Strong functors, strong monads

Strong functors

- A *(left) strong functor* on a monoidal category $(\mathcal{C}, I, \otimes)$ is given by
  - an endofunctor F on $\mathcal{C}$
  - with a nat. transf. $\theta_{A,B} : A \otimes FB \to F(A \otimes B)$ (the *(left) strength*)

  satisfying

$$
\begin{array}{ccc}
I \otimes FA & \xrightarrow{\theta_{I,A}} & F(I \otimes A) \\
\downarrow{\scriptstyle \lambda_{FA}} & & \downarrow{\scriptstyle F\lambda_A} \\
FA & = & FA
\end{array}
$$

$$
\begin{array}{ccccc}
(A \otimes B) \otimes FC & & \xrightarrow{\theta_{A \otimes B, C}} & & F((A \otimes B) \otimes C) \\
\downarrow{\scriptstyle \alpha_{A,B,FC}} & & & & \downarrow{\scriptstyle F\alpha_{A,B,C}} \\
A \otimes (B \otimes FC) & \xrightarrow{A \otimes \theta_{B,C}} & A \otimes F(B \otimes C) & \xrightarrow{\theta_{A,B \otimes C}} & F(A \otimes (B \otimes C))
\end{array}
$$

- Similarly, one defines a *right strong functor*.

- On a monoidal closed category $(\mathcal{C}, I, \otimes, \multimap)$, a strong functor can also be defined as an endofunctor on $\mathcal{C}$ equipped with

  - a natural transformation $F(A \multimap B) \to A \multimap FB$

  or

  - a natural transformation $A \multimap B \to FA \multimap FB$

  subject to appopriate conditions.

- A *bistrong functor* is a functor F with a left strength θ and a right strength ϑ such that

$$\begin{array}{ccccc}
(A \otimes FB) \otimes C & \xrightarrow{\theta_{A,B}\otimes C} & F(A \otimes B) \otimes C & \xrightarrow{\vartheta_{A\otimes B,C}} & F((A \otimes B) \otimes C) \\
\downarrow{\scriptstyle \alpha_{A,FB,C}} & & & & \downarrow{\scriptstyle F\alpha_{A,B,C}} \\
A \otimes (FB \otimes C) & \xrightarrow{A\otimes\vartheta_{B,C}} & A \otimes F(B \otimes C) & \xrightarrow{\theta_{A,B\otimes C}} & F(A \otimes (B \otimes C))
\end{array}$$

- A bistrong functor on a symmetric monoidal category $(\mathcal{C}, I, \otimes)$ category is *symmetric* bistrong, if

$$\begin{array}{ccc}
A \otimes FB & \xrightarrow{\theta_{A,B}} & F(A \otimes B) \\
\downarrow{\scriptstyle \sigma_{A,FB}} & & \downarrow{\scriptstyle F\sigma_{A,B}} \\
FB \otimes A & \xrightarrow{\vartheta_{B,A}} & F(B \otimes A)
\end{array}$$

- In this situation θ determines ϑ via

$$\vartheta_{A,B} =_{\mathrm{df}} FA\otimes B \xrightarrow{\sigma_{FA,B}} B\otimes FA \xrightarrow{\theta_{B,A}} F(B\otimes A) \xrightarrow{F\sigma_{B,A}} F(A\otimes B)$$

Strong natural transformations

- A *(left) strong natural transformation* between two (left) strong functors (F, θ), (G, θ') is a natural transformation $\tau : F \stackrel{\cdot}{\to} G$ satisfying

$$\begin{array}{ccc} A \otimes FB & \xrightarrow{\theta_{A,B}} & F(A \otimes B) \\ {\scriptstyle \mathrm{id}_A \otimes \tau_B} \downarrow & & \downarrow {\scriptstyle \tau_{A \otimes B}} \\ A \otimes GB & \xrightarrow[\theta'_{A,B}]{} & G(A \otimes B) \end{array}$$

- *Right strong* and *bistrong natural transformations* are defined analogously.

Set functors, nat. transfs. are uniquely strong

- On $(\mathbf{Set}, 1, \times)$, any functor F has a unique left strength given by

$$\theta_{A,B}(a, c) = F(\lambda b.(a, b))\, c$$

  and are therefore uniquely bistrong.
- Any natural transformation is left strong and bistrong.

Strong monads

- A *(left) strong monad* on a monoidal category $(\mathcal{C}, I, \otimes)$ is a monad (T, η, μ) with a strength θ for T for which η and μ are strong, i.e., satisfy

$$\begin{array}{ccc}
A \otimes B & = & A \otimes B \\
{\scriptstyle \mathrm{id}_A \otimes \eta_B} \downarrow & & \downarrow {\scriptstyle \eta_{A \otimes B}} \\
A \otimes TB & \xrightarrow[\theta_{A,B}]{} & T(A \otimes B)
\end{array}$$

$$\begin{array}{ccccc}
A \otimes T(TB) & \xrightarrow{\theta_{A,TB}} & T(A \otimes TB) & \xrightarrow{T\theta_{A,B}} & T(T(A \otimes B)) \\
{\scriptstyle \mathrm{id}_A \otimes \mu_B} \downarrow & & & & \downarrow {\scriptstyle \mu_{A \otimes B}} \\
A \otimes TB & & \xrightarrow[\theta_{A,B}]{} & & T(A \otimes B)
\end{array}$$

- (Id is always strong; if F, G are strong, then so is $G \cdot F$.)
- Right strong and bistrong monads are defined analogously.

An alternative: strong Kleisli triples

- A *strong Kleisli triple* is
  - an object mapping $T : |\mathcal{C}| \to |\mathcal{C}|$,
  - for any object A, a map $\eta_A : A \to TA$,
  - for any map $k : X \otimes A \to TB$, a map $k^\star : X \otimes TA \to TB$ (the *strong Kleisli extension* operation)

  such that
  - if $k : X \otimes A \to TB$, then $k^\star \circ (X \otimes \eta_A) = k$,
  - $(\eta_A \circ \lambda_A)^\star = \lambda_{TA} : I \otimes TA \to TA$,
  - if $k : X \otimes A \to TB$, $\ell : Y \otimes B \to TC$, then $(\ell^\star \circ (Y \otimes k) \circ \alpha_{Y,X,A})^\star = \ell^\star \circ (Y \otimes k^\star) \circ \alpha_{Y,X,TA} : (Y \otimes X) \otimes TA \to TC$.
- (No explicit functoriality and naturality conditions! No explicit strength operation, no explicit laws for strength.)
- Strong monads and strong Kleisli triples are in a bijection.

Set monads are uniquely strong

- On $(\mathbf{Set}, 1, \times)$, every monad is uniquely left strong and bistrong.

Commutative bistrong monads

- A [symmetric] bistrong monad $(T, \eta, \mu, \theta, \vartheta)$ defines two natural transformations

$$\mathrm{m}^{lr}_{A,B} =_{\mathrm{df}} TA \otimes TB \stackrel{\vartheta_{A,TB}}{\longrightarrow} T(A \otimes TB) \stackrel{T\theta_{A,B}}{\longrightarrow} T(T(A \otimes B)) \stackrel{\mu_{A\otimes B}}{\longrightarrow} T(A \otimes B)$$

and

$$\mathrm{m}^{rl}_{A,B} =_{\mathrm{df}} TA \otimes TB \stackrel{\theta_{TA,B}}{\longrightarrow} T(TA \otimes B) \stackrel{T\vartheta_{A,B}}{\longrightarrow} T(T(A \otimes B)) \stackrel{\mu_{A\otimes B}}{\longrightarrow} T(A \otimes B)$$

- The [symmetric] bistrong monad is called a *commutative*, if $\mathrm{m}^{lr} = \mathrm{m}^{rl}$.