ITT9132 – Concrete Mathematics
Midterm test of 27 March 2023

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Exercises

Exercise 1 (10 points)

Solve the recurrence:

$$\begin{aligned} T_0 &= 0\,, \\ nT_n &= 2T_{n-1} + \frac{n \cdot 2^n}{(n+3)!} H_n \quad \text{for every } n \geqslant 1\,. \end{aligned} \tag{1}$$

Hint: use a summation factor to obtain a simpler recurrence, and solve that recurrence with finite calculus. Also, $\frac{n}{(n+3)!} = \frac{1}{(n-1)!} \cdot n^{\underline{-3}}$.

Exercise 2 (6 points)

Let m be a positive integer. Prove that:

$$\sum_{k=1}^{m} \left\lfloor \frac{1}{\sum_{d \backslash k} [d \backslash m]} \right\rfloor = \phi(m), \tag{2}$$

where ϕ is Euler's function.

Exercise 3 (8 points)

Prove that, for every positive integer n, the number:

$$M_n = n^{23} - n^{13} - n^{11} + n$$

is divisible by every prime number smaller than 16.

Exercise 4

For each of the following questions, mark the only correct answer:

1. 72 persons are put in a circle and every second one is eliminated. What is the original position of the last person remaining?
   (a) 15.
   (b) 16.
   (c) 17.

2. Which one of the following is $\sum_{n \geqslant 1} n^{-3} H_n$?
   (a) $\dfrac{1}{24}$.
   (b) $\dfrac{1}{8}$.
   (c) $+\infty$.

3. Let $u(n)$ and $v(n)$ be defined for all $n \in \mathbb{N}$. Suppose that $v(n)$ is positive, strictly increasing, and divergent. Which one of the following is true?
   (a) $\lim_{n \to \infty} \dfrac{u(n)}{v(n)}$ exists.
   (b) If $\lim_{n \to \infty} \dfrac{\Delta u(n)}{\Delta v(n)}$ exists, then $\lim_{n \to \infty} \dfrac{u(n)}{v(n)}$ also exists, and coincides with the previous limit.
   (c) If $\lim_{n \to \infty} \dfrac{u(n)}{v(n)}$ exists, then $\lim_{n \to \infty} \dfrac{\Delta u(n)}{\Delta v(n)}$ also exists, and coincides with the previous limit.

4. Which one of the following pairs of sets forms a partition of $\mathbb{Z}^+$?
   (a) $\mathrm{Spec}(\sqrt{2})$ and $\mathrm{Spec}(2 + \sqrt{2})$.
   (b) $\mathrm{Spec}(\sqrt{3})$ and $\mathrm{Spec}(3 + \sqrt{3})$.
   (c) $\mathrm{Spec}(3)$ and $\mathrm{Spec}\left(\dfrac{3}{2}\right)$

5. Let the rational numbers x and y be represented by the strings $LLLLL$ and LR, respectively, as paths from $\frac{1}{1}$ in the Stern-Brocot tree. Which one of the following is true?

(a) $x < y$.

(b) $x = y$.

(c) $x > y$.

6. Let ϕ be Euler's function and let $m \geqslant 2$ be an integer. Which one of the following statements is equivalent to the statement that m is prime?

   (a) $\phi(m) = m - 1$.

   (b) $\phi(m) = m + 1$.

   (c) $\phi(m) = \sum_{d \backslash m} \mu(d) \frac{m}{d}$, where μ is the Möbius function.

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Solutions

Solution to Exercise 1

We need to fins a sequence s_n of numbers such that $s_n b_n = s_{n-1} a_{n-1}$ for every $n \geqslant 1$, where $a_n = n$ and $b_n = 2$. Following the textbook, we put $s_0 = a_0 = 1$ and:

$$s_n = \frac{a_0 \cdots a_{n-1}}{b_1 \cdots b_n} = \frac{(n-1)!}{2^n} \text{ for every } n \geqslant 1\,.$$

Multiplying (1) by s_n and renaming $U_n = s_n a_n T_n = \frac{n!}{2^n} T_n$, we obtain the simpler recurrence:

$$\begin{aligned} U_0 &= 0\,, \\ U_n &= U_{n-1} + \frac{(n-1)!}{2^n} \cdot \frac{n \cdot 2^n}{(n+3)!} H_n \\ &= U_{n-1} + n^{\underline{-3}} H_n \text{ for every } n \geqslant 1\,. \end{aligned}$$

The solution of the new recurrence is, of course:

$$U_n = \sum_{k=0}^{n} k^{\underline{-3}} H_k = \sum_0^{n+1} x^{\underline{-3}} H_x \, \delta x\,, \tag{3}$$

where we use the convention $H_0 = 0$ as an empty sum.

To solve (3) with finite calculus, we must find suitable $u(x)$ and $v(x)$ such that the right-hand side becomes $\sum_0^{n+1} u(x)\, \Delta v(x)\, \delta x$. A natural choice is $u(x) = H_x$, so that $\Delta u(x) = x^{\underline{-1}}$ when we apply summation by parts. It must then be $\Delta v(x) = x^{\underline{-3}}$, which allows us to choose $v(x) = -\frac{1}{2} x^{\underline{-2}}$. Then:

$$\begin{aligned} \sum_0^{n+1} x^{\underline{-3}} H_x \, \delta x\,, &= \left[-\frac{1}{2} x^{\underline{-2}} H_x \right]_0^{n+1} - \sum_0^{n+1} \left(-\frac{1}{2} \right) (x+1)^{\underline{-2}} x^{\underline{-1}} \, \delta x \\ &= -\frac{1}{2} (n+1)^{\underline{-2}} H_{n+1} + \frac{1}{2} \sum_0^{n+1} x^{\underline{-3}} \, \delta x \\ &= -\frac{1}{2} (n+1)^{\underline{-2}} H_{n+1} - \frac{1}{4} \left[x^{\underline{-2}} \right]_0^{n+1} \\ &= -\frac{1}{2} (n+1)^{\underline{-2}} H_{n+1} - \frac{1}{4} (n+1)^{\underline{-2}} + \frac{1}{4} \cdot \frac{1}{2!} \\ &= \frac{1}{8} - \frac{1}{4} (n+1)^{\underline{-2}} (1 + 2H_{n+1})\,. \end{aligned}$$

Dividing by $s_n a_n$ we obtain the solution to our original recurrence:

$$T_n = \frac{2^n}{n!} \cdot \left(\frac{1}{8} - \frac{1}{4} (n+1)^{\underline{-2}} (1 + 2H_{n+1}) \right) . \qquad (4)$$

Solution to Exercise 2

Let $S_{k,m} = \sum_{d \backslash k} [d \backslash m]$. As d ranges over the divisor of k, the summand $[d \backslash m]$ is 1 if d is a common divisor of k and m, and 0 otherwise. Then the sum $S_{k,m}$ is 1 if $k \perp m$, and larger than 1 otherwise: in the first case, $\lfloor 1/S_{k,m} \rfloor = \lfloor 1 \rfloor = 1$; in the second case, $1/S_{k,m}$ is positive and smaller than 1, so $\lfloor 1/S_{k,m} \rfloor = 0$. In other words, $\lfloor 1/S_{k,m} \rfloor = [k \perp m]$, from which we conclude:

$$\begin{aligned} \sum_{k=1}^{m} \left\lfloor \frac{1}{\sum_{d \backslash k} [d \backslash m]} \right\rfloor &= \sum_{k=1}^{m} [k \perp m] \\ &= \phi(m) . \end{aligned}$$

Solution to Exercise 3

The number M_n is the sum of four numbers which are either all even or all odd, so it is even. To check divisibility by 3, 5, 7, 11, and 13, we factorize:

$$\begin{aligned} n^{23} - n^{13} - n^{11} + n &= n \cdot (n^{22} - n^{12} - n^{10} + 1) \\ &= n \cdot (n^{12} - 1) \cdot (n^{10} - 1) \\ &= (n^{12} - n) \cdot (n^{11} - 1) = (n^{13} - n) \cdot (n^{10} - 1) . \end{aligned}$$

From the last line we find that M_n is divisible by 11 and by 13. To check divisibility by 3, 5, and 7, we consider the factor $n^{13} - n$, which we factorize further:

$$\begin{aligned} n^{13} - n &= (n^3 - n) \cdot (n^{10} + n^8 + n^4 + n^2 + 1) \\ &= (n^5 - n) \cdot (n^8 + n^4 + 1) \\ &= (n^7 - n) \cdot (n^6 + 1) . \end{aligned}$$

From the first, second, and third line we find that $n^{13} - n$ is divisible by 3, 5, and 7, respectively; and so is M_n.

Solution to Exercise 4

1. 72 persons are put in a circle and every second one is eliminated. What is the original position of the last person remaining?

   (a) 15.

   No: see below.

   (b) 16.

   No: in this situation, all persons who are initially in even positions are eliminated in the first round.

   (c) 17.

   Yes: $72 = 2^6 + 8$, so the last person remaining was originally in position $2 \cdot 8 + 1 = 17$.

2. Which one of the following is $\sum_{n \geqslant 1} n^{\underline{-3}} H_n$?

   (a) $\frac{1}{24}$.

   No: this is just the first summand.

   (b) $\frac{1}{8}$.

   Yes: you can either go by exclusion, or use your solution to Exercise 1.

   (c) $+\infty$.

   No: the summand is bounded from above by $\frac{1}{n^2}$, and $\sum_{n \geqslant 1} \frac{1}{n^2}$ converges.

3. Let $u(n)$ and $v(n)$ be defined for all $n \in \mathbb{N}$. Suppose that $v(n)$ is positive, strictly increasing, and divergent. Which one of the following is true?

   (a) $\lim_{n \to \infty} \frac{u(n)}{v(n)}$ exists.

   No: for example, if $u(n) = (-1)^n n$ and $v(n) = n$, then the mentioned limit does not exist.

   (b) If $\lim_{n \to \infty} \frac{\Delta u(n)}{\Delta v(n)}$ exists, then $\lim_{n \to \infty} \frac{u(n)}{v(n)}$ also exists, and coincides with the previous limit.

   True: this is the thesis of the Stolz-Cesàro lemma.

(c) If $\lim_{n\to\infty} \frac{u(n)}{v(n)}$ exists, then $\lim_{n\to\infty} \frac{\Delta u(n)}{\Delta v(n)}$ also exists, and coincides with the previous limit.

No: for example, if $u(n) = [n \text{ is even}]$ and $v(n) = n$, then $\Delta u(n) = (-1)^{n+1}$ and $\Delta v(n) = 1$, so the first limit is 0 but the second limit does not exist.

4. Which one of the following pairs of sets forms a partition of $\mathbb{Z}^+$?

(a) $\mathrm{Spec}(\sqrt{2})$ and $\mathrm{Spec}(2+\sqrt{2})$.

Yes: $\sqrt{2}$ and $2+\sqrt{2}$ are both irrational and $\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} = 1$.

(b) $\mathrm{Spec}(\sqrt{3})$ and $\mathrm{Spec}(3+\sqrt{3})$.

No: although $\sqrt{3}$ and $3+\sqrt{3}$ are both irrational, it is easy to see that $\frac{1}{\sqrt{3}} + \frac{1}{3+\sqrt{3}} < 1$.

(c) $\mathrm{Spec}(3)$ and $\mathrm{Spec}\left(\frac{3}{2}\right)$

No: although $\frac{1}{3} + \frac{1}{3/2} = 1$, the two numbers are rational.

5. Let the rational numbers x and y be represented by the strings $LLLLL$ and LR, respectively, as paths from $\frac{1}{1}$ in the Stern-Brocot tree. Which one of the following is true?

(a) $x < y$.

Yes: for any node of the Stern-Brocot tree, every element of the left subtree is smaller than every element of the right subtree. In fact, $x = 1/6$ and $y = 2/3$.

(b) $x = y$.

No: see above.

(c) $x > y$.

No: see above.

6. Let ϕ be Euler's function and let $m \geqslant 2$ be an integer. Which one of the following statements is equivalent to the statement that m is prime?

(a) $\phi(m) = m - 1$.

Yes.

(b) $\phi(m) = m + 1$.

No: the value of the Euler function cannot be larger than its argument.

(c) $\phi(m) = \sum_{d \backslash m} \mu(d) \frac{m}{d}$, where μ is the Möbius function.

No: this is the Möbius inversion formula applied to $m = \sum_{d \backslash m} \phi(d)$, so it is true for *every* positive integer m.