Fantastic Recurrences and How to Solve Them

Day 1: Sums

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http://www.cs.ioc.ee/~silvio/fantasticrecurrences

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Contents

Notation: Iverson brackets, ceiling, floor, slices

The Iverson brackets are the function from the set $\{\mathrm{True}, \mathrm{False}\}$ to the set $\{0, 1\}$ defined as follows:

1. $[\mathrm{True}] = 1$ and $[\mathrm{False}] = 0$.
2. If a is either infinite or undefined, then $a \cdot [\mathrm{False}] = 0$.

The ceiling of a real number x is the integer:

$$\lceil x \rceil = \min\{k \in \mathbb{Z} \mid x \leqslant k\}$$

Dually, the floor of a real number x is the integer:

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leqslant x\}$$

For $m, n \in \mathbb{Z}$ the slice from m to n is the set:

$$[m : n] = \{x \in \mathbb{Z} \mid m \leqslant x \leqslant n\} = [m, n] \cap \mathbb{Z}$$

Next section

Recurrence equations

- A sequence of complex numbers $\langle a_n \rangle = \langle a_0, a_1, a_2, \ldots \rangle$ is called recurrent if for $n \geqslant 1$ its generic term a_n satisfies a recurrence equation

$$a_n = f_n(a_{n-1}, \ldots, a_0),$$

with initial condition $a_0 = \alpha \in \mathbb{C}$, where $f_n : \mathbb{C}^n \to \mathbb{C}$ for every $n \geqslant 1$.

- If there exists $f : \mathbb{N} \times \mathbb{C}^k \to \mathbb{C}$ such that:

$$f_n = f(n; a_{n-1}, \ldots, a_{n-k}) \text{ for every } n \geqslant k,$$

the number k is called the order of the recurrence equation. In this case,

$$a_0 = \alpha_0, a_1 = \alpha_1, \ldots, a_{n-1} = \alpha_{n-1}$$

for suitable $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ are the initial conditions of the recurrence.

- Solving a recurrence means determining a function $f : \mathbb{N} \to \mathbb{C}$, called a closed form, such that $a_n = f(n)$ for every $n \geqslant 0$.

Two examples of recurrences

A recurrence equation of order 2

$$\begin{aligned} a_0 &= 0\,;\, a_1 = 1\,; \\ a_n &= a_{n-1} + a_{n-2} \text{ for every } n \geqslant 2 \end{aligned}$$

This recurrence defines the Fibonacci numbers.

A recurrence equation without a well-defined order

$$\begin{aligned} a_0 &= 1\,; \\ a_n &= a_0\, a_{n-1} + a_1\, a_{n-2} + \ldots + a_{n-1}\, a_0 \text{ for every } n \geqslant 1 \end{aligned}$$

This recurrence defines the Catalan numbers.

Notation

For a finite set $K = \{k_1, k_2, \cdots, k_m\}$ and a given sequence $\langle a_n \rangle$ of complex numbers:

$$\sum_K a_k = \sum_{i=1}^{m} a_{k_i} = \sum_{1 \leqslant i \leqslant m} a_{k_i} = a_{k_1} + a_{k_2} + \cdots + a_{k_m}$$

As addition of complex numbers is commutative, for every permutation p of the slice $[1:m]$ we have:

$$\sum_{i=1}^{m} a_{k_i} = \sum_{i=1}^{m} a_{k_{p(i)}}$$

Next section

The Simplest Recurrences

The simplest nontrivial recurrences are those of the first order:

$$\begin{aligned} S_0 &= a_0 ; \\ S_n &= S_{n-1} + a_n \text{ for every } n \geqslant 1 . \end{aligned}$$

Solving such a recurrence is the same as finding a closed form for the (partial) sum:

$$S_n = \sum_{k=0}^{n} a_k = \sum_{k \in [0:n]} a_k$$

First: Don't panic!

A scary sum?

For $n \geqslant 1$ compute:

$$\sum_{k=1}^{n} \left\lceil \sqrt{\left\lfloor \sum_{j=0}^{k} \frac{1}{j!} \right\rfloor + \left[\sqrt[3]{k} \in \mathbb{Z} \right]} \right\rceil$$

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$$\sum_{k=1}^{n} \left\lfloor \sqrt{\left\lfloor \sum_{j=0}^{k} \frac{1}{j!} \right\rfloor + \left[\sqrt[3]{k} \in \mathbb{Z} \right]} \right\rfloor$$

- First, we note that, as $j! > 2^{j-1}$ for $j > 2$, it is $\sum_{j=0}^{k} \frac{1}{j!} \leqslant \frac{5}{2} + \sum_{i=3}^{k} 2^{1-i} < 3$: Then the floor in the square root is always 2.

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  Then the floor in the square root is always 2.
- Next, we observe that $\left[\sqrt[3]{k} \in \mathbb{Z} \right]$ is always either 0 or 1:
  Then the sum under the square root is always either 2 or 3.

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  Then the floor in the square root is always 2.
- Next, we observe that $\left[\sqrt[3]{k} \in \mathbb{Z} \right]$ is always either 0 or 1:
  Then the sum under the square root is always either 2 or 3.
- But $\lceil \sqrt{2} \rceil = \lceil \sqrt{3} \rceil = 2$. We conclude:

$$\sum_{k=1}^{n} \left\lceil \sqrt{\left\lfloor \sum_{j=0}^{k} \frac{1}{j!} \right\rfloor + \left[\sqrt[3]{k} \in \mathbb{Z} \right]} \right\rceil = 2n.$$

Next subsection

The Repertoire Method: Idea

Consider a recurrence of the first order of the form:

$$\begin{aligned} f(0) &= \alpha_1 \,, \\ f(n) &= \Phi(f(n-1)) + \Psi(n; \alpha_2, \alpha_3, \ldots, \alpha_m) \text{ for every } n \geqslant 1 \,. \end{aligned} \tag{1}$$

Suppose that the following happens:

1. Φ is linear; and
2. Ψ is linear in each α_i (not necessarily in n).

Then can search a solution of the recurrence in the form:

$$f(n) = a_1 A_1(n) + a_2 A_2(n) + \cdots + a_m A_m(n) \tag{2}$$

where $A_1(n), A_2(n), \ldots, A_m(n)$ are determined by a system of equations

$$\begin{aligned} \alpha_{1,1} A_1(n) + \alpha_{1,2} A_2(n) + \ldots + \alpha_{1,m} A_m(n) &= g_1(n) \\ \alpha_{2,1} A_1(n) + \alpha_{2,2} A_2(n) + \ldots + \alpha_{2,m} A_m(n) &= g_2(n) \\ \vdots \quad & \quad \vdots \\ \alpha_{m,1} A_1(n) + \alpha_{m,2} A_2(n) + \ldots + \alpha_{m,m} A_m(n) &= g_m(n) \end{aligned} \tag{3}$$

where the $\alpha_{j,k}$ are suitable constants and the $g_j(n)$ are suitable functions.

The Repertoire Method: Realization

In the hypotheses of the previous slides, suppose that m $(m+1)$-tuples $(\alpha_{j,1},\ldots,\alpha_{j,m},g_j(n))$ exist such that:

1. For every j from 1 to m, the function $g_j(n)$ is the solution of the original recurrence with coefficients $\alpha_k = \alpha_{j,k}$, that is:

$$\begin{aligned} g_j(0) &= \alpha_{j,1}\,, \\ g_j(n) &= \Phi(g_j(n-1)) + \Psi(n;\alpha_{j,2},\ldots,\alpha_{j,m}) \text{ for every } n \geqslant 1\,. \end{aligned}$$

2. The matrix $A = \left(a_{j,k}\right)_{j,k\in[1:m]}$ is nonsingular.

Then:

1. There exist functions $A_1(n),\ldots,A_m(n)$ such that, for every choice of the parameters $\alpha_1,\ldots,\alpha_m$, the recurrence (1) has the unique solution (2).
2. For every $n \geqslant 0$, the m-tuple $(A_1(n),\ldots,A_m(n))$ is the unique solution of the linear system (3).

To find the m $(m+1)$-tuples $(\alpha_{j,1},\ldots,\alpha_{j,m},g_j(n))$ one can proceed in two ways:

1. Choose the parameters $\alpha_{j,1},\ldots,\alpha_{j,m}$ and determine the solution $g_j(n)$.
2. Choose the function $g_j(n)$ and determine the parameters $\alpha_{j,1},\ldots,\alpha_{j,m}$ for which $g_j(n)$ is the solution.

Simplifying by complicating

The repertoire method is a first example of a generic method of "simplifying by complicating":

- See your problem as a specific instance of a more general problem.
- This more general problem can be treated with more general methods.
- It might be simpler to solve the general problem with the general methods, than to solve the specific problem with the specialized methods.
- The solution to the general problem can be reused to solve other specific problems which are also specific instances of the general problem.

Example: Sums of an arithmetic progression

Solve the recurrence equation:

$$\begin{aligned} a_0 &= a\,, \\ a_n &= a_{n-1} + a + bn \text{ for every } n \geqslant 1\,. \end{aligned} \tag{4}$$

Note that a_n is the sum of the first $n+1$ terms of the arithmetic progression $\langle a + nb \rangle$.

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Example: Sums of an arithmetic progression

Solve the recurrence equation:

$$\begin{aligned} a_0 &= a\,, \\ a_n &= a_{n-1} + a + bn \text{ for every } n \geqslant 1\,. \end{aligned} \tag{4}$$

Let us solve instead the more general system:

$$\begin{aligned} a_0 &= \alpha_1\,, \\ a_n &= a_{n-1} + \alpha_2 + \alpha_3 n \text{ for every } n \geqslant 1\,. \end{aligned}$$

This recurrence has the form $a_n = \Phi(a_{n-1}) + \Psi(n; \alpha_2, \alpha_3)$ with $\Phi(x) = x$ linear and $\Psi(x) = \alpha_2 + \alpha_3 n$ linear in α_2 and α_3.
We can then try to apply the repertoire method.

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$$\begin{aligned} a_0 &= \alpha_1\,, \\ a_n &= a_{n-1} + \alpha_2 + \alpha_3 n \text{ for every } n \geqslant 1\,. \end{aligned}$$

For this we use the repertoire method:

- **1.** For $\alpha_{1,1} = 1, \alpha_{1,2} = \alpha_{1,3} = 0$ we have:

$$g_1(n) = 1 \text{ for every } n \geqslant 0\,.$$

Example: Sums of an arithmetic progression

Solve the recurrence equation:

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For this we use the repertoire method:

- **2.** For $\alpha_{2,1} = 0, \alpha_{2,2} = 1, \alpha_{2,3} = 0$ we have:

$$g_2(n) = n \ \text{ for every } n \geqslant 0\,.$$

Example: Sums of an arithmetic progression

Solve the recurrence equation:

$$\begin{aligned} a_0 &= a\,, \\ a_n &= a_{n-1} + a + bn \text{ for every } n \geqslant 1\,. \end{aligned} \tag{4}$$

Let us solve instead the more general system:

$$\begin{aligned} a_0 &= \alpha_1\,, \\ a_n &= a_{n-1} + \alpha_2 + \alpha_3\, n \text{ for every } n \geqslant 1\,. \end{aligned}$$

For this we use the repertoire method:

- **3.** For $g_3(n) = n^2$, as $n^2 = (n-1)^2 + 2(n-1) + 1 = (n-1)^2 + 2n - 1$, we have:

$$\alpha_{3,1} = 0\,,\ \alpha_{3,2} = -1\,,\ \alpha_{3,3} = 2$$

Example: Sums of an arithmetic progression

Solve the recurrence equation:

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Let us solve instead the more general system:

$$\begin{aligned} a_0 &= \alpha_1\,, \\ a_n &= a_{n-1} + \alpha_2 + \alpha_3 n \text{ for every } n \geqslant 1\,. \end{aligned}$$

The repertoire method leads us to the family of linear systems:

$$\begin{array}{cccccc} A_1(n) & & & = & 1 \\ & A_2(n) & & = & n \\ & -A_2(n) & +2A_3(n) & = & n^2 \end{array}$$

which has the unique solution:

$$A_1(n) = 1\,;\;\; A_2(n) = n\,;\;\; A_3(n) = \frac{n^2+n}{2}\,.$$

Example: Sums of an arithmetic progression

Solve the recurrence equation:

$$\begin{aligned} a_0 &= a\,, \\ a_n &= a_{n-1} + a + bn \text{ for every } n \geqslant 1\,. \end{aligned} \tag{4}$$

Let us solve instead the more general system:

$$\begin{aligned} a_0 &= \alpha_1\,, \\ a_n &= a_{n-1} + \alpha_2 + \alpha_3 n \text{ for every } n \geqslant 1\,. \end{aligned}$$

The repertoire method tells that the general solution is:

$$a_n = \alpha_0 + \alpha_1 n + \alpha_2 \cdot \frac{n^2+n}{2}\,.$$

The recurrence (4) corresponds to $\alpha_1 = a, \alpha_2 = a, \alpha_3 = b$. We conclude:

$$a_n = (n+1)\cdot a + \frac{n^2+n}{2}\cdot b\,.$$

Next subsection

The perturbation method

This method is useful to compute a closed form for the sequence of prefix sums of a given sequence $\langle a_n \rangle$:

$$S_n = \sum_{k=0}^{n} a_k$$

1. Perturb the equality by isolating the last summand on the left-hand side, and the first summand on the right-hand side:

$$S_n + a_{n+1} = a_0 + \sum_{k=1}^{n+1} a_k$$

2. Rewrite the right-hand side so that it becomes a function of S_n.
3. Solve with respect to S_n.

Example 1: Sums of a geometric progression

For $a \neq 1$ compute: $S_n = \sum_{k=0}^{n} a^k$.

1. Perturb the sum:

$$S_n + a^{n+1} = 1 + \sum_{k=1}^{n+1} a^k$$

2. Rewrite the right-hand side so that it depends on S_n:

$$1 + \sum_{k=1}^{n+1} a^k = 1 + a \sum_{k=0}^{n} a^k = 1 + aS_n$$

3. Solve with respect to S_n:

$$\begin{aligned} S_n + a^{n+1} &= 1 + aS_n \\ (1-a)S_n &= 1 - a^{n+1} \\ S_n &= \frac{1-a^{n+1}}{1-a} = \frac{a^{n+1}-1}{a-1} \end{aligned}$$

Example 2: $S_n = \sum_{k=0}^{n} ka^k$ with $a \neq 1$

- For $x \neq 1$:

$$
\begin{aligned}
S_n + (n+1)a^{n+1} &= 0 + \sum_{0 \leqslant k \leqslant n} (k+1)a^{k+1} \\
&= \sum_{0 \leqslant k \leqslant n} ka^{k+1} + \sum_{0 \leqslant k \leqslant n} a^{k+1} \\
&= aS_n + \frac{a(1-a^{n+1})}{1-a}
\end{aligned}
$$

- From this we get:

$$\sum_{k=0}^{n} ka^k = \frac{a-(n+1)a^{n+1}+na^{n+2}}{(a-1)^2}$$

Example 3: When perturbation doesn't work ...

Compute: $S_n = \sum_{k=0}^{n} k^2$.

1. Perturb the sum:

$$S_n + n^2 = 0 + \sum_{k=1}^{n+1} k^2$$

Um ... that shifted k^2 sounds bad ...

Example 3: When perturbation doesn't work ...

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2. Rewrite the right-hand side so that it depends on S_n:

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=0}^{n} (k+1)^2 \\
&= \sum_{k=0}^{n} (k^2 + 2k + 1) \\
&= S_n + \sum_{k=0}^{n} (2k+1) \\
&= S_n + 2\frac{n(n+1)}{2} + n + 1
\end{aligned}$$

Example 3: When perturbation doesn't work ...

Compute: $S_n = \sum_{k=0}^{n} k^2$.

1. Perturb the sum:

$$S_n + n^2 = 0 + \sum_{k=1}^{n+1} k^2$$

   Um ... that shifted k^2 sounds bad ...

2. Rewrite the right-hand side so that it depends on S_n:

$$\sum_{k=1}^{n+1} k^2 \quad = \quad S_n + 2\frac{n(n+1)}{2} + n + 1$$

3. Solve with respect to S_n:

$$\begin{aligned} S_n + (n+1)^2 &= S_n + (n+1) + 2\frac{n(n+1)}{2} \\ (n+1)^2 &= (n+1) + 2\frac{n(n+1)}{2} \end{aligned}$$

   ... which is true, but where is S_n?

. . . try perturbing *another* sum!

In addition to S_n, consider the sum: $T_n = \sum_{k=0}^{n} k^3$.

1. Perturb T_n:

$$T_n + (n+1)^3 = 0 + \sum_{k=1}^{n+1} k^3$$

. . . try perturbing *another* sum!

In addition to S_n, consider the sum: $T_n = \sum_{k=0}^{n} k^3$.

1. Perturb T_n:

$$T_n + (n+1)^3 = 0 + \sum_{k=1}^{n+1} k^3$$

2. Rewrite the right-hand side so that it depends on T_n and on S_n:

$$\begin{aligned} \sum_{k=1}^{n+1} k^3 &= \sum_{k=0}^{n} (k+1)^3 \\ &= \sum_{k=0}^{n} (k^3 + 3k^2 + 3k + 1) \\ &= T_n + 3S_n + \sum_{k=0}^{n} (3k+1) \end{aligned}$$

. . . try perturbing *another* sum!

In addition to S_n, consider the sum: $T_n = \sum_{k=0}^{n} k^3$.

1. Perturb T_n:

$$T_n + (n+1)^3 = 0 + \sum_{k=1}^{n+1} k^3$$

2. Rewrite the right-hand side so that it depends on T_n and on S_n:

$$\sum_{k=1}^{n+1} k^3 = T_n + 3S_n + \sum_{k=0}^{n} (3k+1)$$

3. Solve with respect to S_n:

$$\begin{aligned}
(n+1)^3 &= 3S_n + (n+1) + 3\frac{n(n+1)}{2} \\
&= 3S_n + (n+1)\left(1 + \frac{3}{2}n\right) \\
3S_n &= (n+1)\left(n^2 + 2n + 1 - 1 - \frac{3}{2}n\right) \\
S_n &= \frac{1}{3}(n+1)\left(n^2 + \frac{n}{2}\right) = \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Next subsection

Solving $a_n T_n = b_n T_{n-1} + c_n$ with initial condition T_0

The idea:

Find a summation factor s_n satisfying the following property:

$$s_n b_n = s_{n-1} a_{n-1} \text{ for every } n \geqslant 1$$

If such a factor exists, one can do following transformations:

1. Multiply by s_n and get: $s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n = s_{n-1} a_{n-1} T_{n-1} + s_n c_n$.
2. Set $S_n = s_n a_n T_n$ and rewrite the equation as:

$$S_0 = s_0 a_0 T_0$$
$$S_n = S_{n-1} + s_n c_n$$

3. Obtain an "almost closed" formula for the solution:

$$T_n = \frac{1}{s_n a_n} \left(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k \right) = \frac{1}{s_n a_n} \left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k \right)$$

Finding a summation factor

Assuming that $b_n \neq 0$ for every n:

1. Set $s_0 = 1$.
2. Compute the next elements using the property $s_n b_n = s_{n-1} a_{n-1}$:

$$
\begin{aligned}
s_1 &= \frac{a_0}{b_1} \\
s_2 &= \frac{s_1 a_1}{b_2} = \frac{a_0 a_1}{b_1 b_2} \\
s_3 &= \frac{s_2 a_2}{b_3} = \frac{a_0 a_1 a_2}{b_1 b_2 b_3} \\
&= \ldots \\
s_n &= \frac{s_{n-1} a_{n-1}}{b_n} = \frac{a_0 a_1 \cdots a_{n-1}}{b_1 b_2 \cdots b_n}
\end{aligned}
$$

Example: application of summation factor

$a_n = c_n = 1$ and $b_n = 2$ gives the Hanoi Tower sequence:

Evaluate the summation factor:

$$s_n = \frac{s_{n-1}\,a_{n-1}}{b_n} = \frac{a_0\,a_1 \cdots a_{n-1}}{b_1\,b_2 \cdots b_n} = \frac{1}{2^n}$$

The solution is:

$$T_n = \frac{1}{s_n a_n}\left(s_1\,b_1\,T_0 + \sum_{k=1}^{n} s_k\,c_k\right) = 2^n \sum_{k=1}^{n} \frac{1}{2^k} = 2^n(1 - 2^{-n}) = 2^n - 1$$

Yet Another Example: constant coefficients

Equation $Z_n = aZ_{n-1} + b$

Taking $a_n = 1$, $b_n = a$ and $c_n = b$:

- Evaluate summation factor:

$$s_n = \frac{s_{n-1}\, a_{n-1}}{b_n} = \frac{a_0\, a_1 \ldots a_{n-1}}{b_1\, b_2 \ldots b_n} = \frac{1}{a^n}$$

- The solution is:

$$\begin{aligned} Z_n &= \frac{1}{s_n a_n}\left(s_1\, b_1\, Z_0 + \sum_{k=1}^{n} s_k\, c_k\right) = a^n\left(Z_0 + b\sum_{k=1}^{n}\frac{1}{a^k}\right) \\ &= a^n Z_0 + b\left(1 + a + a^2 + \cdots + a^{n-1}\right) \\ &= a^n Z_0 + \frac{a^n - 1}{a - 1}\, b \end{aligned}$$

Yet Another Example: check up on results

Equation $Z_n = aZ_{n-1} + b$

$$\begin{aligned} Z_n &= aZ_{n-1} + b \\ &= a^2 Z_{n-2} + ab + b \\ &= a^3 Z_{n-3} + a^2 b + ab + b \end{aligned}$$

$\ldots\ldots$

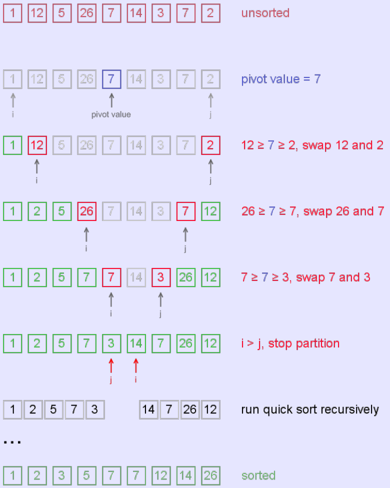
$$\begin{aligned} &= a^k Z_{n-k} + (a^{k-1} + a^{k-2} + \ldots + 1)b \\ &= a^k Z_{n-k} + \frac{a^k - 1}{a - 1} b \qquad \text{(assuming } a \neq 1\text{)} \end{aligned}$$

Continuing until $k = n$:

$$\begin{aligned} Z_n &= a^n Z_{n-n} + \frac{a^n - 1}{a - 1} b \\ &= a^n Z_0 + \frac{a^n - 1}{a - 1} b \end{aligned}$$

Efficiency of Quicksort

Average number of comparisons: $C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$, $C_0 = 0$.



Efficiency of Quicksort: Obtaining the recurrence

The following transformations reduce this equation

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1}$$

Write the last equation for $n-1$:

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

and subtract to eliminate the sum:

$$\begin{aligned} nC_n - (n-1)C_{n-1} &= n^2 + n + 2C_{n-1} - (n-1)^2 - (n-1) \\ nC_n - nC_{n-1} + C_{n-1} &= n^2 + n + 2C_{n-1} - n^2 + 2n - 1 - n + 1 \\ nC_n - nC_{n-1} &= C_{n-1} + 2n \\ nC_n &= (n+1)C_{n-1} + 2n \end{aligned}$$

Efficiency of Quicksort: Solving the recurrence

Equation $nC_n = (n+1)C_{n-1} + 2n$

- Evaluate summation factor with $a_n = n$, $b_n = n+1$ and $c_n = 2n$:

$$s_n = \frac{a_1 a_2 \cdots a_{n-1}}{b_2 b_3 \cdots b_n} = \frac{1 \cdot 2 \cdots (n-1)}{3 \cdot 4 \cdots (n+1)} = \frac{2}{n(n+1)}$$

- Then the solution of the recurrence is:

$$\begin{aligned}
C_n &= \frac{1}{s_n a_n} \left(s_1 b_1 C_0 + \sum_{k=1}^{n} s_k c_k \right) \\
&= \frac{n+1}{2} \sum_{k=1}^{n} \frac{4k}{k(k+1)} \text{ because } C_0 = 0 \\
&= 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1} = 2(n+1) \left(\sum_{k=1}^{n} \frac{1}{k} + \frac{1}{n+1} - 1 \right) \\
&= 2(n+1)H_n - 2n
\end{aligned}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \ln n$ is the nth harmonic number.

Next section

Basic properties

If the set K is finite, then the usual properties of addition hold:

- Distributivity: $\sum_{k \in K} ca_k = c \sum_{k \in K} a_k$.
- Associativity: $\sum_{k \in K}(a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k$.
- Commutativity: $\sum_{k \in K} a_k = \sum_{k \in K} a_{p(k)}$ where $p : K \to K$ is a permutation.

For example, the following derivation is valid:

$$
\begin{aligned}
S &= \sum_{0 \leqslant k \leqslant n} (a + bk) \\
&= \sum_{0 \leqslant k \leqslant n} (a + b(n-k)) \text{ by commutativity} \\
2S &= \sum_{0 \leqslant k \leqslant n} (2a + b(k + n - k)) \text{ by associativity} \\
&= (2a + bn) \sum_{0 \leqslant k \leqslant n} 1 \text{ by distributivity} \\
S &= (n+1)a + \frac{n(n+1)}{2} b
\end{aligned}
$$

The Inclusion-Exclusion Principle

Theorem

Let K and K' be finite sets of indices. Then:

$$\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \sum_{k \in K \cup K'} a_k + \sum_{k \in K \cap K'} a_k$$

Special cases:

a. For $1 \leqslant m \leqslant n$:

$$\sum_{k=1}^{m} a_k + \sum_{k=m}^{n} a_k = a_m + \sum_{k=1}^{n} a_k$$

b. For $n \geqslant 0$:

$$\sum_{0 \leqslant k \leqslant n} a_k = a_0 + \sum_{1 \leqslant k \leqslant n} a_k$$

c. For $n \geqslant 0$:

$$S_n + a_{n+1} = a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1}$$

Next section

Multiple sums

Definition

If K_1 and K_2 are index sets, then:

$$\sum_{i \in K_1, j \in K_2} a_{i,j} = \sum_i \left(\sum_j a_{i,j} [P(i,j)] \right)$$

where P is the predicate $P(i,j) = (i \in K_1) \wedge (j \in K_2)$.

The following law of interchange of the order of summation holds:

$$\sum_j \sum_k a_{j,k} [P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k} [P(j,k)]$$

If $a_{j,k} = a_j b_k$, then:

$$\sum_{j \in J, k \in K} a_j b_k = \left(\sum_{j \in J} a_j \right) \left(\sum_{k \in K} b_k \right)$$

. . . but what is a recurrence with multiple indices?

We can rephrase the definition of recurrent sequences by saying that:

Each term is a function of the previous ones.

With two (or more) indices, we cannot say anymore of any two pairs of indices, which one come first:

Which one between $(1,2)$ or $(2,1)$ should be "larger"?

However, we still have a notion of for every pair of indices, which pairs are previous:

$$(j_1, k_1) \leqslant (j_2, k_2) \text{ if and only if } j_1 \leqslant j_2 \text{ and } k_1 \leqslant k_2$$

Multiple sums with independent indices

If $P(j,k) = Q(j) \wedge R(k)$, where Q and R are properties and $\wedge$ indicates the logical conjunction (AND), then the indices j and k are independent and the double sum can be rewritten:

$$\begin{aligned}
\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k}\left([Q(j) \wedge R(k)]\right) \\
&= \sum_{j,k} a_{j,k}[Q(j)][R(k)] \\
&= \sum_{j}[Q(j)] \sum_{k} a_{j,k} R(k) = \sum_{j}\sum_{k} a_{j,k} \\
&= \sum_{k} a_{j,k}[R(k)] \sum_{j}[Q(j)] = \sum_{k}\sum_{j} a_{j,k}
\end{aligned}$$

Multiple sums with dependent indices

In general, the indices are not independent, but we can write:

$$P(j,k) = Q(j) \wedge R'(j,k) = R(k) \wedge Q'(j,k)$$

In this case, we can proceed as follows:

$$\begin{aligned}
\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k}[Q(j)][R'(j,k)] \\
&= \sum_{j}[Q(j)]\sum_{k} a_{j,k}[R'(j,k)] = \sum_{j \in J}\sum_{k \in K'} a_{j,k} \\
&= \sum_{k}[R(k)]\sum_{j} a_{j,k}[Q'(j,k)] = \sum_{k \in K}\sum_{j \in J'} a_{j,k}
\end{aligned}$$

where:

- $J = \{j \mid Q(j)\}, K' = \{k \mid R'(j,k)\} = K'(j)$
- $K = \{k \mid R(k)\}, J' = \{j \mid Q'(j,k)\} = J'(k)$

TAL TECH

What's wrong with this sum?

$$\begin{aligned}\left(\sum_{j=1}^{n} a_j\right)\cdot\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) &= \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k}\\ &= \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k}\\ &= \sum_{k=1}^{n}\sum_{k=1}^{n} 1\\ &= n^2\end{aligned}$$

What's wrong with this sum?

$$\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k}$$

$$= \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{a_k}{a_k}$$

$$= \sum_{k=1}^{n} \sum_{k=1}^{n} 1$$

$$= n^2$$

Solution

The second passage is seriously wrong:
It is not licit to turn two independent variables into two dependent ones.

Examples of multiple summation: Mutual upper bounds

How are the two sums below related?

$$\sum_{j=1}^{n} \sum_{k=j}^{n} a_{j,k} \text{ and } \sum_{k=1}^{n} \sum_{j=1}^{k} a_{j,k}$$

Examples of multiple summation: Mutual upper bounds

How are the two sums below related?

$$\sum_{j=1}^{n} \sum_{k=j}^{n} a_{j,k} \quad \text{and} \quad \sum_{k=1}^{n} \sum_{j=1}^{k} a_{j,k}$$

Step 1: Rewrite with Iverson brackets

We move the conditions on the indices from the sums to the summands:

$$\begin{aligned} \sum_{j=1}^{n} \sum_{k=j}^{n} a_{j,k} &= \sum_{1 \leqslant j,k \leqslant n} a_{j,k} \, [1 \leqslant j \leqslant n] \, [j \leqslant k \leqslant n] \\ \sum_{k=1}^{n} \sum_{j=1}^{k} a_{j,k} &= \sum_{1 \leqslant j,k \leqslant n} a_{j,k} \, [1 \leqslant k \leqslant n] \, [1 \leqslant j \leqslant k] \end{aligned}$$

Examples of multiple summation: Mutual upper bounds

How are the two sums below related?

$$\sum_{j=1}^{n}\sum_{k=j}^{n} a_{j,k} \text{ and } \sum_{k=1}^{n}\sum_{j=1}^{k} a_{j,k}$$

Step 2: Observe that the summands are equal term by term

We only need to do so for the Iverson brackets:

$$[1 \leqslant j \leqslant n]\,[j \leqslant k \leqslant n] = [1 \leqslant j \leqslant k \leqslant n] = [1 \leqslant k \leqslant n]\,[1 \leqslant j \leqslant k]$$

We conclude that the two sums must be equal.

A nice trick for symmetric summands

Write the summands in a matrix:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} \end{pmatrix}$$

so that j is the row index and k is the column index.

- Then the sum of the values of the upper triangular part of the matrix is:

$$S_U = \sum_{1 \leqslant j \leqslant k \leqslant n} a_{j,k}$$

- Dually, the sum of the values of the lower triangular part of the matrix is:

$$S_L = \sum_{1 \leqslant k \leqslant j \leqslant n} a_{j,k}$$

- Adding S_U to S_L and applying the inclusion-exclusion principle:

$$\sum_{1 \leqslant j \leqslant k \leqslant n} a_{j,k} + \sum_{1 \leqslant k \leqslant j \leqslant n} a_{j,k} = \sum_{1 \leqslant j,k \leqslant n} a_{j,k} + \sum_{1 \leqslant k \leqslant n} a_{k,k}$$

A nice trick for symmetric summands

Write the summands in a matrix:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} \end{pmatrix}$$

so that j is the row index and k is the column index.

- Adding S_U to S_L and applying the inclusion-exclusion principle:

$$\sum_{1 \leqslant j \leqslant k \leqslant n} a_{j,k} + \sum_{1 \leqslant k \leqslant j \leqslant n} a_{j,k} = \sum_{1 \leqslant j,k \leqslant n} a_{j,k} + \sum_{1 \leqslant k \leqslant n} a_{k,k}$$

- If $a_{j,k} = a_{k,j}$ for every j and k, then $S_U = S_L$ and we have:

$$\sum_{1 \leqslant j \leqslant k \leqslant n} a_{j,k} = \frac{1}{2} \left(\sum_{1 \leqslant j,k \leqslant n} a_{j,k} + \sum_{k=1}^{n} a_{k,k} \right)$$

A nice trick for symmetric summands

Write the summands in a matrix:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

so that j is the row index and k is the column index.

- Adding S_U to S_L and applying the inclusion-exclusion principle:

$$\sum_{1 \leqslant j \leqslant k \leqslant n} a_{j,k} + \sum_{1 \leqslant k \leqslant j \leqslant n} a_{j,k} = \sum_{1 \leqslant j,k \leqslant n} a_{j,k} + \sum_{1 \leqslant k \leqslant n} a_{k,k}$$

- In the special case $a_{j,k} = a_j a_k$ we can apply distributivity and obtain:

$$\sum_{1 \leqslant j \leqslant k \leqslant n} a_j a_k = \frac{1}{2} \left(\left(\sum_{k=1}^{n} a_k \right)^2 + \sum_{k=1}^{n} a_k^2 \right)$$

Multiple sums for ordinary sums

Suppose we have a sum of the form $\sum_{k=1}^{n} a_k$ which is "difficult" to compute with the methods from the previous section.

- Write the term a_k in the form $b_k \cdot \left(\sum_{j=1}^{k} c_j\right)$. Then the original sum becomes:

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k \sum_{j=1}^{k} c_j = \sum_{j=1}^{n} c_j \sum_{k=j}^{n} b_k$$

If the summand $d_j = c_j \sum_{k=j}^{n} b_k$ is "easy to manage", we may obtain a new sum $\sum_{j=1}^{n} d_j$ which is easier to compute.

Example 2: $\sum_{k=1}^{n} ka^k$ with $a \neq 1$

Clearly $k = \sum_{j=1}^{k} 1$, so we can expand:

$$\sum_{k=1}^{n} ka^k = \sum_{k=1}^{n} a^k \sum_{j=1}^{k} 1 = \sum_{j=1}^{n} \sum_{k=j}^{n} a^k$$

The sum over j is easy to manage:

$$\sum_{k=j}^{n} a^k = \sum_{k=0}^{n} a^k - \sum_{k=0}^{j-1} a^k = \frac{a^{n+1} - a^j}{a-1}$$

Then:

$$\begin{aligned}
\sum_{k=1}^{n} ka^k &= \frac{1}{a-1} \sum_{j=1}^{n} \left(a^{n+1} - a^j \right) \\
&= \frac{1}{a-1} \left(na^{n+1} - a \sum_{j=0}^{n-1} a^j \right) \\
&= \frac{1}{a-1} \left(na^{n+1} - \frac{a^{n+1} - a}{a-1} \right) \\
&= \frac{na^{n+2} - (n+1)a^{n+1} + a}{(a-1)^2}
\end{aligned}$$