Monoidal functors (aka idioms!)
[Symmetric] monoidal functors

- A **lax monoidal functor** between monoidal categories $(\mathcal{C}, I, \otimes)$ and $(\mathcal{C}, I', \otimes')$ is
  - a functor $F$ from $\mathcal{C}$ to $\mathcal{C}'$
  - with natural transformations $e : I' \to FI$ and $m_{A,B} : FA \otimes' FB \to F(A \otimes B)$ such that

\[
\begin{align*}
I' \otimes' FA & \xrightarrow{e \otimes' e} FI \otimes' FA \xrightarrow{m_{I,A}} F(I \otimes A) & FA & \xrightarrow{id} FA \\
\lambda'_{FA} & \downarrow F\lambda_A & \rho'_{FA} & \downarrow F\rho_A \\
FA & \xrightarrow{id} FA & FA \otimes' I' & \xrightarrow{F\alpha_{A,B} e} FA \otimes' FI \xrightarrow{m_{A,I}} F(A \otimes I) \\
(FA \otimes' FB) \otimes' FC & \xrightarrow{m_{A,B,\otimes' C}} F(A \otimes B) \otimes' FC \xrightarrow{m_{A\otimes B,\otimes C}} F((A \otimes B) \otimes' C) \\
\alpha'_{FA,FB,FC} & \downarrow F\alpha_{A,B,C} \\
FA \otimes' (FB \otimes' FC) & \xrightarrow{FA \otimes' m_{B,C}} FA \otimes' F(B \otimes C) \xrightarrow{m_{A,\otimes B,\otimes C}} F(A \otimes (B \otimes C))
\end{align*}
\]
A lax monoidal functors between symmetric monoidal categories is *lax symmetric monoidal*, if also

\[
FA \otimes' FB \xrightarrow{m_{A,B}} F(A \otimes B)
\]

\[
\sigma_{FA,FB}' \quad \downarrow \quad F\sigma_{A,B}
\]

\[
FB \otimes' FA \xrightarrow{m_{B,A}} F(B \otimes A)
\]

An *oplax [symmetric] monoidal functor* is like a lax [symmetric] monoidal functor, but e, m go in the opposite direction.

A *monoidal [symmetric] functor* is like a lax [symmetric] monoidal functor, but e, m are required to be natural isomorphisms.
A lax [symmetric] monoidal natural transformation between two lax [symmetric] monoidal functors \((F, e, m), (G, e', m')\) is a natural transformation \(\tau : F \xrightarrow{} G\) satisfying

\[
\begin{align*}
I' & \xrightarrow{e} FI \quad FA \otimes' FB \xrightarrow{m_{A,B}} F(A \otimes B) \\
| & \quad | \quad | \\
I' & \xrightarrow{e'} GI \quad GA \otimes' GB \xrightarrow{m'_{A,B}} G(A \otimes B)
\end{align*}
\]

Oplax [symmetric] monoidal and [symmetric] monoidal natural transformations are defined similarly.
Any functor $F$ between Cartesian categories is canonically oplax symmetric monoidal via

- $e = F1 \xrightarrow{!} 1$,
- $m_{A,B} = F(A \times B) \xrightarrow{(F\text{fst},F\text{snd})} FA \times FB$.

Any natural transformation between functors $F, G$ between Cartesian categories is oplax symmetric monoidal for the canonical oplax symmetric monoidalities on $F$ and $G$. 
Lax monoidal functors $\cap$ containers
Containers whose interpretation carries a lax monoidality are given by a container \((S, P)\) with

- \(e : S\)
- \(\bullet : S \to S \to S\)
- \(q_0 : \Pi\{s_0 : S\}. \Pi s_1 : S. P (s_0 \bullet s_1) \to P s_0\)
- \(q_1 : \Pi s_0 : S. \Pi\{s_1 : S\}. P (s_0 \bullet s_1) \to P s_1\)

where we write

- \(q_0 \{s_0\} s_1 p\) as \(s_1 \leftarrow_{s_0} p\)
- \(q_1 s_0 \{s_1\} p\) as \(p \rightarrow_{s_1} s_0\)

such that

- \(e \bullet s = s\)
- \(s = s \bullet e\)
- \((s \bullet s') \bullet s'' = s \bullet (s' \bullet s'')\)

and \ldots
... and

- \( e \downarrow_s p = p \)
- \( p \uparrow_s e = p \)
- \( s' \downarrow_s (s'' \downarrow_{s \bullet s'} p) = (s' \bullet s'') \downarrow_s p \)
- \( (s'' \downarrow_{s \bullet s'} p) \uparrow_{s'} s = s'' \downarrow_{s'} (p \uparrow_{s' \bullet s''} s) \)
- \( p \uparrow_{s''} (s \bullet s') = (p \uparrow_{s' \bullet s''} s) \uparrow_{s''} s' \)

\((S, e, \bullet)\) make a monoid.

\((\downarrow, \uparrow)\) resemble a biaction of \((S, e, \bullet)\).

Those containers whose interpretation carries a lax symmetric monoidality satisfy also

- \( s \bullet s' = s' \bullet s \),
- \( s' \downarrow_s p = p \uparrow_{s'} s \)

i.e., the monoid \((S, e, \bullet)\) is commutative and one action determines the other.
Monoidal monads
A lax [symmetric] monoidal monad on a [symmetric] monoidal category \( (\mathcal{C}, I, \otimes) \) is a monad \( (T, \eta, \mu) \) with a lax [symmetric] monoidality \((e, m)\) of \( T \) for which \( \eta \) and \( \mu \) are lax [symmetric] monoidal, i.e., satisfy

\[
\begin{align*}
I & \xrightarrow{\eta I} TI \\
I & \xrightarrow{I} TI
\end{align*}
\]

\[
\begin{align*}
I & \xrightarrow{\eta I} TI \\
I & \xrightarrow{I} TI
\end{align*}
\]

\[
\begin{align*}
T(I) & \xrightarrow{e} TI \\
T(I) & \xrightarrow{T(e)} T(TI)
\end{align*}
\]

\[
\begin{align*}
A \otimes B & \xrightarrow{\eta A \otimes \eta B} T(A \otimes B) \\
A \otimes B & \xrightarrow{T \eta_A \otimes \eta_B} T(T(A \otimes B))
\end{align*}
\]

\[
\begin{align*}
T(A \otimes B) & \xrightarrow{Tm_{A,B}} T(T(A \otimes B)) \\
T(A \otimes B) & \xrightarrow{\mu_{A \otimes B}} T(T(A \otimes B))
\end{align*}
\]

\[
\begin{align*}
T(A \otimes B) & \xrightarrow{Tm_{A,B}} T(T(A \otimes B)) \\
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\end{align*}
\]

(Note that \( \text{Id} \) is lax [symmetric] monoidal and, if \( F, G \) are lax [symmetric] monoidal, then so is \( G \cdot F \).)
- The 1st law forces that $e = \eta_I$ and the 2nd law follows from one of the monad laws, so we only need $m$ and the 3rd and 4th laws.

- On a Cartesian category, every monad is canonically oplax symmetric monoidal.
Lax monoidal monads $\equiv$ Comm. bistrong monads

There is a bijection of lax [symmetric] monoidalities $m$ on a monad $(T, \eta, \mu)$ on a [symmetric] monoidal category $(\mathcal{C}, I, \otimes)$ and commutative [symmetric] bistrengths $(\theta, \varphi)$.

It is defined by

- $m_{A,B} = m^{lr}_{A,B} = m^{rl}_{A,B}$
- $\theta_{A,B} = A \otimes TB \xrightarrow{\eta_A \otimes TB} TA \otimes TB \xrightarrow{m_{A,B}} T(A \otimes B)$,
- $\varphi_{A,B} = TA \otimes B \xrightarrow{TA \otimes \eta_B} TA \otimes TB \xrightarrow{m_{A,B}} T(A \otimes B)$.

On $(\textbf{Set}, 1, \times)$, as any monad has a unique left strength and [symmetric] bistrength, it is lax [symmetric] monoidal in at most one way.
Exception idioms

- Lax [symmetric] monoidalities \((e, m)\) on the exception functor for \(E\)
- \(TA = E + A\)

are in a bijection with [commutative] semigroup structures \(\otimes\) on \(E\) via

- \(e \ast = \text{inr} \ast\),
  
  \[
  m_{A,B} (\text{inl } e_0, \text{inl } e_1) = \text{inl} (e_0 \otimes e_1),
  
  m_{A,B} (\text{inl } e, \text{inr } b) = \text{inl } e
  
  m_{A,B} (\text{inr } a, \text{inl } e) = \text{inl } e
  
  m_{A,B} (\text{inr } a, \text{inr } b) = \text{inr} (a, b);
  
- \(e_0 \otimes e_1 = \text{case } m_{0,0} \text{ of } \text{inl } e \mapsto e\).

Two special cases are \(e_0 \otimes e_1 = e_0\) (the left zero semigroup) and \(e_0 \otimes e_1 = e_1\) (the right zero semigroup).

The exception monad for \(E\) is not lax [symmetric] monoidal except for the special case \(E = 1\).
Writer idioms

- Lax [symmetric] monoidalities $(e, m)$ on the writer functor for a set $P$
  - $TA = P \times A$

are in a bijection with [commutative] monoid structures $(i, \otimes)$ on $P$.

- Lax [symmetric] monoidalities $m$ on the writer monad for a monoid $(P, o, \oplus)$ are in a bijection with those [commutative] monoid structures $(i, \otimes)$ on $P$ that satisfy
  - $i = o$
  - $(e_0 \oplus e_1) \otimes (e_2 \oplus e_3) = (e_0 \otimes e_2) \oplus (e_1 \otimes e_3)$
    (middle-four interchange)
Under the 1st condition, the 2nd condition implies

\[ e_0 \otimes e_1 = (e_0 \oplus o) \otimes (o \oplus e_1) = (e_0 \otimes i) \oplus (i \otimes e_1) = e_0 \oplus e_1 \]

and further

\[ e_0 \oplus e_1 = (o \oplus e_0) \oplus (e_1 \oplus o) = (o \oplus e_1) \oplus (e_0 \oplus o) = e_1 \oplus e_0 \]

as well as follows from these conditions.

Hence the writer monad is lax [symmetric] monoidal if and only if \( \oplus \) is commutative.