\infty$, for every p large enough we have:

$$L-\varepsilon < \frac{u(n+p)}{v(n+p)} < L+\varepsilon$$

As $\varepsilon > 0$ is arbitrary, the thesis follows.

A useful corollary

Arithmetic mean theorem

If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k = L$ too.

That is:

If a sequence converges,
then the sequence of its arithmetic means converges to the same limit.

Proof:

- Let $u(x) = \sum_{k=0}^{x-1} a_k$ and $v(x) = x$.
- Then $\Delta u(x) = a_x$ and $\Delta v(x) = 1$.
- Apply the Stolz-Cesàro lemma.

Next subsection

Summation by Parts

Infinite analogue: integration by parts

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x$$

Difference of a product

$$\begin{aligned}
\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\
&= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \\
&= \Delta u(x)v(x+1) + u(x)\Delta v(x) \\
&= u(x)\Delta v(x) + Ev(x)\Delta u(x)
\end{aligned}$$

where E is the shift operator $Ef(x) = f(x+1)$. We then have the:

Rule for summation by parts

$$\sum u\Delta v\,\delta x = uv - \sum Ev\Delta u\,\delta x$$

Why the shift?

If we repeat our derivation with two continuous functions f and g of one real variable x, we find for any increment $h \neq 0$:

$$\begin{aligned} f(x+h)g(x+h) - f(x)g(x) &= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x) \\ &= f(x)(g(x+h) - g(x)) + g(x+h)(f(x+h) - f(x)) \end{aligned}$$

The incremental ratio is thus:

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = f(x) \cdot \frac{g(x+h) - g(x)}{h} + g(x+h) \cdot \frac{f(x+h) - f(x)}{h}$$

So there is a shift: but it is infinitesimal—and disappears by continuity of g.

TAL TECH

Example: $S_n = \sum_{k=0}^{n} kc^k$ with $c \neq 1$

- We want to write $S_n = \sum_0^{n+1} u(x)\,\Delta v(x)\,\delta x$ for suitable $u(x)$ and $v(x)$.
- Let $u(x) = x$ and $v(x) = c^x/(c-1)$.
- Then $\Delta u(x) = 1$, $\Delta v(x) = c^x$, and $Ev(x) = c^{x+1}/(c-1)$.
- Summing by parts:

$$
\begin{aligned}
\sum_0^{n+1} x\,c^x\,\delta x &= \left.\frac{xc^x}{c-1}\right|_0^{n+1} - \sum_0^{n+1} \frac{c^{x+1}}{c-1}\,\delta x \\
&= \frac{(n+1)c^{n+1}}{c-1} - \frac{c}{c-1}\sum_0^{n+1} c^x\,\delta x \\
&= \frac{(n+1)c^{n+1}}{c-1} - \frac{c}{(c-1)^2}\left(c^{n+1}-1\right) \\
&= \frac{nc^{n+2}-(n+1)c^{n+1}+c}{(c-1)^2}
\end{aligned}
$$

TALTECH

Example: $S_n = \sum_{k=0}^{n} kH_k$

- We want to write $S_n = \sum_0^{n+1} u(x)\,\Delta v(x)\,\delta x$ for suitable $u(x)$ and $v(x)$.
- Let $u(x) = H_x$ and $v(x) = x^{\underline{2}}/2$.
- Then $\Delta u(x) = x^{\underline{-1}}$, $\Delta v(x) = x$, and $Ev(x) = (x+1)^{\underline{2}}/2$.
- Summing by parts:

$$
\begin{aligned}
\sum_0^{n+1} x H_x \,\delta x &= \frac{x^{\underline{2}}}{2} H_x \Big|_0^{n+1} - \sum_0^{n+1} \frac{(x+1)^{\underline{2}}}{2} x^{\underline{-1}} \,\delta x \\
&= \frac{(n+1)n}{2} H_{n+1} - \frac{1}{2} \sum_0^{n+1} x^{\underline{-1}} (x-(-1))^{\underline{2}} \,\delta x \\
&= \frac{(n+1)n}{2} H_{n+1} - \frac{1}{2} \sum_0^{n+1} x^{\underline{1}} \,\delta x \\
&= \frac{(n+1)n}{2} H_{n+1} - \frac{(n+1)n}{4} \\
&= \frac{(n+1)n}{2} \left(H_{n+1} - \frac{1}{2} \right)
\end{aligned}
$$

Next section

How to sum infinite number sequences?

Setting $\sum_{k\in\mathbb{N}} a_k = \lim_{n\to\infty}\sum_{k=0}^{n} a_k$ seems meaningful ...

Example 1

Let

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots .$$

Then

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2 + S,$$

and

$$S = 2$$

How to sum infinite number sequences?

Setting $\sum_{k\in\mathbb{N}} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$ seems meaningful ...

Example 1

Let

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots .$$

Then

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2 + S,$$

and

$$S = 2$$

But can we manipulate such sums like we do with finite sums?

How to sum infinite number sequences?

Example 2

Let

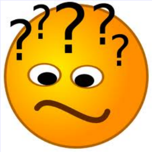
$$T = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots$$

Then

$$2T = 2 + 4 + 8 + 16 + 32 + 64 + 128 \dots = T - 1$$

and

$$T = -1$$



How to sum infinite number sequences?

Example 2

Let

$$T = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots$$

Then

$$2T = 2 + 4 + 8 + 16 + 32 + 64 + 128 \dots = T - 1$$

and

$$T = -1$$

Problem:

- The sum T is infinite . . .
- and we cannot subtract an infinite quantity from another infinite quantity.

How to sum infinite number sequences?

Example 3

Let

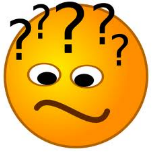
$$\sum_{k \geqslant 0} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

Different ways to sum

$$(1-1)+(1-1)+(1-1)+(1-1)+\ldots = 0+0+0+0+\ldots = 0$$

and

$$1-(1-1)-(1-1)-(1-1)-(1-1)-\ldots = 1-0-0-0-0-0-\ldots = 1$$



How to sum infinite number sequences?

Example 3

Let

$$\sum_{k \geqslant 0} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

Different ways to sum

$$(1-1)+(1-1)+(1-1)+(1-1)+\dots = 0+0+0+0+\dots = 0$$

and

$$1-(1-1)-(1-1)-(1-1)-(1-1)-\dots = 1-0-0-0-0-0-\dots = 1$$

Problem:

- The sequence of the partial sums does not converge ...
- and we cannot manipulate something that does not exist.

Defining Infinite Sums: Nonnegative Summands

Definition 1

If $a_k \geqslant 0$ for every $k \geqslant 0$, then:

$$\sum_{k \geqslant 0} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k = \sup_{K \subseteq \mathbb{N}, |K| < \infty} \sum_{k \in K} a_k$$

Note that:

- The definition as a limit is (sort of) a *Riemann integral*.
- The definition as a least upper bound is a *Lebesgue integral* with respect to the counting measure

$$\mu(X) = \text{if } |X| = n \in \mathbb{N} \text{ then } n \text{ else } +\infty$$

- The limit / least upper bound above can be finite or infinite, but are always equal.
  Exercise: Prove this fact.

TAL TECH

Defining Infinite Sums: Riemann Summation

Definition 2 (Riemann sum of a series)

A series $\sum_{k \geqslant 0} a_k$ with complex coefficients converges to a complex number S, called the sum of the series, if:

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k = S\,.$$

In this case, we write: $\sum_{k \geqslant 0} a_k = S$.
The values $S_n = \sum_{k=0}^{n} a_k$ are called the partial sums of the series.
The series $\sum_{k \geqslant 0} a_k$ converges absolutely if $\sum_{k \geqslant 0} |a_k|$ converges.

Note that the series $\sum_{k \geqslant 0} a_k = \sum_{k \geqslant 0} (b_k + ic_k)$ converges to $S = T + iU$ if and only if $\sum_{k \geqslant 0} b_k$ converges to T and $\sum_{k \geqslant 0} c_k$ converges to U.

A series that converges, but not absolutely

Let $a_k = \dfrac{(-1)^{k-1}}{k} [k > 0]$. Then $\sum_{k \geqslant 0} a_k = \ln 2$.

However, it is easy to prove by induction that $\sum_{k=0}^{2^n} |a_k| = H_{2^n} > \dfrac{n}{2}$ for every $n \geqslant 1$.

Infinite Sums: Associativity *within* a series

Associativity

A series $\sum_{k \geqslant 0} a_k$ has the associative property if for every two strictly increasing sequences

$$i_0 = 0 < i_1 < i_2 < \ldots < i_k < i_{k+1} < \ldots$$
$$j_0 = 0 < j_1 < j_2 < \ldots < j_k < j_{k+1} < \ldots$$

we have:

$$\sum_{k \geqslant 0} \left(\sum_{i=i_k}^{i_{k+1}-1} a_i \right) = \sum_{k \geqslant 0} \left(\sum_{j=j_k}^{j_{k+1}-1} a_j \right)$$

We have seen that the series $\sum_{k \geqslant 0} (-1)^k$ *does not* have the associative property.

Theorem

A series has the associative property if and only if it has a sum (finite or infinite).

Proof: Regrouping as in the definition means taking a subsequence of the sequence of partial sums, which can converge to any of the latter's limit points.

Defining Infinite Sums: Lebesgue Summation

Every real number can be written as $x = x^+ - x^-$, where:

$$x^+ = x \cdot [x > 0] = \max(x, 0) \text{ and } x^- = -x \cdot [x < 0] = \max(-x, 0)$$

Note that: $x^+ \geqslant 0$, $x^- \geqslant 0$, and $x^+ + x^- = |x|$.

Definition 3 (Lebesgue sum of a series)

Let $\{a_k\}_k$ be an absolutely convergent sequence of real numbers. Then:

$$\sum_k a_k = \sum_k a_k^+ - \sum_k a_k^-$$

The series $\sum_k a_k$:

- converges absolutely if $\sum_k a_k^+ < +\infty$ and $\sum_k a_k^- < +\infty$;
- diverges positively if $\sum_k a_k^+ = +\infty$ and $\sum_k a_k^- < +\infty$;
- diverges negatively if $\sum_k a_k^+ < +\infty$ and $\sum_k a_k^- = +\infty$.

If both $\sum_k a_k^+ = +\infty$ and $\sum_k a_k^- = +\infty$ then "Bad Stuff happens".

Infinite Sums: Bad Stuff

Riemann series theorem

Let $\sum_k a_k$ be a series with real coefficients which converges, but not absolutely. For every real number L there exists a permutation p of $\mathbb{N}$ such that:

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{p(k)} = L$$

Example: The harmonic series

If we rearrange the terms of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ as follows:

$$\begin{aligned} 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots &= \ldots + \frac{1}{2k-1} - \frac{1}{2(2k-1)} - \frac{1}{4k} + \ldots \\ &= \ldots + \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k}\right) + \ldots \end{aligned}$$

we obtain:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2 \quad \text{but} \quad 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots = \ln\sqrt{2}$$

Infinite Sums: Commutativity

Commutativity

A series $\sum_{k \geqslant 0} a_k$ has the commutative property if for every permutation p of $\mathbb{N}$,

$$\sum_{k \geqslant 0} a_{p(k)} = \sum_{k} a_k$$

The Riemann series theorem says that any series which is convergent, but not absolutely convergent, does not have the commutative property.

Theorem

A convergent series has the commutative property if and only if it is absolutely convergent.

Proof: (Sketch) Think of Lebesgue summation.

Infinite Sums: Commutativity

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A convergent series has the commutative property if and only if it is absolutely convergent.

If we want to manipulate infinite sums like finite ones,
we must require absolute convergence.

Infinite sums: Associativity *between* two series

Definition

Two series $\sum_k a_k, \sum_k b_k$ satisfy the associative property if:

$$\sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$$

Can we say that any two series have the associative property?

Infinite sums: Associativity *between* two series

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Can we say that any two series have the associative property? In general, no:

- Let $a_k = [k \geqslant m]$ and $b_k = -[k \geqslant n]$ with $m, n \in \mathbb{Z}$.
- Then $\sum_k a_k = +\infty$ and $\sum_k b_k = -\infty$, but $\sum_k (a_k + b_k) = n - m$.

However, we have again the $+\infty - \infty$ issue...

Infinite sums: Associativity *between* two series

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$$\sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$$

Can we say that any two series have the associative property?

Theorem

- If the a_k and the b_k are all nonnegative, then $\sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$.
- If $\sum_k a_k$ and $\sum_k b_k$ both have a limit and at most one of those limits is infinite, then $\sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$.
- If $\sum_k a_k$ and $\sum_k b_k$ both converge absolutely, then $\sum_k (a_k + b_k)$ also converges absolutely.

Next subsection

Limits of sums, sums of limits

Consider the double indexed sequence:

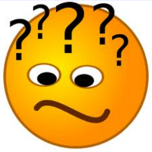
$$a_{j,k} = \frac{1}{j} [1 \leqslant k \leqslant j]$$

Then on the one hand:

$$\sum_k a_{j,k} = 1 \text{ for every } j \,, \text{ hence } \lim_{j \to \infty} \sum_k a_{j,k} = 1$$

But on the other hand:

$$\lim_{j \to \infty} a_{j,k} = 0 \text{ for every } k \,, \text{ hence } \sum_k \lim_{j \to \infty} a_{j,k} = 0$$



A positive result

Monotone Convergence Theorem

If the $a_{j,k}$ are all nonnegative and for every k the sequence $\langle a_{j,k} \rangle_{j \geqslant 0}$ is monotone nondecreasing, then:

$$\lim_{j \to \infty} \sum_k a_{j,k} = \sum_k \lim_{j \to \infty} a_{j,k} \,,$$

regardless of the two sides being finite or infinite.

Proof: Let $a_k = \lim_{j \to \infty} a_{j,k} = \sup_j a_{j,k}$ and $S_j = \sum_k a_{j,k}$.

- Then $\langle S_j \rangle$ is nondecreasing and $\lim_{j \to \infty} S_j = \sup_j S_j \leqslant \sum_k a_k = \sum_k \lim_{j \to \infty} a_{j,k}$.
- If the l.h.s. is $+\infty$ or the r.h.s. is 0, we have nothing else to do.
- Otherwise, suppose $\sum_k a_k > \alpha > 0$: We will prove that $\sup_j S_j > \alpha$ too.
- Fix $\delta > 0$ such that $\sum_k a_k > \alpha + 2\delta$ too.
- Choose $k_1, \ldots, k_n$ such that $\sum_{i=1}^n a_{k_i} > \alpha + \delta$.
- Choose j such that $a_{j,k_i} > a_{k_i} - \delta \cdot 2^{-i}$ for every $1 \leqslant i \leqslant n$. Then:

$$S_j \geqslant \sum_{i=1}^n a_{j,k_i} > \sum_k a_{k_i} - \delta \cdot \sum_{i=1}^n \frac{1}{2^i} > \alpha + \delta - \delta = \alpha$$

What can we be sure of, in general?

Fatou's Lemma

If the $a_{j,k}$ are all nonnegative, then:

$$\sum_k \liminf_j a_{j,k} \leqslant \liminf_j \sum_k a_{j,k}$$

Proof: (Sketch) Apply the monotone convergence theorem to $b_{j,k} = \inf_{i \geqslant j} a_{i,k}$.

What can we be sure of, in general?

Fatou's Lemma

If the $a_{j,k}$ are all nonnegative, then:

$$\sum_k \liminf_j a_{j,k} \leqslant \liminf_j \sum_k a_{j,k}$$

Dominated Convergence Theorem

If $a_k = \lim_{j\to\infty} a_{j,k}$ exists for every k and in addition there exists a sequence $\langle b_k \rangle$ such that:

1. $|a_{j,k}| \leqslant b_k$ for every $j \geqslant 0$, and
2. $\sum_k b_k < \infty$,

then:

$$\lim_{j\to\infty} \sum_k |a_{j,k} - a_k| = 0\,;$$

consequently,

$$\lim_{j\to\infty} \sum_k a_{j,k} = \sum_k a_k = \sum_k \lim_{j\to\infty} a_{j,k}\,.$$

Proof: (Sketch) Apply Fatou's lemma to $c_{j,k} = 2b_k - |a_{j,k} - a_k|$.

TAL TECH

Divergence of the harmonic series[1]

By contradiction, assume $\sum_{k \geqslant 1} \dfrac{1}{k} = S < +\infty$.

- For, $j, k \geqslant 1$ put $a_{j,k} = \dfrac{1}{j} [1 \leqslant k \leqslant j]$ and $b_k = \dfrac{1}{k}$.
- Then for every j and k, $\left| a_{j,k} \right| \leqslant b_k$, and $\sum_{k \geqslant 1} b_k$ converges.
- Now, $\lim_{j \to \infty} a_{j,k} = 0$ for every k, so $\sum_{k \geqslant 1} \lim_{j \to \infty} a_{j,k} = 0$.
- But $\sum_{k \geqslant 1} a_{j,k} = 1$ for every j, so $\lim_{j \to \infty} \sum_{k \geqslant 1} a_{j,k} = 1$.
- This contradicts the Dominated Convergence Theorem.

[1] This proof is taken from the MathExchange thread "Awfully sophisticated proofs of simple facts".

Next subsection

Multiple infinite sums

Definition: Double infinite sums

For every $j, k \geqslant 0$ let $a_{j,k} \geqslant 0$.

1. If $a_{j,k} \geqslant 0$ for every j and k, then:

$$\sum_{j,k} a_{j,k} = \sup_{K \subseteq \mathbb{N} \times \mathbb{N}, |K| < \infty} \sum_{K} a_{j,k} = \lim_{n \to \infty} \sum_{0 \leqslant j,k \leqslant n} a_{j,k} \,.$$

   (Recall that $\sum_{0 \leqslant j,k \leqslant n} a_{j,k} = \sum_{j,k} a_{j,k} [0 \leqslant j \leqslant n] [0 \leqslant k \leqslant n]$.)

2. If $\sum_{j,k} |a_{j,k}| < +\infty$, then:

$$\sum_{j,k} a_{j,k} = \sum_{j,k} a^{+}_{j,k} - \sum_{j,k} a^{-}_{j,k} \,.$$

Multiple infinite sums

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$$\sum_{j,k} a_{j,k} = \sum_{j,k} a_{j,k}^{+} - \sum_{j,k} a_{j,k}^{-} \,.$$

Can we use $\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k}$ or $\sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k}$ instead?

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2. If $\sum_{j,k} |a_{j,k}| < +\infty$, then:

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Can we use $\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k}$ or $\sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k}$ instead? In general, no:

- One writing is the limit on j of a limit on k which is a function of j;
- The other writing is the limit on k of a limit on j which is a function of k.
- There are no guarantees that the double limits be equal!

Multiple sums: An example of noncommutativity

From Joel Feldman's notes[2]

Let $a_{j,k} = [j = k = 0] + [k = j+1] - [k = j-1]$:

	0	1	2	3	4	…
0	1	1	0	0	0	…
1	−1	0	1	0	0	…
2	0	−1	0	1	0	…
3	0	0	−1	0	1	…
⋮	⋮	⋮	⋮	⋮	⋮	

Then:

- for every $j \geqslant 0$, $\sum_{k \geqslant 0} a_{j,k} = 2 \cdot [j = 0]$;
- for every $k \geqslant 0$, $\sum_{j \geqslant 0} a_{j,k} = 0$; and
- for every $n \geqslant 0$, $\sum_{0 \leqslant j,k \leqslant n} a_{j,k} = 1$.

Hence:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = 2\,; \quad \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = 0\,; \quad \text{and} \quad \lim_{n \to \infty} \sum_{0 \leqslant j,k \leqslant n} a_{j,k} = 1\,.$$

[2] http://www.math.ubc.ca/~feldman/m321/twosum.pdf retrieved 21.02.2019.

Multiple infinite sums: Swapping indices

Theorem

For $j, k \geqslant 0$ let $a_{j,k}$ be real numbers.

Tonelli: If $a_{j,k} \geqslant 0$ for every j and k, then:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = \sum_{j,k} a_{j,k} \,,$$

regardless of the quantities above being finite or infinite.

Fubini: If $\sum_{j,k} |a_{j,k}| < +\infty$, then:

$$\sum_{j \geqslant 0} \sum_{k \geqslant 0} a_{j,k} = \sum_{k \geqslant 0} \sum_{j \geqslant 0} a_{j,k} = \sum_{j,k} a_{j,k} \,.$$

Fubini's theorem is proved in the textbook.

Multiple infinite sums: Swapping indices

Theorem

For $j, k \geqslant 0$ let $a_{j,k}$ be real numbers.

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Fubini's theorem is proved in the textbook. Again:

If we want to manipulate infinite sums like finite ones,
we must require absolute convergence.

Next subsection

Cesàro summation

Given a series $\sum_k a_k$, consider the sequence $S_n = \sum_{k=0}^{n} a_k$ of the partial sums.

- Put $u(x) = \sum_{k=0}^{x-1} S_k$ and $v(x) = x$. Then $\Delta u(x) = S_x$ and $\Delta v(x) = 1$.
- Suppose $\sum_k a_k$ converges. Put $L = \sum_{k \geqslant 0} a_k = \lim_{n\to\infty} \dfrac{S_n}{1}$.
- We then have by the Stolz-Cesàro lemma: $\lim_{n\to\infty} \dfrac{\sum_{k=0}^{n-1} S_k}{n} = L$.

Given a (not necessarily convergent) series $\sum_k a_k$, the quantity:

$$C\sum_k a_k = \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_k}{n},$$

if it exists, is called the Cesàro sum of the series $\sum_k a_k$.

Cesàro sum without convergence

The series: $a_k = (-1)^k$ does not converge. However:

$$S_n = \sum_{k=0}^{n} (-1)^k = [n \text{ is even}]$$

so for every $n \geqslant 1$:

$$\begin{aligned}
\sum_{k=0}^{n-1} S_k &= \sum_{k=0}^{n-1} [k \text{ is even}] \\
&= \frac{n}{2}[n-1 \text{ is odd}] + \left(\frac{n-1}{2}+1\right)[n-1 \text{ is even}] \\
&= \frac{n}{2}[n \text{ is even}] + \frac{n+1}{2}[n \text{ is odd}] = \frac{n+[n \text{ is odd}]}{2}
\end{aligned}$$

The Cesàro sum of $a_k = (-1)^k$ is thus:

$$C\sum_k (-1)^k = \frac{1}{2}$$

n	0	1	2	3	4	5	6	7	8	9
a_n	1	-1	1	-1	1	-1	1	-1	1	-1
S_n	1	0	1	0	1	0	1	0	1	0
$\sum_{k=0}^{n-1} S_k$	0	1	1	2	2	3	3	4	4	5

Abel summation

Abel's summation theorem

Let the series $S(x) = \sum_{k \geqslant 0} a_k x^k$ converge for every $0 \leqslant x < 1$. If:

$$S(1) = \sum_{k \geqslant 0} a_k$$

converges, then:

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k x^k = S(x) \text{ uniformly in } [0, 1]$$

In particular:

$$L = \lim_{x \to 1^-} S(x) = S(1)$$

We can then define the Abel sum of a series as:

$$A \sum_k a_k = \lim_{x \to 1^-} \sum_k a_k x^k \text{ if the right-hand side exists}$$

Abel sum without convergence

The series $a_k = (-1)^k$ does not converge. However, for $0 \leqslant x < 1$ the series:

$$S(x) = \sum_{k \geqslant 0} (-1)^k x^k = \sum_{k \geqslant 0} (-x)^k$$

converges to $\dfrac{1}{1+x}$, and:

$$\lim_{x \to 1^-} S(x) = \lim_{x \to 1^-} \frac{1}{1+x} = \frac{1}{2}$$

The Abel sum of $a_k = (-1)^k$ is thus:

$$A\sum_{k} (-1)^k = \frac{1}{2}$$

Tauber's theorems

Tauber's first theorem (partial converse of Abel's summation theorem)

Let $S(x) = \sum_{k \geqslant 0} a_k x^k$ be such that $L = \lim_{x \to 1^-} S(x)$ exists. If:

$$\lim_{k \to \infty} k a_k = 0$$

then $S(1) = \sum_{k \geqslant 0} a_k = L$.

The condition here is that a_k is infinitesimal of order greater than first.

Tauber's second theorem (full converse of Abel's summation theorem)

Let $S(x) = \sum_{k \geqslant 0} a_k x^k$ be such that $L = \lim_{x \to 1^-} S(x)$ exists. Then $\sum_{k \geqslant 0} a_k$ converges if and only if:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k a_k = 0$$

In this case, $\sum_{k \geqslant 0} a_k = L$.

The condition here is that $k a_k$ converges to zero in arithmetic mean.
This is more general than the previous one because of the Stolz-Cesàro lemma.