Notes on Restriction Categories*

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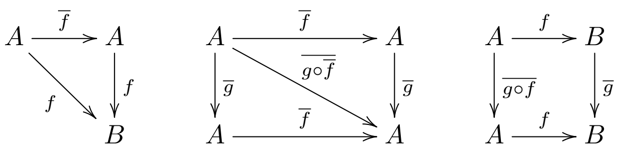
Abstract

This is a summary of some notes taken by me during the course given by Prof. Robin Cockett at the Institute of Cybernetics in February 2013.

1 Restriction categories

We assume familiarity with the basics of category theory. Given a category $\mathbb{X}$, we write $\mathrm{Ob}(\mathbb{X})$ or $|\mathbb{X}|$ for the collection of its objects, and $\mathrm{Hom}_{\mathbb{X}}(A, B)$ or $\mathbb{X}(A, B)$ for the collection of the maps in $\mathbb{X}$ from A to B. We put $\mathrm{Hom}_{\mathbb{X}} = \bigcup_{A,B\in|\mathbb{X}|} \mathbb{X}(A, B)$. If $\mathbb{X}$ is clear from the context, we may omit it from the Hom-notation.

Definition 1.1 (Restriction category)**.** A *restriction category* is a category $\mathbb{X}$ together with a *restriction operator* $\overline{(\cdot)}$ which associates to every $f \in \mathbb{X}(A, B)$ an $\overline{f} \in \mathbb{X}(A, A)$ so that the following diagrams commute:



(1)

i.e., the following properties are satisfied:

1. $f \circ \overline{f} = f$ for every $f : A \to B$.
2. $\overline{g} \circ \overline{f} = \overline{f} \circ \overline{g}$ for every $f : A \to B$, $g : A \to C$.
3. $\overline{g} \circ \overline{f} = \overline{g \circ \overline{f}}$ for every $f : A \to B$, $g : A \to C$.
4. $\overline{g} \circ f = f \circ \overline{g \circ f}$ for every $f : A \to B$, $g : B \to C$.

Let us interpret the rules. Rule 1 says that the result of f does not change if its restriction $\overline{f}$ is applied first. Rule 2 says that restricted maps with same domain commute with each other. Rule 3 says that composing two restrictions is the same as applying a restricted map first, then an ordinary one, and finally restrict the composition. Rule 4 says that a restriction can be moved from "after" to "before" an ordinary map, provided the originally restricted map is pre-composed with the other one.

Example 1.2. The category **Par** whose objects are sets and whose maps are partial functions, is a restriction category: the restriction $\overline{f} : A \to A$ of $f : A \to B$ satisfies $\overline{f}(x) = x$ if $f(x)$ is defined, and undefined otherwise.

Lemma 1.3. *Let $\mathbb{X}$ be a restriction category. For every $f : A \to B$, $g : B \to C$ the following hold:*

1. $\overline{\overline{f}} = \overline{f}$*. That is: the restriction operator on $\mathbb{X}$ is idempotent.*
2. $\overline{f} \circ \overline{f} = \overline{f}$*. That is: restricted maps are idempotents in $\mathbb{X}$.*
3. *If f is monic then $\overline{f} = \mathrm{id}_A$. In particular, $\overline{\mathrm{id}_A} = \mathrm{id}_A$.*
4. $\overline{g \circ f} = \overline{\overline{g} \circ f}$ *for every $f : A \to B$, $g : B \to C$.*

Proof. We first prove point 2. By putting $g = f$ in rule 3 of restriction categories we get $\overline{f} \circ \overline{f} = \overline{f \circ \overline{f}}$, which is $\overline{f}$ by rule 1.

We now prove point 3. By rule 1, $f \circ \overline{f} = f = f \circ \mathrm{id}_A$: as f is monic, $\overline{f} = \mathrm{id}_A$.

To prove point 4 we use all the rules of restriction categories:

$$\overline{\overline{g} \circ f} = \overline{f \circ \overline{g \circ f}} = \overline{f} \circ \overline{g \circ f} = \overline{g \circ f} \circ \overline{f} = \overline{g \circ f \circ \overline{f}} = \overline{g \circ f}\,.$$

Finally,

$$\overline{\overline{f}} = \overline{\mathrm{id}_A \circ \overline{f}} = \overline{\mathrm{id}_A} \circ \overline{f} = \mathrm{id}_A \circ \overline{f} = \overline{f} :$$

which proves point 1. □

Definition 1.4 (Total map)**.** Let $\mathbb{X}$ be a restriction category. A map $f : A \to B$ in $\mathbb{X}$ is *total* if $\overline{f} = \mathrm{id}_A$.

Monic maps are total. In addition, we have the following

Lemma 1.5. *Let $\mathbb{X}$ be a restriction category, $f \in \mathbb{X}(A, B)$, $g \in \mathbb{X}(B, C)$. If $g \circ f$ is total, then so is f.*

Proof. If $\overline{g \circ f} = \mathrm{id}_A$, then

$$\overline{f} = \mathrm{id}_A \circ \overline{f} = \overline{g \circ f} \circ \overline{f} = \overline{g \circ f \circ \overline{f}} = \overline{g \circ f} = \mathrm{id}_A\,.$$

□

Lemma 1.6. *For a restriction category $\mathbb{X}$, the objects of $\mathbb{X}$ together with the total maps of $\mathbb{X}$ form a subcategory* **Total**$(\mathbb{X})$*.*

Proof. By point 3 of Lemma 1.3, identities are total maps. By point 4, if $f : A \to B$ and $g : B \to C$ are total maps, then

$$\overline{g \circ f} = \overline{\overline{g} \circ f} = \overline{\mathrm{id}_B \circ f} = \overline{f} = \mathrm{id}_A\,,$$

and $g \circ f : A \to C$ is a total map. □

Definition 1.7. Let $\mathbb{X}$ be a restriction category and let $f, g : A \to B$ maps in $\mathbb{X}$. We write $f \leq g$ if $g \circ \overline{f} = f$, *i.e.*, if the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle \overline{f}}\searrow \quad \nearrow{\scriptstyle g} & \\ & A & \end{array} \tag{2}$$

In the case $A = B$ observe that $\overline{f} \leq \mathrm{id}_A$ for every $f : A \to A$.

Example 1.8. Let $\mathbb{X} = \mathbf{Par}$. Then $f \leq g$ if and only if g is an extension of f.

Lemma 1.9. *The relation introduced in Definition 1.7 is a partial order. Moreover, restriction is monotone with respect to such partial order, i.e., if $f \leq g$ then $\overline{f} \leq \overline{g}$.*

Proof. Reflexivity follows by putting $g = f$ and applying the first rule of restriction categories. For transitivity, if $f \leq g$ and $g \leq h$, then

$$h \circ \overline{f} = h \circ \overline{g \circ \overline{f}} = h \circ \overline{g} \circ \overline{f} = g \circ \overline{f} = f\,,$$

so that $f \leq h$ as well. For antisymmetry, if $f \leq g$ and $g \leq f$, then

$$f = g \circ \overline{f} = f \circ \overline{g} \circ \overline{f} = f \circ \overline{f} \circ \overline{g} = f \circ \overline{g} = g\,.$$

Finally, if $g \circ \overline{f} = f$ then $\overline{g} \circ \overline{\overline{f}} = \overline{g} \circ \overline{f} = \overline{g \circ \overline{f}} = \overline{f}$. □

Recall that a binary relation $\mathcal{R}$ defined on a partial monoid M is *enriching* if, for every $f, g, h, k \in M$ such that hfk and hgk are defined, if $f \,\mathcal{R}\, g$ then $hfk \,\mathcal{R}\, hgk$.

Lemma 1.10. *The ordering from Definition 1.7 is enriching.*

Proof. A relation on maps is enriching if $f \,\mathcal{R}\, g$ implies $(k \circ f \circ h) \,\mathcal{R}\, (k \circ g \circ h)$ whenever $f, g \in \mathbb{X}(A, B)$, $h \in \mathbb{X}(Z, A)$, and $k \in \mathbb{X}(B, C)$ as $Z, A, B, C \in |\mathbb{X}|$. We will prove separately the two cases $h = \mathrm{id}_A$ and $k = \mathrm{id}_B$, from which the general case follows easily.

First, suppose $C = B$ and $k = \mathrm{id}_B$: if $g \circ \overline{f} = f$, then by rule 4 of restriction categories $g \circ h \circ \overline{f \circ h} = g \circ \overline{f} \circ h = f \circ h$. Thus, if $f \leq g$ then $f \circ h \leq g \circ h$.

Next, suppose $Z = A$ and $h = \mathrm{id}_A$: if $g \circ \overline{f} = f$, then

$$\begin{aligned} k \circ g \circ \overline{k \circ f} &= k \circ g \circ \overline{k \circ f \circ \overline{f}} \ \text{ by Rule 1} \\ &= k \circ g \circ \overline{k \circ f} \circ \overline{f} \ \text{ by Rule 3} \\ &= k \circ g \circ \overline{f} \circ \overline{k \circ f} \ \text{ by Rule 2} \\ &= k \circ f \circ \overline{k \circ f} \\ &= k \circ f \ \text{ by Rule 1}\,. \end{aligned}$$

Thus, if $f \leq g$ then $k \circ f \leq k \circ g$. □

Corollary 1.11. *Let $f : A \to B$, $h : A \to C$, $k : B \to D$. Then $\overline{k} \circ f \circ \overline{h} \leq f$.*

Proof. Follows from Lemma 1.10 and $\overline{h} \leq \mathrm{id}_A$, $\overline{k} \leq \mathrm{id}_B$. □

Definition 1.12 (Compatible maps)**.** Two maps $f, g : A \to B$ in a restriction category $\mathbb{X}$ are *compatible*, written $f \smile g$, if $g \circ \overline{f} = f \circ \overline{g}$.

Example 1.13. Two maps in **Par** are compatible if and only if they are equal on the intersection of their domains.

Lemma 1.14. *Let $f, g : A \to B$, $h : Z \to A$, $k : B \to C$ be maps in a restriction category $\mathbb{X}$.*

1. *$f \smile g$ if and only if $g \circ \overline{f} \leq f$ and $f \circ \overline{g} \leq g$. In fact, if either inequality holds, then $f \smile g$, and the other one holds too.*
2. *If $f \smile g$ then $k \circ f \circ h \smile k \circ g \circ h$.*

Proof. If $f \smile g$, then

$$f \circ \overline{g \circ \overline{f}} = f \circ \overline{g} \circ \overline{f} = f \circ \overline{f}\,\overline{g} = f \circ \overline{g} = g \circ \overline{f}\,,$$

that is, $g \circ \overline{f} \leq f$; similarly, $f \circ \overline{g} \leq g$. On the other hand, if $g \circ \overline{f} \leq f$, then

$$g \circ \overline{f} = f \circ \overline{g \circ \overline{f}} = f \circ \overline{g} \circ \overline{f} = f \circ \overline{f} \circ \overline{g} = f \circ \overline{g}\,,$$

that is, $f \smile g$; similarly if $f \circ \overline{g} \leq g$. This proves point 1.

Point 2 is so tricky that we provide two proofs. Recall that $f \smile g$ means $g \circ \overline{f} = f \circ \overline{g}$.

- *First proof:* (by James Chapman) On the one hand,

$$\begin{aligned} k \circ g \circ h \circ \overline{k \circ f \circ h} &= k \circ g \circ \overline{k \circ f} \circ h \\ &= k \circ g \circ \overline{g} \circ \overline{k \circ f} \circ h \\ &= k \circ g \circ \overline{k \circ f} \circ \overline{g} \circ h \\ &= k \circ g \circ \overline{k \circ f \circ \overline{g}} \circ h \\ &= k \circ g \circ \overline{k \circ g \circ \overline{f}} \circ h \\ &= k \circ g \circ \overline{k \circ g} \circ \overline{f} \circ h \\ &= k \circ g \circ \overline{f} \circ h\ : \end{aligned}$$

  on the other hand,

$$\begin{aligned} k \circ f \circ h \circ \overline{k \circ g \circ h} &= k \circ f \circ \overline{k \circ g} \circ h \\ &= k \circ f \circ \overline{f} \circ \overline{k \circ g} \circ h \\ &= k \circ f \circ \overline{k \circ g} \circ \overline{f} \circ h \\ &= k \circ f \circ \overline{k \circ g \circ \overline{f}} \circ h \\ &= k \circ f \circ \overline{k \circ f \circ \overline{g}} \circ h \\ &= k \circ f \circ \overline{k \circ f} \circ \overline{g} \circ h \\ &= k \circ f \circ \overline{g} \circ h\,, \end{aligned}$$

  and it follows from the hypothesis that $k \circ g \circ \overline{f} \circ h = k \circ f \circ \overline{g} \circ h$.

- *Second proof:* (by Robin Cockett) We prove separately that, if $f \smile g$, then $f \circ h \smile g \circ h$ and $k \circ f \smile k \circ g$: from which the general case follows immediately. In fact, if $g \circ \overline{f} = f \circ \overline{g}$, then

$$g \circ h \circ \overline{f \circ h} = g \circ \overline{f} \circ h = f \circ \overline{g} \circ h = f \circ h \circ \overline{g \circ h}\,,$$

  and

$$\begin{array}{rcl}
k \circ g \circ \overline{k \circ f} & = & k \circ g \circ \overline{g} \circ \overline{k \circ f \circ \overline{f}} \\
& = & k \circ g \circ \overline{g} \circ \overline{k \circ f} \circ \overline{f} \\
& = & k \circ g \circ \overline{f} \circ \overline{g} \circ \overline{k \circ f} \\
& = & k \circ f \circ \overline{g} \circ \overline{g} \circ \overline{k \circ f} \\
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& = & k \circ f \circ \overline{g} \circ \overline{k \circ f \circ \overline{g}} \\
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& = & k \circ f \circ \overline{k \circ g} \circ \overline{g} \circ \overline{f} \\
& = & k \circ f \circ \overline{k \circ g \circ \overline{g}} \circ \overline{f} \\
& = & k \circ f \circ \overline{k \circ g} \circ \overline{f} \\
& = & k \circ f \circ \overline{f} \circ \overline{k \circ g} \\
& = & k \circ f \circ \overline{k \circ g}\,.
\end{array}$$

□

In the last chain of equalities, the leftmost k is only a placeholder that never enters the transformations. We thus get

Corollary 1.15. *Let $\mathbb{X}$ be a restriction category. For every $A, B, C \in |\mathbb{X}|$, $f, g \in \mathbb{X}(A, B)$, and $k \in \mathbb{X}(B, C)$, if $g \circ \overline{f} = f \circ \overline{g}$, then $g \circ \overline{k \circ f} = f \circ \overline{k \circ g}$.*

2 Examples of restriction categories

Example 2.1 (The trivial restriction)**.** Let $\mathbb{X}$ be *any* category and let $\overline{f} = \mathrm{id}_A$ for every $f \in \mathbb{X}(A, B)$.

Example 2.1 tells that being a restriction category is additional structure, not a limitation.

Example 2.2 (The restriction monoid)**.** Let $\mathbb{X}$ have $\mathbb{N}$ as its only object and the partial recursive functions as maps. Define restriction as in **Par**, *i.e.*, if f is a partial recursive function, then $\overline{f}(n)$ is n if $f(n)$ is defined, and undefined otherwise.

Example 2.3 (Meet semilattices)**.** Let $\mathbb{X}$ be a meet semilattice, *i.e.*, a set with a binary operation $\wedge$ satisfying $x \wedge y = y \wedge x$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, and $x \wedge x = x$. Then $(\mathbb{X}, \wedge)$ is easily given a monoid structure, thus making it a category: define restriction by $\overline{x} = x$. The axioms of restricted category follow then directly from those of meet semilattice.

Example 2.4 (Trunked trees)**.** Let $G = (V, E)$ be a directed graph. Consider the *path category* (or *free category*) on G, whose objects are the nodes of G, and whose maps are the finite directed paths from a node to the other, with identities being the empty paths. We can then indicate the maps in $\mathbf{Path}(G)$ as $(A, [a_1, \ldots, a_n], B)$; the composition of $(A, [a_1, \ldots, a_n], B)$ and $(B, [b_1, \ldots, b_m], C)$ will be $(A, [a_1, \ldots, a_n, b_1, \ldots, b_m], C)$. In a similar way, the *trunked tree* on G is defined as the category $\mathbf{TrunkT}(G)$ whose objects are the nodes of G, and where a map from A to B is a tuple $(A, (p, P), B)$ where

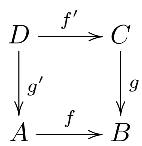
- p is a finite path in G, and
- P is a *finite, prefix-closed* set of finite paths on G that contains p.

Then $(A, (p, P), B)(B, (q, Q), C) = (A, (pq, P \cup PQ), C)$, and $\mathrm{id}_A = (A, ([], \{[]\}), A)$. The restriction of $(A, (p, P), B)$ is defined as $(A, ([], P), A)$: thus, restriction in trunked trees is not trivial. The total maps of $\mathbf{TrunkT}(G)$ are those where P is reduced to $\{[]\}$, *i.e.*, the identities.

We recall the definition of pullback.

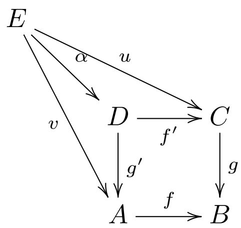
Definition 2.5 (Pullback)**.** Let $f : A \to B$ and $g : C \to B$ be two convergent arrows in a category $\mathbb{X}$. A *pullback of f along g* is a map $f' : D \to C$, for which a map $g' : D \to A$ exists such that the following hold:

1. $g \circ f' = f \circ g'$, *i.e.*, the following diagram commutes:



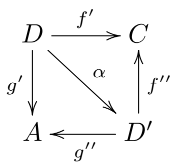
(3)

2. For every object E and maps $u : E \to C$, $v : E \to A$ such that $g \circ u = f \circ v$ there exists a unique map $\alpha : E \to D$ such that $f' \circ \alpha = u$ and $g' \circ \alpha = v$, *i.e.*, the following diagram commutes:



(4)

Pullbacks are unique, up to the following equivalence: $f' \sim f''$ if and only if there is an isomorphism α such that the following diagram commutes:

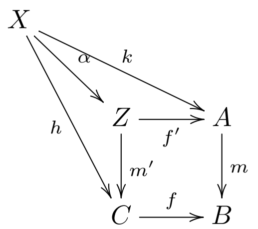


(5)

Lemma 2.6. *Let $\mathbb{X}$ be a category and let $m \in \mathbb{X}(A, B)$.*

1. *If m is monic, then every pullback of m along any map is monic.*
2. *m is monic if and only if* $\begin{array}{ccc} A & = & A \\ \| & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{m} & B \end{array}$ *is a pullback diagram.*

Proof. Let m be monic and let $\begin{array}{ccc} Z & \xrightarrow{f'} & A \\ \downarrow{\scriptstyle m'} & & \downarrow{\scriptstyle m} \\ C & \xrightarrow{f} & B \end{array}$ be a pullback; let then $u, v : X \to Z$ satisfy $m' \circ u = m' \circ v$. Then $m \circ f' \circ u = f \circ m' \circ u = f \circ m' \circ v = m \circ f' \circ v$, so $f' \circ u = f' \circ v$ as m is monic. As m' is a pullback, for $h = m' \circ u = m' \circ v$ and $k = f' \circ u = f' \circ v$, there exists a unique map $\alpha : X \to Z$ such that the diagram



commutes: which implies $u = \alpha = v$.

Now, surely $\begin{array}{ccc} A & = & A \\ \| & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{m} & B \end{array}$ commutes. If m is monic and $u, v : Z \to A$ also make $\begin{array}{ccc} Z & \xrightarrow{v} & A \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{m} & B \end{array}$ commute, then $u = v$ by monicness of m, and clearly $\begin{array}{ccc} A & = & A \\ \| & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{m} & B \end{array}$ is a pullback. On the other hand, if the latter is a pullback and $m \circ u = m \circ v$, then $u = \alpha \, \mathrm{id}_A = v$ for a unique $\alpha : Z \to A$: which is only possible for $u = \alpha = v$. □

Lemma 2.7. *Let*

$$\begin{array}{ccccc} X & \xrightarrow{h'} & D & \xrightarrow{f'} & A \\ \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{h} & B & \xrightarrow{f} & C \end{array} \tag{6}$$

be a commutative diagram.

1. *If the inner squares in (6) are pullback diagrams, then so is the outer rectangle.*

That is: if g' is the pullback of g along f, and g'' is the pullback of g' along h, then g'' is the pullback of g along $f \circ h$.

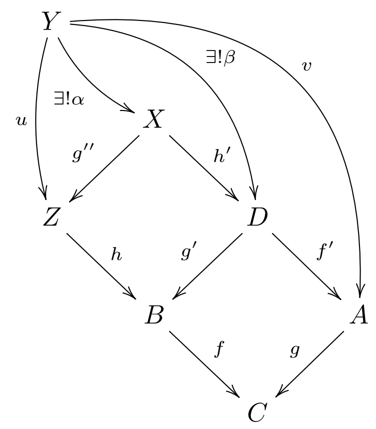
2. *If the outer rectangle and the rightmost square in (6) are pullback diagrams, then so is the leftmost square.*

   That is: if g' is the pullback of g along f, and g'' is the pullback of g along $f \circ h$, then g'' is the pullback of g' along h.

Point 1 can be stated as such: the pullback of a pullback is a pullback.

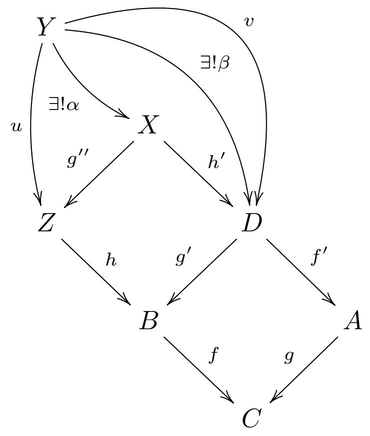
Proof. First, suppose that g' is the pullback of g along f, and g'' is the pullback of g' along h. Suppose that there exist an object Y and two maps $u : Y \to Z$, $v : Y \to A$ such that $f \circ h \circ u = g \circ v$. As g' is the pullback of g along f, there exists a unique $\beta : Y \to D$ such that $v = f \circ \beta$ and $h \circ u = g' \circ \beta$; as g'' is the pullback of g' along h, there exists a unique $\alpha : Y \to X$ such that $u = g'' \circ \alpha$ and $\beta = h' \circ \alpha$, which in turn implies $v = f \circ \beta = f' \circ h' \circ \alpha$. This proves that g'' is the pullback of g along $f \circ h$.

The situation is summarized by the following commutative diagram:



Next, suppose that g' is the pullback of g along f, and that g'' is the pullback of g along $f \circ h$. Consider an object Y and two maps $u : Y \to Z$, $v : Y \to D$ such that $h \circ u = g' \circ v$: then $g \circ f' \circ v = f \circ h \circ u$ as well. As g'' is the pullback of g along $f \circ h$, there exists a unique $\alpha : Y \to X$ such that $u = g'' \circ \alpha$ and $f' \circ v = f' \circ h' \circ \alpha$; as g' is the pullback of g along f, there exists a unique $\beta : Y \to D$ such that $h \circ u = g' \circ \beta$ and $f' \circ v = f' \circ \beta$. Then $\beta = v = h' \circ \alpha$, which proves that g'' is the pullback of g' along f.

The situation is summarized by the following commutative diagram:



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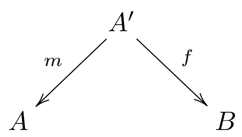
We recall that $\mathrm{Hom}_{\mathbb{X}} = \bigcup_{A,B \in |\mathbb{X}|} \mathbb{X}(A, B)$.

Definition 2.8 (Stable set of monics)**.** A family $\mathcal{M} \subseteq \mathrm{Hom}_{\mathbb{X}}$ is a *stable set of monics* if it satisfies the following four properties:

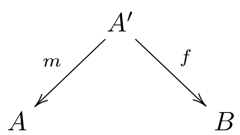
1. Every isomorphism of $\mathbb{X}$ belongs to $\mathcal{M}$; in particular, $\mathrm{id}_A \in \mathcal{M}$ for every $A \in |\mathbb{X}|$.
2. Every $m \in \mathcal{M}$ is monic.
3. $\mathcal{M}$ is closed by composition.
4. For every $m \in \mathcal{M}$, the pullbacks of m along any f exist and belong to $\mathcal{M}$.

Definition 2.9 (Partial map category)**.** Let $\mathbb{X}$ be a category and let $\mathcal{M} \subseteq \mathrm{Hom}_{\mathbb{X}}$ be a stable set of monics. The *partial map category* on $\mathbb{X}$ by $\mathcal{M}$ is the category $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ defined as follows:

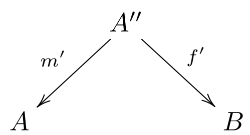
- The objects of $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ are the objects of $\mathbb{X}$.
- The maps of $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ from A to B are the spans



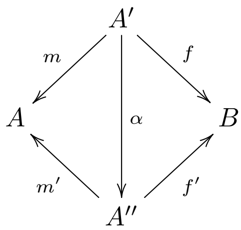
in $\mathbb{X}$ from A to B with $m \in \mathcal{M}$, modulo the following equivalence relation:



is equivalent to



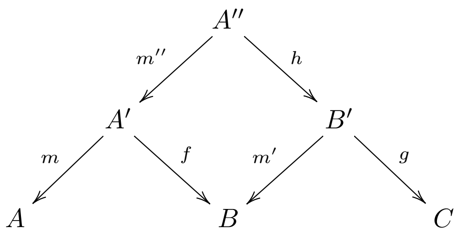
if and only if there exists an isomorphism $\alpha : A' \to A''$ such that $m = m' \circ \alpha$ and $f = f' \circ \alpha$, *i.e.*, the diagram



(7)

commutes.[1]

- The identities of $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ are the upper left corners of the form $(\mathrm{id}_A, \mathrm{id}_A)$.
- The composition of $(m, f) : A \to B$ with $(m', g) : B \to C$ is defined (up to equivalence) as $(m \circ m'', g \circ h)$, where h is the pullback of f along m' and $m'' : A'' \to A'$ is the corresponding map:

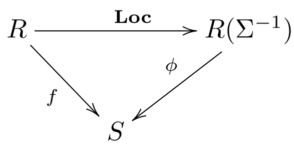


(8)

That $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ is indeed a category, follows from Lemma 2.7 and uniqueness of pullbacks up to equivalence.

Example 2.10. Let $\mathbb{X} = \mathbf{Set}$ and let $\mathcal{M}$ be the family of all injective functions. Then $\mathbf{Par}(\mathbb{X}, \mathcal{M}) = \mathbf{Par}$.

Example 2.11. Let R be a commutative ring; let $\Sigma \subseteq R$ not contain the zero, and let $R(\Sigma^{-1})$ be the smallest commutative ring that contains R and where every element of Σ has a multiplicative inverse. The embedding of R into $R(\Sigma^{-1})$ is called *localization*, and indicated by **Loc**: it is known that, if $f : R \to S$ is a ring homomorphism such that $f(x)$ is a unit for every $x \in \Sigma$, then there is a unique $\phi : R(\Sigma^{-1}) \to S$ such that the diagram



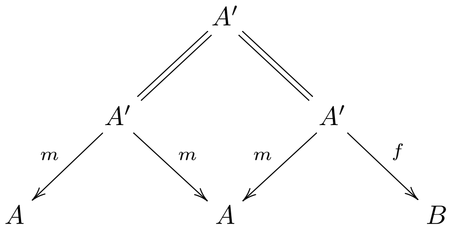
commutes. Moreover, **Loc** is an *epic* map. It turns out that the category $\mathbf{Par}(\mathbf{CRing}^{\mathrm{op}}, \mathbf{Loc})$ is the opposite category of *commutative rings with rational functions.*

Example 2.12. Let $\mathbb{X} = \mathbf{Top}$ and let $\mathcal{M}$ be the family of *continuous pushouts of open sets*: that is, a map $m \in \mathcal{M}$ from X to Y is a continuous function from an open subset U of X into Y. Then $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ is a restriction category.

[1] The definition in the February 27, 2013 version of these notes incorrectly assumed the spans to be pullbacks. Thanks to James Chapman for pointing out this error.

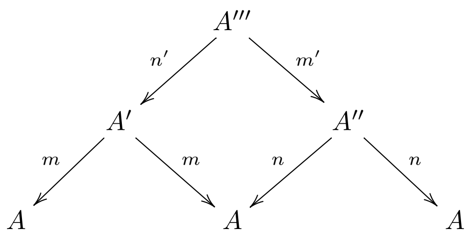
Theorem 2.13. $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ *is a restriction category, where the restriction of* (m, f) *is* (m, m).

Proof. If (m, f) is a map in $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ from A to B, then monicness of m allows



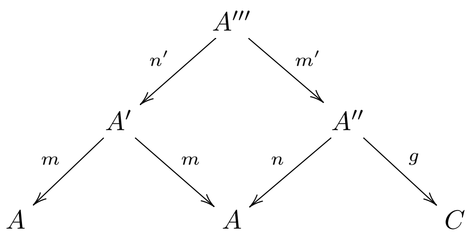
so that $(m, f) \circ (m, m) = (m, f)$: thus, $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ satisfies Rule 1.

If $(m, f) : A \to B$ and $(n, g) : A \to C$ in $\mathbf{Par}(\mathbb{X}, \mathcal{M})$, then



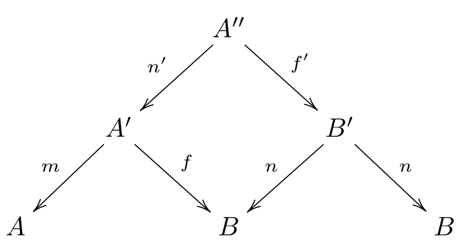
can be read indifferently from left to right, or from right to left: hence, $(n, n) \circ (m, m) = (m, m) \circ (n, n)$. This proves that Rule 2 holds in $\mathbf{Par}(\mathbb{X}, \mathcal{M})$.

As pullbacks are unique modulo equivalence, the diagram for $\overline{(n, g)} \circ \overline{(m, f)}$ says that $(n, g) \circ \overline{(m, f)}$ can be constructed as

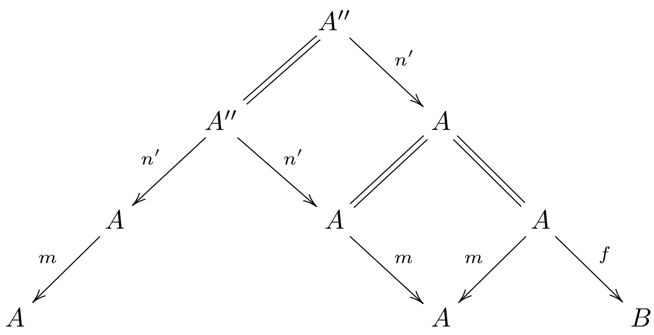


whose restriction is $(m \circ n', m \circ n') = (m \circ n', n \circ m')$, which we know to be $\overline{(n, g)} \circ \overline{(m, f)}$. Thus, $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ satisfies Rule 3.

Finally, let $(m, f) : A \to B$ and $(n, g) : B \to C$. Then $\overline{(n, g)} \circ (m, f)$ is given by



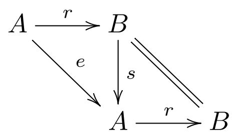
By Lemma 2.6 and Lemma 2.7, we can write $(m, f) \circ \overline{(n, g) \circ (m, f)}$ as follows:



which yields $(m, f) \circ \overline{(n, g) \circ (m, f)} = (m \circ n', f \circ n')$: but $f \circ n' = n \circ f'$ by the diagram for $\overline{(n, g) \circ (m, f)}$, which proves that $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ satisfies Rule 4. $\square$

3 Idempotents

Definition 3.1 (Split)**.** Let $\mathbb{X}$ be a category and let $e : A \to A$ be an *idempotent*, *i.e.*, $e^2 = e \circ e = e$. We say that e *splits* if there exist an object B, a monomorphism $s : B \to A$, and an epimorphism $r : A \to B$ such that $s \circ r = e$ and $r \circ s = \mathrm{id}_B$, *i.e.*, the diagram

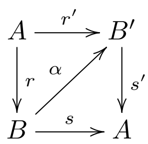


(9)

commutes. We call the map s a *split* of e, and the equality $e = s \circ r$ a *splitting* of e.

Observe that r is a *retraction i.e.*, an epimorphism with a one-side converse; dually, s is a *section*. Also observe that, if e has such a decomposition, then e is an idempotent.

Lemma 3.2. *Let $e : A \to A$ be an idempotent and let $e = s \circ r = s' \circ r'$ be two splittings of e. Then there exists a unique isomorphism $\alpha : B \to B'$ such that $s = s' \circ \alpha$ and $r' = \alpha \circ r$, i.e., the diagram*



(10)

commutes.

Proof. Set $\alpha = r' \circ s$. Then $r \circ s' = \alpha^{-1}$, as

$$r \circ s' \circ r' \circ s = r \circ e \circ s = r \circ s \circ r \circ s = \mathrm{id}_B \circ \mathrm{id}_B = \mathrm{id}_B$$

and similarly $r' \circ s \circ r \circ s' = \mathrm{id}_{B'}$. Monicness of s' ensures uniqueness of α. $\square$

Lemma 3.2 states that splittings of idempotents are unique, up to (a reasonable notion of) an isomorphism. Therefore, in the rest of these notes, we will indicate $e = s \circ r$ as "the" splitting of e.

Recall that the *equalizer* of $f, g : A \to B$ is a map $h : Z \to A$ such that $f \circ h = g \circ h$, and that, for every $k : Y \to A$ such that $f \circ k = g \circ k$, there exists a unique $\alpha : Y \to Z$ such that $k = h \circ \alpha$. Dually, the *coequalizer* of $f, g : A \to B$ is a map $p : B \to C$ such that $p \circ f = p \circ g$ and that, for every $q : B \to D$ such that $q \circ f = q \circ g$, there exists a unique $\beta : C \to D$ such that $q = \beta \circ p$. Equalizers and coequalizers are unique, up to isomorphisms. It follows from the definition that equalizers are monic, and coequalizers are epic.

Corollary 3.3. *If $e = s \circ r$ is the splitting of an idempotent $e : A \to A$, then s is the equalizer of e and id_A, and r is the coequalizer of e and id_A.*

Proof. Clearly, $e \circ s = s \circ r \circ s = s \circ \mathrm{id}_B = s = s \circ \mathrm{id}_A$. If $h : Z \to A$ also satisfies $e \circ h = h$, then $\alpha = r \circ h$ satisfies $s \circ \alpha = s \circ r \circ h = e \circ h = h$; also, if $h = s \circ \alpha'$, then $s \circ \alpha' = e \circ h = s \circ r \circ h$, and $\alpha' = r \circ h = \alpha$ by monicness of s.

Dually, $r \circ e = r \circ s \circ r = \mathrm{id}_B \circ r = r = r \circ \mathrm{id}_A$, and if $k : A \to C$ also satisfies $k \circ e = k \circ \mathrm{id}_A$, then $\beta = k \circ s$ is the unique map from B to C such that $k = \beta \circ r$. □

In arbitrary categories, idempotents need not split. However, it is always possible to split idempotents.

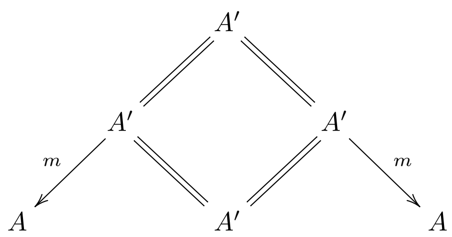
Definition 3.4. Let $\mathbb{X}$ be an arbitrary category and let E be a class of idempotents of $\mathbb{X}$ such that $\mathrm{id}_A \in E$ for every $e \in E \cap \mathbb{X}(A, A)$. The category $\mathbf{Split}_E(\mathbb{X})$ is defined as follows:

- The objects of $\mathbf{Split}_E(\mathbb{X})$ are the elements of E.
- A map in $\mathbf{Split}_E(\mathbb{X})$ from $e : A \to A$ to $e' : A' \to A'$, is a map $f : A \to A'$ in $\mathbb{X}$ such that $e' \circ f \circ e = f$.
- Composition is defined as in $\mathbb{X}$.
- The identity id_e of $e : A \to A$ in $\mathbf{Split}_E(\mathbb{X})$ is the map e of $\mathbb{X}$.

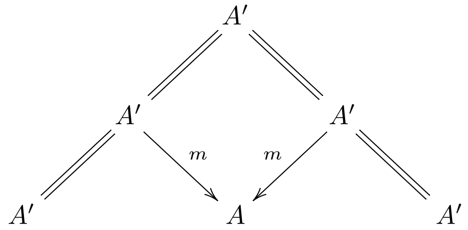
Then all the elements of E are split in $\mathbf{Split}_E(\mathbb{X})$. If $\mathbb{X}$ is a restriction category, the restriction of $f : e \to e'$ in $\mathbf{Split}_E(\mathbb{X})$ is the map $\overline{f} \circ e$ of $\mathbb{X}$.

Example 3.5. Let $\mathbb{X}$ be a category and let $\mathcal{M}$ be a stable set of monics for $\mathbb{X}$. Then every restriction in $\mathbf{Par}(\mathbb{X}, \mathcal{M})$ splits.

To see this, let $(m, f) : A \to B$ be a map in $\mathbf{Par}(\mathbb{X}, \mathcal{M})$—*i.e.*, let $m : A' \to A$ and $f : A' \to B$ constitute the pullback of a monic map—and let (m, m) be its restriction. Put $s = (\mathrm{id}_{A'}, m)$ and $r = (m, \mathrm{id}_{A'})$: then the pullback diagrams



and



show that indeed $s \circ r = (m, m)$ and $r \circ s = (\mathrm{id}_{A'}, \mathrm{id}_{A'})$.

Definition 3.6 (Split restriction category)**.** A restriction category where every restriction map splits is called a *split restriction category.*

Let $\mathbb{X}$ be a split restriction category. We call $\mathcal{M}_{\mathbb{X}}$ the class of splits of restrictions: that is, $m : A \to B$ belongs to $\mathcal{M}_{\mathbb{X}}$ if and only if there exist a map $f : B \to A$ and an epic map $r = r_m : B \to A$ such that $m \circ r = \overline{f}$ and $r \circ m = \mathrm{id}_A$. Observe that, in this case, $\overline{m \circ r} = \overline{\overline{f}} = \overline{f} = m \circ r$: but then, $m \circ r = \overline{m \circ r} = \overline{\overline{m} \circ r} = \overline{r}$ as m is monic.

In words: if $\overline{f} = m \circ r$ is the splitting of a restriction, then restricting the retraction is the same as postcomposing it with the split.

Let us consider some properties of $\mathcal{M}_{\mathbb{X}}$. First of all, it is a class of monic maps of $\mathbb{X}$—actually, of $\mathbf{Total}(\mathbb{X})$, because monic maps are total. Also, as every isomorphism is clearly the split of (the restriction of) an identity, $\mathcal{M}_{\mathbb{X}}$ contains every isomorphism. Moreover, if $\overline{f} = m \circ r_m$ and $\overline{g} = n \circ r_n$ are splittings of restrictions, then $n \circ m$ is the split of a restriction idempotent and $r_m \circ r_n$ is the corresponding retraction, as

$$
\begin{aligned}
n \circ m \circ r_m \circ r_n &= n \circ \overline{r_m} \circ r_n \\
&= n \circ r_n \circ \overline{r_m \circ r_n} \\
&= \overline{r_n} \circ \overline{r_m \circ r_n} \\
&= \overline{r_m \circ r_n} \circ \overline{r_n} \\
&= \overline{r_m \circ r_n \circ \overline{r_n}} \\
&= \overline{r_m \circ r_n}
\end{aligned}
$$

and $r_m \circ r_n \circ n \circ m = r_m \circ \mathrm{id}_B \circ m = r_m \circ m = \mathrm{id}_A$.

So, $\mathcal{M}_{\mathbb{X}}$ is close to being a stable set of monics. The main obstacle to this is that, in general, elements of $\mathcal{M}_{\mathbb{X}}$ might not have pullbacks along arbitrary maps of $\mathbb{X}$. Notably, by relaxing this condition a little bit, we get

Theorem 3.7. *If $\mathbb{X}$ is a* split *restriction category, then $\mathcal{M}_{\mathbb{X}}$ is a stable set of monics in* $\mathbf{Total}(\mathbb{X})$*.*

Proof. We must prove that, if $m : A \to B$ is the split of a restriction idempotent $\overline{u} = e = e_m : B \to B$ of $\mathbb{X}$, $r = r_m$ is the corresponding retraction, and $f : C \to B$ is a total map in $\mathbb{X}$, then the pullback $m' : D \to C$ of m along f exists, and is the split of a restriction idempotent of $\mathbb{X}$.

We observe that $e' = \overline{e \circ f}$ is a restriction idempotent: by hypothesis, e' has a splitting $e' = m' \circ r'$ with $m' : D \to C$ monic. Then

$$f' = r \circ f \circ m' \tag{11}$$

is such that $m \circ f' = f \circ m'$, as

$$
\begin{aligned}
m \circ f' &= m \circ r \circ f \circ m' \\
&= e \circ f \circ m' \\
&= \overline{e} \circ f \circ m' \\
&= f \circ \overline{e \circ f} \circ m' \\
&= f \circ e' \circ m' \\
&= f \circ m' \circ r' \circ m' \\
&= f \circ m' \circ \mathrm{id}_D \\
&= f \circ m' \,.
\end{aligned}
$$

We then only need to prove:

1. that f' is a total map, and
2. that m' is a pullback in $\mathbf{Total}(\mathbb{X})$.

For point 1, by monicness of m and the fact that $\overline{e} = e$ we have:

$$
\begin{aligned}
\overline{f'} &= \overline{r \circ f \circ m'} \\
&= \overline{\overline{m} \circ r \circ f \circ m'} \\
&= \overline{r \circ \overline{m \circ r} \circ f \circ m'} \\
&= \overline{r \circ e \circ f \circ m'} \\
&= \overline{\overline{\overline{r} \circ e \circ f} \circ m'} \ :
\end{aligned}
$$

but r is the retraction in the splitting of $e = m \circ r$, so $\overline{r} = m \circ r = e$, and

$$
\begin{aligned}
\overline{f'} &= \overline{\overline{e \circ e \circ f} \circ m'} \\
&= \overline{\overline{e \circ f} \circ m'} \\
&= \overline{m' \circ r' \circ m'} \\
&= \overline{m'} \\
&= \mathrm{id}_D \,.
\end{aligned}
$$

as m' is monic too.

For point 2, suppose $x : X \to C$ and $y : X \to A$ satisfy $f \circ x = m \circ y$. As m' is monic, if $\alpha : X \to D$ exists such that $m' \circ \alpha = x$ and $f' \circ \alpha = y$, then it is unique. Our candidate is thus $\alpha = r' \circ x$, for which we prove the second equation by showing that $m \circ f' \circ r' \circ x = m \circ y$ and exploiting monicness of m:

$$
\begin{aligned}
m \circ f' \circ r' \circ x &= m \circ r \circ f \circ m' \circ r' \circ x \\
&= e \circ f \circ \overline{e \circ f} \circ x \\
&= e \circ f \circ x \\
&= m \circ r \circ m \circ y \\
&= m \circ y \,.
\end{aligned}
$$

Given this, we get

$$
\begin{aligned}
m' \circ r' \circ x &= e' \circ x \\
&= x \circ \overline{e' \circ x} \\
&= x \circ \overline{\overline{e \circ f} \circ x} \\
&= x \circ \overline{e \circ f \circ x} \\
&= x \circ \overline{e \circ m \circ y} \ :
\end{aligned}
$$

but $e \circ m \circ y = m \circ r \circ m \circ y = m \circ y$, thus

$$
\begin{aligned}
m' \circ r' \circ x &= x \circ \overline{m \circ y} \\
&= x \circ \overline{f \circ x} \\
&= \overline{f} \circ x \\
&= x
\end{aligned}
$$

as f is total. □

Theorem 3.8 (The Completeness Theorem)**.** *Let* $\mathbb{X}$ *be a split restriction category. Then*

$$\mathbb{X} \cong \mathbf{Par}\left(\mathbf{Total}(\mathbb{X}), \mathcal{M}_{\mathbb{X}}\right) \tag{12}$$

via the equivalence F *that sends every object into itself, and every* $f \in \mathrm{Hom}(A, B)$ *into*

$$Ff = (m_{\overline{f}}, f \circ m_{\overline{f}}) \in \mathbf{Par}(\mathbf{Total}(\mathbb{X}), \mathcal{M}_{\mathbb{X}})(A, B) \tag{13}$$

where $\overline{f} = m_{\overline{f}} \circ r_{\overline{f}}$ *is the splitting. Moreover,*

$$\overline{Ff} = F\overline{f} \ \ \forall f \in \mathrm{Hom}_{\mathbb{X}} \,. \tag{14}$$

That is: every split restriction category is a full subcategory of a partial map category, up to equivalence.

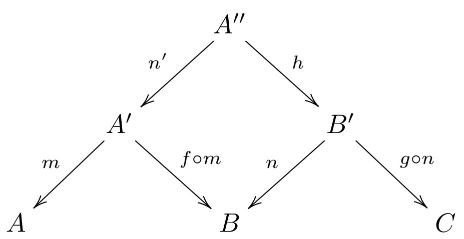
Proof. For brevity and clarity, let us put $m = m_{\overline{f}}$ and $r = r_{\overline{f}}$.

Observe that $f \circ m : A' \to B$ is total, because

$$\overline{f \circ m} = \overline{\overline{f} \circ m} = \overline{m \circ r \circ m} = \overline{m} = \mathrm{id}_{A'} \,.$$

If $f \in \mathrm{Hom}(A, B)$, then $A \xrightarrow{r_{\overline{f}}} A' \xrightarrow{f \circ m_{\overline{f}}} B$: but $f \circ m_{\overline{f}} \circ r_{\overline{f}} = f \circ \overline{f} = f$.

To show that F is actually a functor, let $Ff = (m, f \circ m)$ and $Fg = (n, g \circ n)$, where $\overline{g} = m_{\overline{g}} \circ r_{\overline{g}} = n \circ s$ is the splitting: then $(Ff)(Fg) = (m \circ n', g \circ n \circ h)$, where n' is the pullback of n along f and h is the pullback of f along n':



Now, we know from the proof of Theorem 3.7 that n' is the split of $\overline{\overline{g} \circ f \circ m} = \overline{\overline{\overline{g} \circ f} \circ m} = \overline{\overline{g \circ f} \circ m}$: if s' is the retraction corresponding to n', then

$$
\begin{array}{rcl}
m \circ n' \circ s' \circ r & = & m \circ \overline{\overline{g \circ f} \circ m} \circ r \\
& = & \overline{g \circ f} \circ m \circ r \\
& = & \overline{g \circ f} \circ \overline{f} \\
& = & \overline{g \circ f \circ \overline{f}} \\
& = & \overline{g \circ f} \,,
\end{array}
$$

so that $m \circ n'$ is indeed the split of $\overline{g \circ f}$ (and $s' \circ r$ is the corresponding retraction). But $h = s \circ f \circ m \circ n'$, so that

$$
\begin{array}{rcl}
g \circ n \circ h & = & g \circ n \circ s \circ f \circ m \circ n' \\
& = & g \circ \overline{g} \circ f \circ m \circ n' \\
& = & g \circ f \circ m \circ n' \ :
\end{array}
$$

thus, $(m \circ n', g \circ n \circ h) = (m \circ n', g \circ f \circ m \circ n')$, *i.e.*, $Fg \circ Ff = F(g \circ f)$. That $F\mathrm{id}_A = (\mathrm{id}_A, \mathrm{id}_A)$ for every object A follows immediately from the definitions.

The inverse functor is defined by sending $(m, g) \in \mathbf{Par}(\mathbf{Total}(\mathbb{X}), \mathcal{M}_{\mathbb{X}})(A, B)$ into $g \circ r \in \mathbb{X}(A, B)$, where $m \circ r = \overline{f}$ is the splitting of a restriction. In fact, $(m, f \circ m)$ is sent to $f \circ m \circ r = f \circ \overline{f} = f$, while $g \circ r$ is sent to $(m, g \circ r \circ m) = (m, g)$.

Finally, for every $f \in \mathrm{Hom}(A, B)$ we have

$$
\begin{array}{rcl}
\overline{Ff} & = & (m_{\overline{f}}, m_{\overline{f}}) \\
& = & (m_{\overline{f}}, m_{\overline{f}} \circ r_{\overline{f}} \circ m_{\overline{f}}) \\
& = & (m_{\overline{\overline{f}}}, \overline{f} \circ m_{\overline{\overline{f}}}) \\
& = & (m_{\overline{f}}, \overline{f} \circ m_{\overline{f}}) \\
& = & F\overline{f} \,.
\end{array}
$$

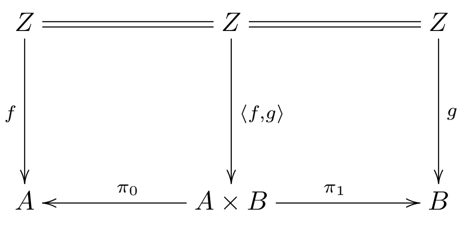
□

4 Cartesian restriction categories

Definition 4.1 (Restriction product)**.** Let $\mathbb{X}$ be a restriction category and let A and B be two objects in $\mathbb{X}$. The *restriction product* of A and B is an object $A \times_R B$ together with two *total* maps $\pi_0 : A \times_R B \to A, \pi_1 : A \times_R B \to B$ such that, for every object Z and pair of maps $f : Z \to A$, $g : Z \to B$ there exists a unique map $\langle f, g \rangle_R : Z \to A \times_R B$ such that

$$
\pi_0 \circ \langle f, g \rangle_R = f \circ \overline{g} \ \text{ and } \ \pi_1 \circ \langle f, g \rangle_R = g \circ \overline{f} \,. \tag{15}
$$

Recall that the standard product requires commutativity of the rectangle:



Instead, the restriction product demands commutativity of the following rectangle:

$$\begin{array}{ccccc} Z & \xleftarrow{\overline{g}} & Z & \xrightarrow{\overline{f}} & Z \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \langle f,g\rangle_R} & & \big\downarrow{\scriptstyle g} \\ A & \xleftarrow{\pi_0} & A \times_R B & \xrightarrow{\pi_1} & B \end{array} \tag{16}$$

We stress that the "projections" π_0 and π_1 in Definition 4.1 are total maps. On the other hand, the "pairing" $\langle f, g\rangle_R$ is *not* required to be total, as this would imply f and g to be total too (cf. Lemma 4.2 later on). However, if f and g are total, then so is $\langle f, g\rangle_R$ by Lemma 1.5, as in this case

$$\overline{\pi_0 \circ \langle f, g\rangle_R} = \overline{f \circ \overline{g}} = \overline{f} = \mathrm{id}_Z$$

and similarly for π_1. The noteworthy feature of restriction product is that it moves non-totality on the component which is *not* involved in the projection.

Observe that restriction products are unique, up to a unique isomorphism. In fact, if $(C, p_0.p_1)$ is another candidate to the restriction product of A and B, then the pairings $\langle \pi_0, \pi_1\rangle_R$ and $\langle p_0, p_1\rangle_R$ must be each other's inverse.

Lemma 4.2. *Let $\mathbb{X}$ be a restriction category. Every time the compositions and restriction products are defined, the following hold:*

1. $\overline{\langle f, g\rangle_R} = \overline{f} \circ \overline{g} = \overline{g} \circ \overline{f}$.
   As a consequence: if $\langle f, g\rangle_R$ is total, then so are f and g.
2. $\langle f \circ \overline{h}, g\rangle_R = \langle f, g\rangle_R \circ \overline{h} = \langle f, g \circ \overline{h}\rangle_R$.
3. $\langle f, g\rangle_R \circ h = \langle f \circ h, g \circ h\rangle_R$.
4. *Define $f \times_R g$ as $\langle f \circ \pi_0, g \circ \pi_1\rangle_R$. Then $\overline{f \times_R g} = \overline{f} \times_R \overline{g}$.*

$f \times_R g$ for restriction products is the perfect analogous of $f \times g$ for products. Indeed, $f \times_R g$ is the unique map ϕ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xleftarrow{\pi_0} & A \times_R B & \xrightarrow{\pi_1} & B \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle g} \\ A' & \xleftarrow{\pi_0'} & A' \times_R B' & \xrightarrow{\pi_1'} & B' \end{array}$$

which is the same as requiring that the following one does:

$$\begin{array}{ccccc} A \times_R B & \xleftarrow{\overline{f \circ \pi_0}} & A \times_R B & \xrightarrow{\overline{g \circ \pi_1}} & A \times_R B \\ {\scriptstyle f \circ \pi_0}\big\downarrow & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle g \circ \pi_1} \\ A' & \xleftarrow{\pi_0'} & A' \times_R B' & \xrightarrow{\pi_1'} & B' \end{array}$$

Proof of Lemma 4.2. As π_1 is total,

$$\overline{\langle f,g\rangle_R} = \overline{\overline{\pi_1} \circ \langle f,g\rangle_R} = \overline{\pi_1 \circ \langle f,g\rangle_R} = \overline{g \circ \overline{f}} = \overline{g} \circ \overline{f} \, .$$

If, in addition, $\langle f,g\rangle_R$ is total, then $\overline{f}$ and $\overline{g}$ are each other's inverse, so they are monic, and $\overline{f} = \overline{\overline{f}} = \mathrm{id}_Z = \overline{\overline{g}} = \overline{g}$. Point 1 is thus proved.

For point 2, observe that

$$\pi_0 \circ \langle f,g\rangle_R \circ \overline{h} = f \circ \overline{g} \circ \overline{h} = f \circ \overline{h} \circ \overline{g} \, ,$$

while

$$\pi_1 \circ \langle f,g\rangle_R \circ \overline{h} = g \circ \overline{f} \circ \overline{h} = g \circ \overline{f \circ \overline{h}} \, :$$

the thesis follows by definition and uniqueness of $\left\langle f \circ \overline{h}, g\right\rangle_R$.

Moreover,

$$\pi_0 \circ \langle f,g\rangle_R \circ h = f \circ \overline{g} \circ h = f \circ h \circ \overline{g \circ h} = \pi_0 \circ \langle f \circ h, g \circ h\rangle_R \, ,$$

and similarly,

$$\pi_1 \circ \langle f,g\rangle_R \circ h = \pi_1 \circ \langle f \circ h, g \circ h\rangle_R \, :$$

point 3 then follows from uniqueness of $\langle f \circ h, g \circ h\rangle_R$.

Finally, as clearly $\langle \pi_0, \pi_1\rangle_R = \mathrm{id}_{A \times_R B}$, from point 1 follows

$$\begin{aligned}
\overline{f \times_R g} &= \overline{g \circ \pi_1} \circ \overline{f \circ \pi_0} \\
&= \langle \pi_0, \pi_1\rangle_R \circ \overline{g \circ \pi_1} \circ \overline{f \circ \pi_0} \\
&= \left\langle \pi_0 \circ \overline{f \circ \pi_0}, \pi_1 \circ \overline{g \circ \pi_1}\right\rangle_R \\
&= \left\langle \overline{f} \circ \pi_0, \overline{g} \circ \pi_1\right\rangle_R \\
&= \overline{f} \times_R \overline{g} \, :
\end{aligned}$$

which proves point 4. □

Definition 4.3 (Restriction functor)**.** Let $\mathbb{X}$ and $\mathbb{Y}$ be restriction categories. A *restriction functor* from $\mathbb{X}$ to $\mathbb{Y}$ is a functor $F : \mathbb{X} \to \mathbb{Y}$ such that $\overline{Ff} = F\overline{f}$ for every $f \in \mathrm{Hom}_{\mathbb{X}}$.

Point 4 of Lemma 4.2 can thus be read as such: the functor $\times_R : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ that associates to every pair of objects (A, B) the restriction product $A \times_R B$ and to every pair of maps $f : A \to A'$, $g : B \to B'$ the restriction product map $\langle f \circ \pi_0, g \circ \pi_1\rangle_R : A \times_R B \to A' \times_R B'$, is a restriction functor.

Definition 4.4 (Restriction final object)**.** A *restriction final object* in a restriction category $\mathbb{X}$ is an object $\mathbf{1}_R$ such that every object has a unique *total* map $!_A : A \to \mathbf{1}_R$ with the following property: for every map $f : A \to \mathbf{1}_R$ the equality $f = !_A \circ \overline{f}$ holds, *i.e.*, $f \leq !_A$.

If $\mathbb{X}$ has a restriction final object $\mathbf{1}_R$, then $\mathbb{X}(A, \mathbf{1}_R)$ is equivalent to $\mathcal{O}(A) = \{e : A \to A \mid \overline{e} = e\}$, the lattice of restriction idempotents at A, which may be seen as a family of "open sets".

Definition 4.5 (Cartesian restriction category)**.** A *cartesian restriction category* is a restriction category with a restriction final object $\mathbf{1}_R$ where every two objects (thus, every n objects for arbitrary $n \in \mathbb{N}$) have a restriction product.

Definition 4.6 (Partial isomorphism)**.** A map $f : A \to B$ in a restriction category $\mathbb{X}$ is a *partial isomorphism* if there exists a map $g = f^{(-1)} : B \to A$ such that $g \circ f = \overline{f}$ and $f \circ g = \overline{g}$.

Lemma 4.7. *Let $\mathbb{X}$ be a restriction category and let $f : A \to B$, $g : B \to C$ be partial isomorphisms.*

1. *$f^{(-1)}$ is unique (and so is $g^{(-1)}$).*
2. *$g \circ f$ is a partial isomorphism.*

Proof. If g and g' satisfy $g \circ f = g' \circ f = \overline{f}$, $f \circ g = \overline{g}$, $f \circ g' = \overline{g'}$, then

$$g = g \circ \overline{g} = g \circ f \circ g = \overline{f} \circ g = g' \circ f \circ g = g' \circ \overline{g} \ :$$

on the other hand,

$$\begin{aligned} g' \circ \overline{g} &= g' \circ \overline{g'} \circ \overline{g} \\ &= g' \circ \overline{g} \circ \overline{g'} \\ &= g' \circ f \circ g \circ f \circ g' \\ &= g' \circ f \circ \overline{f} \circ g' \\ &= g' \circ f \circ g' \\ &= g' \circ \overline{g'} \\ &= g' \, . \end{aligned}$$

Thus, $g = g'$, and point 1 is proved. For point 2, the partial inverse of $g \circ f$ is simply $f^{(-1)} \circ g^{(-1)}$, as

$$\begin{aligned} f^{(-1)} \circ g^{(-1)} \circ g \circ f &= f^{(-1)} \circ \overline{g} \circ f \\ &= f^{(-1)} \circ f \circ \overline{g \circ f} \\ &= = \overline{f} \circ \overline{g \circ f} \\ &= \overline{g \circ f} \circ \overline{g} \\ &= \overline{g \circ f \circ \overline{f}} \\ &= \overline{g \circ f} \, , \end{aligned}$$

and similarly, $g \circ f \circ f^{(-1)} g^{(-1)} = \overline{f^{(-1)} \circ g^{(-1)}}$. □

Lemma 4.8. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two restriction categories and let $F : \mathbb{X} \to \mathbb{Y}$ be a restriction functor. If $f \in \mathrm{Hom}_{\mathbb{X}}(A, B)$ is a partial isomorphism, then so is $Ff \in \mathrm{Hom}_{\mathbb{Y}}(FA, FB)$.*

Proof. If $g = f^{(-1)}$ is the partial inverse of f in $\mathbb{X}$, then

$$Fg \circ Ff = F(g \circ f) = F(\overline{f}) = \overline{Ff}$$

and similarly $Ff \circ Fg = \overline{Fg}$, so Fg is the partial inverse of Ff in $\mathbb{Y}$. □

Let f be a partial isomorphism: if f is total, $f^{(-1)}$ needs not be so.

Example 4.9. Let $\mathbb{X} = \mathbf{Par}$, $A = \{a\}$, $B = \{b_1, b_2\}$, $f(a) = b_1$, $g(b_1) = a$, $g(b_2)$ undefined. Then f is a partial isomorphism, g is its partial inverse, f is total, and g is not.

Definition 4.10 (Discrete restriction category)**.** A cartesian restriction category $\mathbb{X}$ is *discrete* if for every object A the *diagonal* $\Delta_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle_R \in \mathbb{X}(A, A \times_R A)$ is a partial isomorphism.

Observe that Δ_A is total: hence, $\Delta_A^{(-1)} \circ \Delta_A = \mathrm{id}_A$. The partial inverses $\Delta_A^{(-1)} : A \times_R A \to A$ are called *equalities.*[2,3]

Example 4.11. Let $\mathbb{X}$ be a discrete cartesian restriction category. Then **Total**($\mathbb{X}$) has ordinary products, which coincide with restriction products in $\mathbb{X}$; and final object, which is just the restriction final object of $\mathbb{X}$. But **Total**($\mathbb{X}$) has pullbacks, and the equalizer m of $f, g : X \to A$ in **Total**($\mathbb{X}$) can be constructed as the pullback of Δ_A along $\langle f, g \rangle_R$ as follows:

$$\begin{array}{ccc} [[f = g]] & \xrightarrow{\ m\ } & X \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \langle f,g \rangle_R} \\ A & \xrightarrow{\ \Delta_A\ } & A \times_R A \end{array}$$

From the *existence theorem for limits* [6] it then follows that **Total**($\mathbb{X}$) has all limits. So:

if $\mathbb{X}$ is a discrete cartesian restriction category,
then **Total**($\mathbb{X}$) is a "nice" category.

Definition 4.12 (Meets in a restriction category)**.** A restriction category $\mathbb{X}$ has *meets* if there exists a binary operator $\wedge$ which associates to every pair of parallel maps $f, g : A \to B$ a map $f \wedge g : A \to B$ so that the following properties are satisfied:

1. $f \wedge f = f$.
2. $f \wedge g \leq f, g$. That is: $f \circ \overline{f \wedge g} = g \circ \overline{f \wedge g} = f \wedge g$.
3. $(f \wedge g) \circ h = (f \circ h) \wedge (g \circ h)$ for every $h : Z \to A$.

Lemma 4.13. *Let $\mathbb{X}$ be a restriction category with meets and let $f, g \in \mathbb{X}(A, B)$.*

1. *For every $h \in \mathrm{Hom}(A, B)$, if $h \leq f, g$, then $h \leq f \wedge g$.*

   That is: the meet in a restriction category is actually a greatest lower bound.

   In particular: the meet operation is commutative.

2. *For every $f' \in \mathrm{Hom}(A, B)$, if $f \leq f'$, then $f \wedge g \leq f' \wedge g$.*

   From this and the previous point: for every $g' \in \mathrm{Hom}(A, B)$, if $g \leq g'$, then $f \wedge g \leq f \wedge g'$.

---

[2] Versions of these notes earlier that July 16, 2014 incorrectly stated: "It follows from Lemma 4.8 and point 4 of Lemma 4.2 that in a discrete cartesian restriction category $\mathbb{X}$ the diagonals have partial inverses $\Delta_A^{(-1)} : A \times_R A \to A$, which are called *equalities*."

[3] The version from July 16, 2014 contained a *wrong* "proof" that $\Delta_A^{(-1)}$ is total. On careful examination, such proof relied on the equalities $\pi_0 = \pi_1 = \Delta_A^{(-1)}$: which are, in general, false.

3. *For every* $h \in Hom(A, C)$, $(f \wedge g) \circ \overline{h} = (f \circ \overline{h}) \wedge g = f \wedge (g \circ \overline{h})$

4. *If* $f \smile g$ *then* $f \wedge g = g \circ \overline{f} = f \circ \overline{g}$.

5. *For every* $k \in \mathrm{Hom}(B, C)$, $\overline{k} \circ (f \wedge g) = (\overline{k} \circ f) \wedge (\overline{k} \circ g)$.

Proof. Let us prove the points one by one:

1. If $f \circ \overline{h} = g \circ \overline{h} = h$, then $(f \wedge g) \circ \overline{h} = (f \circ \overline{h}) \wedge (g \circ \overline{h}) = h \wedge h = h$.

   In particular, as $f \wedge g \leq g, f$, we get $f \wedge g \leq g \wedge f$; similarly, $g \wedge f \leq f \wedge g$, and from antisymmetry we get equality.

2. If $f \leq f'$ then $f \wedge g \leq f'$ too, thus $f \wedge g \leq f' \wedge g$ by point 1.

3. By point 2, $(f \wedge g) \circ \overline{h} = (f \circ \overline{h}) \wedge (g \circ \overline{h}) \leq (f \circ \overline{h}) \wedge g$. On the other hand, if $k : A \to D$ is such that $k \leq f \circ \overline{h}, g$, then

$$\begin{aligned} g \circ \overline{h} \circ \overline{k} &= g \circ \overline{k} \circ \overline{h} \\ &= k \circ \overline{h} \\ &= ((f \circ \overline{h}) \circ \overline{k}) \circ \overline{h} \\ &= f \circ \overline{h} \circ \overline{h} \circ \overline{k} \\ &= (f \circ \overline{h}) \circ \overline{k}, \end{aligned}$$

   so that $k \leq g \circ \overline{h}$ too, and $k \leq (f \circ \overline{h}) \wedge (g \circ \overline{h}) = (f \wedge g) \circ \overline{h}$ by point 1. Therefore, $(f \wedge g) \circ \overline{h} = (f \circ \overline{h}) \wedge g$: this holds for every suitable f, g, and h, whence $(f \wedge g) \circ \overline{h} = (g \wedge f) \circ \overline{h} = (g \circ \overline{h}) \wedge f = f \wedge (g \circ \overline{h})$.

4. Suppose $f \smile g$. Then $g \circ \overline{f} \leq g$ and $\underline{g \circ \overline{f}} = f \circ \overline{g} \leq f$, thus $g \circ \overline{f} \leq f \wedge g$. But if $h \leq f, g$, then $\underline{g \circ \overline{f}\,\overline{h}} = g \circ \overline{f \circ \overline{h}} = g \circ \overline{h} = h$, *i.e.*, $h \leq g \circ \overline{f}$: in particular, $f \wedge g \leq g \circ \overline{f}$.

5. By exploiting point 3 and $f \circ \overline{f \wedge g} = g \circ \overline{f \wedge g} = f \wedge g$, we find:[4]

$$\begin{aligned} \overline{k} \circ (f \wedge g) &= (f \wedge g) \circ \overline{k \circ (f \wedge g)} \\ &= (f \wedge g) \circ \overline{k \circ (f \wedge g)} \circ \overline{k \circ (f \wedge g)} \\ &= (f \wedge g) \circ \overline{k \circ g \circ \overline{f \wedge g}} \circ \overline{k \circ f \circ \overline{f \wedge g}} \\ &= (f \wedge g) \circ \overline{k \circ g} \circ \overline{f \wedge g} \circ \overline{k \circ f} \circ \overline{f \wedge g} \\ &= (f \wedge g) \circ \overline{f \wedge g} \circ \overline{f \wedge g} \circ \overline{k \circ g} \circ \overline{k \circ f} \\ &= (f \wedge g) \circ \overline{k \circ g} \circ \overline{k \circ f} \\ &= (f \circ \overline{k \circ f}) \wedge (g \circ \overline{k \wedge g}) \\ &= (\overline{k} \circ f) \wedge (\overline{k} \circ g). \end{aligned}$$

□

Theorem 4.14. *A cartesian restriction category is discrete if and only if it has meets.*

[4] Thanks to Prof. Cockett for pointing out this chain of equalities.

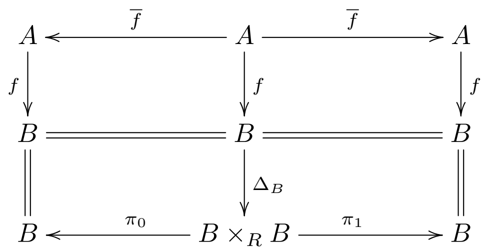
Proof. Let $\mathbb{X}$ be a cartesian restriction category.

If $\mathbb{X}$ is discrete, for any $f, g \in \mathrm{Hom}_{\mathbb{X}}(A, B)$ set

$$f \wedge g = \Delta_B^{(-1)} \circ \langle f, g \rangle_R \ . \tag{17}$$

Then:

1. By uniqueness, $\langle f, f \rangle_R = \Delta_B \circ f$ as the diagram



   clearly commutes: whence, Δ_B being total,

$$f \wedge f = \Delta_B^{(-1)} \circ \langle f, f \rangle_R = \Delta_B^{(-1)} \circ \Delta_B \circ f = \overline{\Delta_B} \circ f = f \ .$$

2. We have[5]

$$\begin{array}{rcl}
f \circ \overline{f \wedge g} & = & f \circ \overline{\Delta_B^{(-1)} \circ \langle f, g \rangle_R} \\
& = & f \circ \overline{\Delta_B^{(-1)} \circ \langle f, g \circ \overline{g} \rangle_R} \\
& = & f \circ \overline{\Delta_B^{(-1)} \circ \langle f, g \rangle_R \circ \overline{g}} \\
& = & f \circ \overline{\Delta_B^{(-1)} \circ \langle f, g \rangle_R} \circ \overline{g} \\
& = & f \circ \overline{g} \circ \overline{\Delta_B^{(-1)} \circ \langle f, g \rangle_R} \\
& = & \pi_0 \circ \langle f, g \rangle_R \circ \overline{\Delta_B^{(-1)} \circ \langle f, g \rangle_R} \\
& = & \pi_0 \circ \overline{\Delta_B^{(-1)}} \circ \langle f, g \rangle_R \\
& = & \pi_0 \circ \Delta_B \circ \Delta_B^{(-1)} \circ \langle f, g \rangle_R \\
& = & \mathrm{id}_B \circ (f \wedge g) \\
& = & f \wedge g \ :
\end{array}$$

   that is, $f \wedge g \leq f$. Similarly, $f \wedge g \leq g$.

3. In our context, $(f \wedge g) \circ h = (f \circ h) \wedge (g \circ h)$ means

$$\Delta_B^{(-1)} \circ \langle f, g \rangle_R \circ h = \Delta_B^{(-1)} \circ \langle f \circ h, g \circ h \rangle_R \ :$$

   which follows from point 3 of Lemma 4.2

If $\mathbb{X}$ has meets, define

$$\Delta_A^{(-1)} = \pi_0 \wedge \pi_1 \ . \tag{18}$$

[5]The proof in the versions before July 17, 2014 contained an error.

(Recall that $\Delta_A \in \mathrm{Hom}_{\mathbb{X}}(A, A \times_R A)$, so that $\pi_0, \pi_1 \in \mathrm{Hom}_{\mathbb{X}}(A \times_R A, A)$.) In fact,

$$(\pi_0 \wedge \pi_1) \circ \Delta_A = (\pi_0 \circ \Delta_A) \wedge (\pi_1 \circ \Delta_A) = \mathrm{id}_A \wedge \mathrm{id}_A = \mathrm{id}_A = \overline{\Delta_A}$$

as Δ_A is total, while

$$\begin{aligned}
\Delta_A \circ (\pi_0 \wedge \pi_1) &= \langle \mathrm{id}_A, \mathrm{id}_A \rangle_R \circ (\pi_0 \wedge \pi_1) \\
&= \langle \pi_0 \wedge \pi_1, \pi_0 \wedge \pi_1 \rangle_R \\
&= \langle \pi_0 \circ \overline{\pi_0 \wedge \pi_1}, \pi_1 \circ \overline{\pi_0 \wedge \pi_1} \rangle_R \\
&= \langle \pi_0, \pi_1 \rangle_R \circ \overline{\pi_0 \wedge \pi_1} \\
&= \mathrm{id}_{A \times_R A} \circ \overline{\pi_0 \wedge \pi_1} \\
&= \overline{\pi_0 \wedge \pi_1}\,.
\end{aligned}$$

□

5 Turing categories

Definition 5.1. A *Turing category* is a cartesian restriction category $\mathbb{X}$ that has a *Turing object* $\mathbf{T}$ with an associate *Turing structure*, *i.e.*, a collections of (partial) maps $\bullet_{X,Y} : \mathbf{T} \times_R X \to Y$ satisfying the following property: for every (partial) map $f : A \times_R X \to Y$ there exists a *total* map $\lambda^* f : A \to \mathbf{T}$ such that

$$\bullet_{X,Y} \circ (\lambda^* f \times_R \mathrm{id}_X) = f\ , \quad \text{i.e.,} \qquad \begin{array}{ccc} \mathbf{T} \times_R X & \xrightarrow{\ \bullet_{X,Y}\ } & Y \\ {\scriptstyle \lambda^* f \times_R \mathrm{id}_X}\uparrow & \nearrow_{f} & \\ A \times_R X & & \end{array} \tag{19}$$

Observe that $\mathbf{T}$ behaves as a *weak exponential object* for every pair of objects. Recall that, in a cartesian category (with standard products) an exponential object for X and Y is an object Y^X together with a map $\mathrm{eval} : Y^X \times X \to Y$, called *evaluation*, such that for every object A and map $f : A \times_R X \to Y$ there exists a unique morphism $\lambda f : A \to Y^X$ such that $\mathrm{eval} \circ (\lambda f \times_R \mathrm{id}_X) = f$. Then $\mathbf{T}$ is behaving like Y^X, $\bullet_{X,Y}$ like eval, and $\lambda^* f$ like λf, with two important differences:

1. The single object $\mathbf{T}$ takes the role of Y^X whatever the objects X and Y are.

2. The map $\lambda^* f$ is required not to be unique, but to be total.

In a Turing category there may be more than one Turing object, and any given Turing object may have more than one Turing structure.

Example 5.2 (The degenerate Turing category)**.** The category with a single object and a single morphism is a Turing category.

Example 5.3 (Kleene's first model)**.** Given a Gödel enumeration of the Turing machines, consider the partial map $\bullet : \mathbb{N} \times_R \mathbb{N} \to \mathbb{N}$ given by $\bullet(n, m) = \{n\}m$, the result of the computation of the nth Turing machine over the input m. In this category, the objects are the powers of $\mathbb{N}$—*i.e.*, the *arities*—and the Turing object is $\mathbf{T} = \mathbb{N}$, while the Turing structure is defined similarly to $\bullet = \bullet_{\mathbb{N},\mathbb{N}}$ described above. This is a *partial* model, because application is defined as computation by Turing machines, and cannot be anything but partial.

Example 5.4 (λ-calculus with β-equality)**.** Consider the category $\mathbb{X}$ whose objects are the natural numbers, 0 is the final object, a map $n \to m$ has the form $(x_1, \ldots, x_n) \mapsto (t_1, \ldots, t_m)$, where the t_i's are λ-*terms* in the x_j's, and composition is defined by substitution, *up to β-equality.*

This is a *total* model, because application between λ-terms (or combinators) is total, even modulo β-reduction. Evaluation, on the other hand, corresponds to *reduction to normal form*: which is only partial.

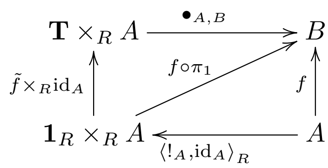
The following statement is immediate.

Proposition 5.5. *The product of two Turing categories is a Turing category.*

Definition 5.6 (Point in a restriction category)**.** Let $\mathbb{X}$ be a restriction category with a restriction final object $\mathbf{1}_R$ and let A be an object in $\mathbb{X}$. A *point* on A is a *total* map $p : \mathbf{1}_R \to A$.

Definition 5.6 mimics the standard definition of point in categories with a final object, with the key difference that we require $\mathbf{1}_R$ to be a restriction final object.

Proposition 5.7. *Let $\mathbb{X}$ be a Turing category with Turing object $\mathbf{T}$ and restriction final object $\mathbf{1}_R$. For every $f : A \to B$ there exists a point $\tilde{f}$ on $\mathbf{T}$ such that the following diagram commutes:*



(20)

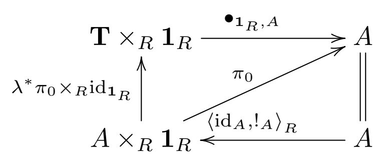
Proof. Just set $\tilde{f} = \lambda^*(f \circ \pi_1)$. Then the upper triangle commutes, while the lower one does by construction. □

Recall that an object A is a *retract* of an object B if there exist an epic map $r : B \to A$ (the *retraction*) and a monic map $s : A \to B$ (the *section*) such that $r \circ s = \mathrm{id}_A$. An object U is *universal* if every object is a retract of U.

Proposition 5.8. *Let $\mathbb{X}$ be a Turing category.*

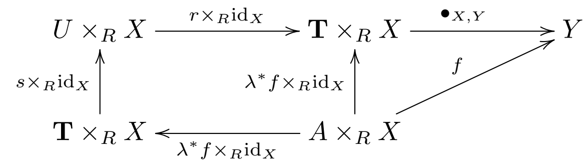
1. *The Turing object is universal.*
2. *Every universal object is a Turing object.*

Proof. In the definition of Turing category, set $X = \mathbf{1}_R$ and $Y = A$. Then the following diagram commutes:



Setting $r_A = \bullet_{\mathbf{1}_R,A} \circ \langle \mathrm{id}_{\mathbf{T}}, !_{\mathbf{T}} \rangle_R$ and $s_A = \lambda^* \pi_0$ proves point 1.

Let now U be an arbitrary universal object. Then $\mathbf{T}$ is a retract of U: let $s : \mathbf{T} \to U$ monic and $r : U \to \mathbf{T}$ epic satisfy $r \circ s = \mathrm{id}_{\mathbf{T}}$. Let X and Y be arbitrary objects: for every object A and map $f : A \times_R X \to Y$ there exists a total map $\lambda^* f : A \to \mathbf{T}$ such that the following diagram commutes:

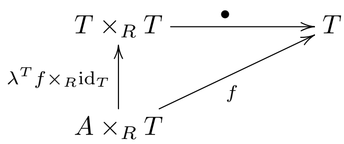


where $\bullet = \bullet^{\mathbf{T}}$ is the Turing structure relative to $\mathbf{T}$. Then a Turing structure $\bullet^U$ relative to U can be defined as $\bullet^U_{X,Y} = \bullet^{\mathbf{T}}_{X,Y} \circ (r \times_R \mathrm{id}_X)$: the total map from A to U corresponding to $f : A \times_R X \to Y$ will be $\lambda^U f = s \circ \lambda^* f$. $\square$

Corollary 5.9. *In a Turing category with Turing object* $\mathbf{T}$*, an object* A *is a Turing object if and only if* $\mathbf{T}$ *is a retract of* A*.*

Corollary 5.10. *In Kleene's first model, every infinite recursively enumerable set is a Turing object.*

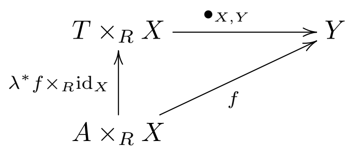
Theorem 5.11 (First characterization theorem)**.** *A cartesian restriction category* $\mathbb{X}$ *is a Turing category if and only if it has a universal object* T *and a* universal application $\bullet : T \times_R T \to T$ *(also called the* Turing map*) with the following* weak exponential property*: for every map* $f : A \times_R T \to T$ *there exists a* total *map* $\lambda^T f : A \to T$ *such that* $\bullet \circ (\lambda^T f \times_R \mathrm{id}_T) = f$*, i.e., the following diagram commutes:*



(21)

Shortly: in a cartesian restriction category, being able to perform evaluation over arbitrary objects, is the same as being able to perform evaluation over a single universal object.

Proof. Let X and Y be arbitrary objects. We want to construct a map $\bullet_{X,Y} : T \times_R X \to Y$ such that, for every object A and map $f : A \times_R X \to Y$, there exists a total map $\lambda^* f : A \to T$ such that the following diagram commutes:



As T is universal, X and Y are retracts of T: let $s_X : X \to T, s_Y : Y \to T$ monic and $r_X : T \to X, r_Y : T \to Y$ epic satisfy $r_X \circ s_X = \mathrm{id}_X$ and $r_Y \circ s_Y = \mathrm{id}_Y$. Set

$g = s_Y \circ f \circ (\mathrm{id}_A \times_R r_X)$. Then the following diagram commutes:

$$
\begin{array}{ccccccc}
T \times_R X & \xrightarrow{\mathrm{id}_T \times_R s_X} & T \times_R T & \xrightarrow{\quad\bullet\quad} & T & \xrightarrow{r_Y} & Y \\
\uparrow{\scriptstyle \lambda^T g \times_R \mathrm{id}_X} & & \uparrow{\scriptstyle \lambda^T g \times_R \mathrm{id}_T} & & & \nwarrow{\scriptstyle s_Y} & \| \\
A \times_R X & \xrightarrow{\mathrm{id}_A \times_R s_X} & A \times_R T & \xrightarrow{\mathrm{id}_A \times_R r_X} & A \times_R X & \xrightarrow{\quad f\quad} & Y
\end{array}
$$

But $r_X \circ s_X = \mathrm{id}_X$, so $(\mathrm{id}_A \times_R r_X) \circ (\mathrm{id}_A \times_R s_X) = \mathrm{id}_A \times_R \mathrm{id}_X = \mathrm{id}_{A \times_R X}$, and the diagram above can be shrunk into

$$
\begin{array}{ccccccc}
T \times_R X & \xrightarrow{\mathrm{id}_T \times_R s_X} & T \times_R T & \xrightarrow{\bullet} & T & \xrightarrow{r_Y} & Y \\
\uparrow{\scriptstyle \lambda^T g \times_R \mathrm{id}_X} & & & & & \nwarrow{\scriptstyle s_Y} & \| \\
A \times_R X & & & \xrightarrow{\qquad\qquad f \qquad\qquad} & & & Y
\end{array}
$$

We then put

$$\bullet_{X,Y} = r_Y \circ \bullet \circ (\mathrm{id}_T \times_R s_X)\ , \tag{22}$$

which correctly depends only on X and Y and not on A, and

$$\lambda^* f = \lambda^T g = \lambda^T (s_Y \circ f \circ (\mathrm{id}_A \times_R r_X))\ : \tag{23}$$

then

$$
\begin{aligned}
\bullet_{X,Y} \circ (\lambda^* f \times_R \mathrm{id}_X) &= r_Y \circ \bullet \circ (\mathrm{id}_T \times_R s_X) \circ (\lambda^T g \times_R \mathrm{id}_X) \\
&= r_Y \circ s_Y \circ f \\
&= \mathrm{id}_Y \circ f \\
&= f\,.
\end{aligned}
$$

□

An object which is the retract of its own self-power is called a *powerful* object: universal objects, in particular Turing objects, are powerful.

Definition 5.12 (Reduction of maps)**.** Let $e = \overline{e} : A \to A$ and $e' = \overline{e'} : A' \to A'$. We say that e *reduces to* e' if there exists a *total* map $f : A \to A'$ such that $\overline{e' \circ f} = e$. We say that e is *complete* if every e' reduces to e.

Definition 5.13 (Halting)**.** Let $\mathbb{X}$ be a restriction category with Turing object $\mathbf{T}$. The *halting set* for $\mathbb{X}$ is defined as

$$\Delta\bullet = \bullet_{\mathbf{T},\mathbf{T}} \circ \Delta_{\mathbf{T}}\,. \tag{24}$$

The restriction $\mathbf{h} = \overline{\Delta\bullet}$ of the halting set is called the *halting predicate.*

Lemma 5.14. *In every Turing category the halting predicate is complete.*

Proof. For simplicity, we write Δ for $\Delta_{\mathbf{T}}$, and $\bullet$ for $\bullet_{\mathbf{T},\mathbf{T}}$.

Let $e = \overline{e} : A \to A$; let $s = s_A : A \to \mathbf{T}$ monic and $r = r_A : \mathbf{T} \to A$ epic such that $r_A \circ s_A = \mathrm{id}_A$. Set $p = s_A \circ \overline{e} \circ \pi_1$ and $k = \lambda^* p$. Then the following diagram commutes:

$$
\begin{array}{ccccc}
\mathbf{T} & \xrightarrow{\quad\Delta\quad} & \mathbf{T} \times_R \mathbf{T} & \xrightarrow{\quad\bullet\quad} & \mathbf{T} \\
\uparrow{\scriptstyle k} & & \uparrow{\scriptstyle k \times_R \mathrm{id}_{\mathbf{T}}} & & \uparrow{\scriptstyle s_A} \\
A & \xrightarrow{\langle \mathrm{id}_A, s_A \rangle_R} & A \times_R \mathbf{T} & \xrightarrow{\overline{e} \circ \pi_1} & A
\end{array}
$$

Consequently, as s_A is monic,

$$
\begin{aligned}
\overline{\overline{\Delta \bullet} \circ k} &= \overline{\Delta \bullet \circ k} \\
&= \overline{s_A \circ \overline{e} \circ \pi_1 \circ \langle \mathrm{id}_A, s_A \rangle_R} \\
&= \overline{\overline{e} \circ \pi_1 \circ \langle \mathrm{id}_A, s_A \rangle_R} \\
&= \overline{\overline{e} \circ s_A} \\
&= \overline{e} \circ \overline{s_A} \\
&= \overline{e}\,.
\end{aligned}
$$

□

Definition 5.15 (Partial combinator algebra)**.** Let $\mathbb{X}$ be a cartesian restriciton category with restriction final object $\mathbf{1}_R$. A *partial combinator algebra* on $\mathbb{X}$ is a quadruple $(A, \bullet, k, s)$ where $A \in |\mathbb{X}|$, $\bullet \in \mathrm{Hom}_{\mathbb{X}}(A \times_R A, A)$, and $k, s \in \mathrm{Hom}_{\mathbb{X}}(\mathbf{1}_R, A)$ are points over A satisfying the following conditions:

1. The following diagram commutes:

$$
\begin{array}{ccc}
\mathbf{1}_R \times_R A \times_R A & \xrightarrow{\pi_1} & A \\
\big\downarrow{\scriptstyle \langle k, \mathrm{id}_{A \times_R A} \rangle_R} & & \big\uparrow{\scriptstyle \bullet} \\
A \times_R A \times_R A & \xrightarrow{\bullet \times_R \mathrm{id}_A} & A \times_R A
\end{array}
\tag{25}
$$

2. The following diagram commutes:

$$
\begin{array}{ccccc}
\mathbf{1}_R \times_R A \times_R A \times_R A & \xrightarrow{\langle \langle \pi_1, \pi_3 \rangle_R, \langle \pi_2, \pi_3 \rangle_R \rangle_R} & A \times_R A \times_R A \times_R A & \xrightarrow{\bullet \times_R \bullet} & A \times_R A \\
\big\downarrow{\scriptstyle \langle s, \mathrm{id}_A \times_R \mathrm{id}_A \times_R \mathrm{id}_A \rangle_R} & & & & \big\downarrow{\scriptstyle \bullet} \\
A \times_R A \times_R A \times_R A & & \xrightarrow{\bullet^{(3)}} & & A
\end{array}
\tag{26}
$$

   where $(a, b, c, d) \stackrel{\bullet^{(3)}}{\longrightarrow} ((a \bullet b) \bullet c) \bullet d$.

3. $\bullet \circ \langle s, \mathrm{id}_A \rangle_R$ is total.

Condition 1 in Definition 5.15 can be translated by saying that $kxy = x$ for every x and y; similarly, condition 2 can be expressed by saying that $sxyz = xz(yz)$ for every x, y and z. The points k and s thus behave as the combinators K and S of Curry's combinatory logic.

Observe that every Turing category is a partial combinator algebra.

Theorem 5.16 (Second characterization theorem)**.** *Every Turing category is the computable functions for some partial combinator algebra.*

The "is" in Theorem 5.16 means that one might need to split some idempotents.

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