Cut-elimination and Proof-search for Bi-Intuitionistic Logic Using Nested Sequents

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Estonian CS Theory Days
Jõulumäe, 3-5 October 2008

Introduction: Sequent Calculus and Proof Search

- Sequent calculus: proof system
- Sequent: $\Gamma \vdash \Delta$ read as $\bigwedge \Gamma \rightarrow \bigvee \Delta$
- Derivation: tree of sequents e.g. $$\dfrac{\dfrac{A, B \vdash A \qquad A, B \vdash B}{A, B \vdash A \wedge B} \wedge_R}{A \vdash B \rightarrow (A \wedge B)} \rightarrow_R$$
  - Leaf nodes: axioms
  - Root: logical consequence to be proven
  - Rules link nodes
- Backward proof search: build derivation from root towards leaves
- Cut rule: encodes transitivity $$\dfrac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$
  - Non-deterministic when applied backwards
- Cut-elimination: proof that uses cut $\rightsquigarrow$ proof that does not
  - If $\Gamma \vdash \Delta$ derivable using cut then $\Gamma \vdash \Delta$ derivable without using cut

Introduction: Bi-Intuitionistic logic

- Bi-Intuitionistic logic: extension of intuitionistic logic
- Exclusion connective $-\!\!<$ dual to implication $\rightarrow$:
    - $$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_R \qquad \frac{A \vdash B, \Delta}{A -\!\!< B \vdash \Delta} -\!\!<_L$$
- Hilbert calculus, algebraic and Kripke semantics (Rauszer 1974)
- Type theoretic interpretation of co-routines (Crolard 2004)
- “Cut-free” sequent calculus for BiInt (Rauszer 1974)
- Rauszer’s cut-elimination fails (Uustalu 2006)
- More recent calculi either have:
    - cut-elimination (Goré 1998), or
    - proof search (Uustalu, Pinto draft; Buisman/Postniece, Goré 2007)
    - but not both
- Goal: a sequent calculus for BiInt that has both cut-elimination and proof search

Syntax and Kripke Models

- Connectives: $\wedge \vee \rightarrow \prec$
- Constants: $\top \bot$
- Kripke semantics: model $\langle W, \leq, V \rangle$
  - $\leq$ is a reflexive and transitive binary relation over W
  - V maps atoms to 2^W
  - $V(\top) = W$ and $V(\bot) = \emptyset$
  - V satisfies: if $w \in V(p)$ and $w \leq u$ then $u \in V(p)$
- $w \Vdash p$ iff $w \in V(p)$
- $w \Vdash A \wedge B$ iff $w \Vdash A \ \ \& \ \ w \Vdash B$
- $w \Vdash A \vee B$ iff $w \Vdash A$ or $w \Vdash B$
- $w \Vdash A \rightarrow B$ iff $\forall u \geq w.u \nVdash A$ or $u \Vdash B$
- $w \Vdash A \prec B$ iff $\exists u \leq w.u \Vdash A \ \& \ u \nVdash B$
- BiInt is a conservative extension of Int
- Persistence: $w \Vdash A \Rightarrow \forall u \geq w.u \Vdash A$
- Reverse Persistence: $w \nVdash A \Rightarrow \forall u \leq w.u \nVdash A$

Validity and Falsifiability

- A valid iff for all models $\langle W, \leq, V\rangle$ and all worlds $w \in W$, $w \Vdash A$
- A falsifiable iff exists a model $\langle W, \leq, V\rangle$, a world $w \in W$, $w \nVdash A$
- $w \Vdash \Gamma$ iff $w \Vdash A$ for all $A \in \Gamma$
- $w \nVdash \Delta$ iff $w \nVdash A$ for all $A \in \Delta$
- $\Gamma \vdash \Delta$ valid iff $\bigwedge \Gamma \to \bigvee \Delta$ valid
  - i.e. for all models $\langle W, \leq, V\rangle$ and all worlds $w \in W$, $w \Vdash \Gamma$ implies $w \Vdash B$ for some $B \in \Delta$
- $\Gamma \vdash \Delta$ falsifiable iff $\bigwedge \Gamma \to \bigvee \Delta$ falsifiable
  - i.e. exists a model $\langle W, \leq, V\rangle$, a world $w \in W$, $w \Vdash \Gamma$ and $w \nVdash \Delta$

Uustalu's Example: Using Cut

- Rauszer's $\to_R$ and $-\!\!<_L$ require singleton succedent/antecedent:
  - $$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \to_R \qquad \frac{A \vdash B, \Delta}{A -\!\!< B \vdash \Delta}\ -\!\!<_L$$
- $p \vdash q, r \to ((p -\!\!< q) \wedge r)$ is not cut-free derivable in Rauszer's G1
- Failed derivation attempt:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\ \text{Id} \qquad q \vdash ?}{p, r \vdash p -\!\!< q}\ -\!\!<_R \qquad \dfrac{}{p, r \vdash r}\ \text{Id}}{p, r \vdash (p -\!\!< q) \wedge r}\ \wedge_R}{p \vdash r \to ((p -\!\!< q) \wedge r)}\ \to_R}{p \vdash q, r \to ((p -\!\!< q) \wedge r)}\ w_R$$

- Derivation using cut:

$$\dfrac{\dfrac{\dfrac{}{p \vdash q, p}\ \text{Id} \qquad \dfrac{}{q \vdash q}\ \text{Id}}{p \vdash q, p -\!\!< q}\ -\!\!<_R \qquad \dfrac{\dfrac{\dfrac{}{p -\!\!< q, r \vdash p -\!\!< q}\ \text{Id} \qquad \dfrac{}{p -\!\!< q, r \vdash r}\ \text{Id}}{p -\!\!< q, r \vdash (p -\!\!< q) \wedge r}\ \wedge_R}{p -\!\!< q \vdash r \to ((p -\!\!< q) \wedge r)}\ \to_R}{p \vdash q, r \to ((p -\!\!< q) \wedge r)}\ \text{cut}$$

Uustalu's Example: Semantic Motivation

- Sequent $p \vdash q, r \to ((p \prec q) \wedge r)$ is valid
- Counter-model construction
  - $w \Vdash p$
  - $w \nVdash q, r \to ((p \prec q) \wedge r)$

Uustalu's Example: Semantic Motivation

- Sequent $p \vdash q, r \rightarrow ((p -\!\!< q) \land r)$ is valid
- Counter-model construction
  - $w \Vdash p$
  - $w \not\Vdash q, r \rightarrow ((p -\!\!< q) \land r)$

$$w \Vdash p$$
$$w \not\Vdash q$$
$$w \not\Vdash r \rightarrow ((p -\!\!< q) \land r)$$

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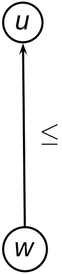


$u \Vdash p$

$u \Vdash r$

$u \nVdash (p -\!\!< q) \wedge r$

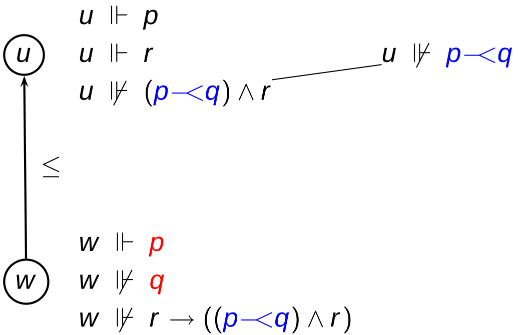
$w \Vdash p$

$w \nVdash q$

$w \nVdash r \to ((p -\!\!< q) \wedge r)$

Uustalu's Example: Semantic Motivation

- Sequent $p \vdash q, r \to ((p \mathbin{-\!\!<} q) \wedge r)$ is valid
- Counter-model construction
  - $w \Vdash p$
  - $w \not\Vdash q, r \to ((p \mathbin{-\!\!<} q) \wedge r)$



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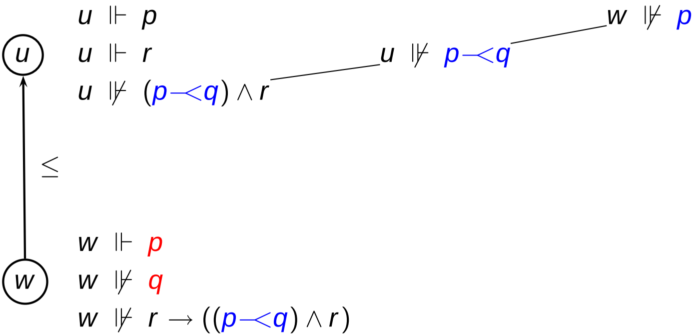

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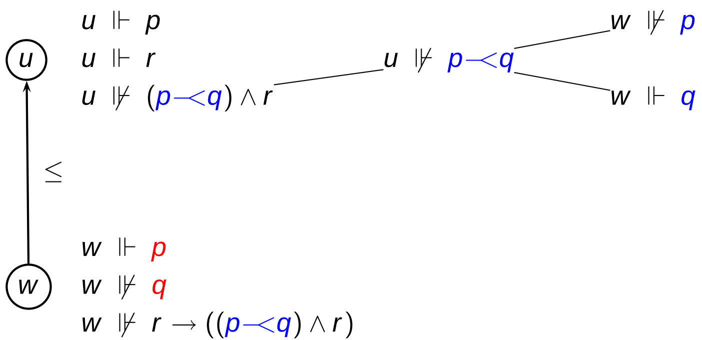

Uustalu's Example: Semantic Motivation

- Sequent $p \vdash q, r \rightarrow ((p \mathbin{-\!\!<} q) \wedge r)$ is valid
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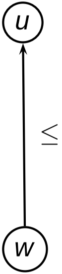

$u \Vdash p$

$u \Vdash r$

$u \not\Vdash (p \mathbin{-\!\!<} q) \wedge r$

$u \not\Vdash p \mathbin{-\!\!<} q$ $u \not\Vdash r$

$w \not\Vdash p$ $w \Vdash q$

$w \Vdash p$

$w \not\Vdash q$

$w \not\Vdash r \rightarrow ((p \mathbin{-\!\!<} q) \wedge r)$

Nested Sequents

- Negative structures: $N := \emptyset \mid A \mid (N, N) \mid N < P$
- Positive structures: $P := \emptyset \mid A \mid (P, P) \mid N > P$
- Sequents: $X \vdash Y$ where
  - X is a negative structure
  - Y is a positive structure
- $\{X\}$ denote top-level formulae of X
- Examples:
  - $r \vdash p \prec q$
  - $(p < q), r \vdash (q > r), ((p < q) > w)$
  - $\{(p < q), r, s\} = \{r, s\}$

LBiInt Rules

$$\frac{}{X, A \vdash A, Y}\ id$$

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$$\frac{}{X, A \vdash A, Y}\ id$$

$$\frac{X, A \rightarrow B \vdash A, Y \quad X, A \rightarrow B, B \vdash Y}{X, A \rightarrow B \vdash Y}\ \rightarrow_L$$

LBiInt Rules

$$\frac{}{X, A \vdash A, Y}\ id$$

$$\frac{X, A \to B \vdash A, Y \quad X, A \to B, B \vdash Y}{X, A \to B \vdash Y}\ \to_L$$

$$\frac{X \vdash A, A \prec B, Y \quad X, B \vdash A \prec B, Y}{X \vdash A \prec B, Y}\ \prec_R$$

LBiInt Rules

$$\frac{}{X, A \vdash A, Y}\ id$$

$$\frac{X, A \to B \vdash A, Y \quad X, A \to B, B \vdash Y}{X, A \to B \vdash Y}\ \to_L$$

$$\frac{X \vdash A, A \prec B, Y \quad X, B \vdash A \prec B, Y}{X \vdash A \prec B, Y}\ \prec_R$$

$$\frac{X_2 \vdash Y_2, \{Y_1\}}{X_1, (X_2 < Y_2) \vdash Y_1}\ < \ \{Y_1\} \not\subseteq \{Y_2\} \qquad \frac{\{X_1\}, X_2 \vdash Y_2}{X_1 \vdash Y_1, (X_2 > Y_2)}\ > \ \{X_1\} \not\subseteq \{X_2\}$$

LBiInt Rules

$$\frac{}{X, A \vdash A, Y}\ id$$

$$\frac{X, A \to B \vdash A, Y \quad X, A \to B, B \vdash Y}{X, A \to B \vdash Y}\ \to_L$$

$$\frac{X \vdash A, A \prec B, Y \quad X, B \vdash A \prec B, Y}{X \vdash A \prec B, Y}\ \prec_R$$

$$\frac{X_2 \vdash Y_2, \{Y_1\}}{X_1, (X_2 < Y_2) \vdash Y_1}\ < \ \{Y_1\} \not\subseteq \{Y_2\} \qquad \frac{\{X_1\}, X_2 \vdash Y_2}{X_1 \vdash Y_1, (X_2 > Y_2)}\ > \ \{X_1\} \not\subseteq \{X_2\}$$

$$\frac{A \vdash B, \{Y\}, (X, A \prec B > Y)}{X, A \prec B \vdash Y}\ \prec_{L2} \qquad \frac{(X < Y, A \to B), \{X\}, A \vdash B}{X \vdash Y, A \to B}\ \to_{R2}$$

Uustalu's Example Revisited

Using cut:

$$\dfrac{\dfrac{\dfrac{}{p \vdash q, p}\,\text{Id} \qquad \dfrac{}{q \vdash q}\,\text{Id}}{p \vdash q, p -\!\!< q}\,-\!\!<_R \qquad \dfrac{\dfrac{\dfrac{}{p -\!\!< q, r \vdash p -\!\!< q}\,\text{Id} \qquad \dfrac{}{p -\!\!< q, r \vdash r}\,\text{Id}}{p -\!\!< q, r \vdash (p -\!\!< q) \wedge r}\,\wedge_R}{p -\!\!< q \vdash r \to ((p -\!\!< q) \wedge r)}\,\to_R}{p \vdash q, r \to ((p -\!\!< q) \wedge r)}\,\text{cut}$$

Using LBiInt:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash q, \cdots, p}\,Id \qquad \dfrac{}{p, q \vdash q, \cdots}\,Id}{p \vdash q, \cdots, p -\!\!< q}\,-\!\!<_R}{(p < q, \cdots), p, r \vdash p -\!\!< q}\,< \qquad \dfrac{}{(p < q, \cdots), p, r \vdash r}\,Id}{(p < q, \cdots), p, r \vdash (p -\!\!< q) \wedge r}\,\wedge_R}{p \vdash q, r \to ((p -\!\!< q) \wedge r)}\,\to_{R2}$$

Save/Restore

- Rauszer's G1 vs LBiInt:

Lose context:

$$\dfrac{\dfrac{X, A \vdash B}{X \vdash A \to B} \to_R}{X \vdash Y, A \to B} w_R$$

Save context:

$$\dfrac{(X < Y, A \to B), \{X\}, A \vdash B}{X \vdash Y, A \to B} \to_{R2}$$

- Restore context:

$$\dfrac{X_2 \vdash Y_2, \{Y_1\}}{X_1, (X_2 < Y_2) \vdash Y_1} < \quad \{Y_1\} \not\subseteq \{Y_2\}$$

- Cut-elimination: instance of cut $\leadsto$ pair of save/restore rules

Soundness

- Relates syntax to semantics
- Formula-translation τ maps sequents to formulae

$$\begin{array}{lcl} \tau_N(\emptyset) & = & \top \\ \tau_N(A) & = & A \\ \tau_N(X, Y) & = & \tau_N(X) \wedge \tau_N(Y) \\ \tau_N(X < Y) & = & \tau_N(X) \prec \tau_P(Y) \end{array} \quad \begin{array}{lcl} \tau_P(\emptyset) & = & \bot \\ \tau_P(A) & = & A \\ \tau_P(X, Y) & = & \tau_P(X) \vee \tau_P(Y) \\ \tau_P(X > Y) & = & \tau_N(X) \rightarrow \tau_P(Y) \end{array}$$

  - $<$ is a proxy for $\prec$
  - $>$ is a proxy for $\rightarrow$
- Soundness: for every rule, show:
  - if τ of premises is valid then τ of conclusion is valid

Saturation and Blocking

Definition

We classify the rules of LBiInt_2 into three groups:

$$\begin{aligned}\text{Static Rules:} &= \{id, \wedge_L, \wedge_R, \vee_L, \vee_R, \to_L, \prec_R, \prec_{L1}, \to_{R1}\};\\ \text{Jump Rules:} &= \{\prec_{L2}, \to_{R2}\};\ \text{and}\\ \text{Return Rules:} &= \{<, >\}.\end{aligned}$$

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We call a sequence of static rule applications a *saturation*.

Definition

A sequent $X_0 \vdash Y_0$ is saturated if applying a static rule would yield at least one premise $X_1 \vdash Y_1$ such that $\{X_1\} \subseteq \{X_0\}$ and $\{Y_1\} \subseteq \{Y_0\}$.

Saturation and Blocking

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$$\begin{aligned} \text{Static Rules: } &= \{id, \wedge_L, \wedge_R, \vee_L, \vee_R, \rightarrow_L, -\!\!<_R, -\!\!<_{L1}, \rightarrow_{R1}\}; \\ \text{Jump Rules: } &= \{-\!\!<_{L2}, \rightarrow_{R2}\}; \text{ and} \\ \text{Return Rules: } &= \{<, >\}. \end{aligned}$$

We call a sequence of static rule applications a *saturation*.

Definition

A sequent $X_0 \vdash Y_0$ is saturated if applying a static rule would yield at least one premise $X_1 \vdash Y_1$ such that $\{X_1\} \subseteq \{X_0\}$ and $\{Y_1\} \subseteq \{Y_0\}$.

Definition

A LBiInt_2 rule ρ is applicable to a sequent $X_0 \vdash Y_0$ if for every premise $X_i \vdash Y_i$ of ρ, $\{X_i\} \not\subseteq \{X_0\}$ or $\{Y_i\} \not\subseteq \{Y_0\}$.

Corollary

Only jump and return rules are applicable to saturated sequents.

Proof Search Strategy

Function Prove
Input: sequent γ_0
Output: *true* (i.e. γ_0 is derivable) or *false* (i.e. γ_0 is not derivable)

1. If *id* is applicable to γ_0 then return *true*
2. Else if a static rule ρ is applicable to γ_0 then
   1. Let $\gamma_1, \cdots, \gamma_n$ be the premises of ρ obtained from γ_0
   2. Return $\bigwedge_{i=1}^{n} Prove(\gamma_i)$
3. Else if $Prove(\gamma_1) = true$ for some premise instance γ_1 obtained from γ_0 by applying $\rho \in \{ -\!\!<_{L2}, \to_{R2}, <, > \}$ then return *true*
4. Else return *false*.

Linear Sequents

Definition

1. If Γ/Δ are sets of formulae, then $\Gamma \vdash \Delta$ is a linear sequent.
2. If $X \vdash Y$ is a linear sequent and Γ/Δ are sets of formulae, then
   1. $(X < Y), \Gamma \vdash \Delta$ and
   2. $\Gamma \vdash \Delta, (X > Y)$

   are linear sequents.

Example

$C \vdash B, A \to B$
$(C < B, A \to B), C, A \vdash B$
$D \vdash E, ((C < B, A \to B), C, A > B)$

Lemma

Every LBiInt-derivation of a linear end-sequent contains only linear sequents.

Lists

Definition (Linear Sequent to List)

$$
\begin{array}{lcl}
list(\Gamma \vdash \Delta) & = & \langle \Gamma, \Delta \rangle \\
list((X < Y), \Gamma \vdash \Delta) & = & list(X \vdash Y) \leq \langle \Gamma, \Delta \rangle \\
list(\Gamma \vdash \Delta, (X > Y)) & = & list(X \vdash Y) \geq \langle \Gamma, \Delta \rangle
\end{array}
$$

Example

$$
\begin{array}{lcl}
list((C < B, A \to B), C, A \vdash B) & = & list(C \vdash B, A \to B) \leq \langle \{C, A\}, \{B\} \rangle \\
 & = & \langle \{C\}, \{B, A \to B\} \rangle \leq \langle \{C, A\}, \{B\} \rangle
\end{array}
$$

Remark

Lists encoded in sequents $\rightsquigarrow$ *branches in counter-model.*

List Operations

Corollary

A backward LBiInt$_2$ rule application to a linear sequent $X \vdash Y$ can be viewed as an operation on list$(X \vdash Y)$:

- *Conclusion/premise is the list before/after the operation*
- *Jump rules: append a node to the list*
- *Static rules: saturate the end node*
- *Return rules: remove end node, update penultimate node.*

Example

$$\frac{(C < B, A \to B), C, A \vdash B}{C \vdash B, A \to B} \to_{R2} \qquad \frac{\langle \{C\}, \{B, A \to B\} \rangle \;\; \leq \;\; \langle \{C, A\}, \{B\} \rangle}{\langle \{C\}, \{B, A \to B\} \rangle}$$

Termination

Lemma (Bounded Lists)

Let $X \vdash Y$ be any sequent encountered during proof search. Using jump rules, list$(X \vdash Y)$ can be extended at most $\mathcal{O}(m^2)$ times.

Lemma (Saturation)

Let $X \vdash Y$ be any sequent encountered during proof search. Then the saturation process for $X \vdash Y$ terminates after $\mathcal{O}(m)$ steps.

Theorem

The proof search strategy terminates.

Proof.

- Nodes/lists are bounded in size/length
- Jump/return rules cannot repeatedly create/remove nodes
  - Every update adds one more (sub)formula to a node
  - Eventually no subformulae can be added to any node
  - Return rules are blocked

Countermodel Construction

- Want to show: if A valid then $\emptyset \vdash A$ derivable
- Usually show contrapositive:
  - if $\emptyset \vdash A$ not derivable then A not valid
  - if $\mathit{Prove}(\emptyset \vdash A) = \mathit{false}$, then $\emptyset \vdash A$ has a counter-model
- More generally:
  - if $\mathit{Prove}(\Gamma \vdash \Delta) = \mathit{false}$, then $w \Vdash \Gamma$ and $w \nVdash \Delta$ for some $\langle W, \leq, V \rangle$ and $w \in W$
- Extract countermodel from failed proof search
- Usually done inductively by combining sub-models
- We use the notion of a pre-model

Pre-model: a “model in progress”

- Tree of nodes
  - Nodes are pairs $\langle \Gamma, \Delta \rangle$ of sets of formulae
  - Nodes are linked by labels $\leq$ or $\geq$
- Nodes are marked either I or C
  - Each inner node is marked C
  - Each C-node is saturated
- Satisfies “pre-model properties”
- When all nodes are marked C, the root has a counter-model
  - By induction on the size of the pre-model
  - Pre-model properties ensure forcing and persistence
- Aim: build a pre-model with all nodes marked C
  - Use failed proof search to guide us

Completeness Proof

- Create pre-model with root I-node $\langle \emptyset, \{A\} \rangle$
- Assume $\mathit{Prove}(\emptyset \vdash A) = \mathit{false}$
- Use failed proof search to update pre-model
  - Static rules: saturate leaf nodes
  - Jump rules: mark node C, create new I-nodes
  - Return rules: remove I-node, update previous node and mark it I
  - No rule applicable: mark leaf C
- Concatenate lists obtained from linear sequents to obtain a tree
- When all leaves are marked C, the root has a model
  - $w \not\Vdash \{A\}$ for some $\langle W, \leq, V \rangle$ and $w \in W$
  - A is not valid
- By contrapositive, if A is valid, then $\mathit{Prove}(\emptyset \vdash A) = \mathit{true}$

Restart Example

$$
\dfrac{
  \dfrac{
    \vdots
    \qquad
    \dfrac{
      \dfrac{
        (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0
        \qquad \vdots
      }{
        (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot
      }\to_L
    }{
      p_0 \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0
    }\to_{R2}
  }{
    \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0
  }\prec_R
}{
  \dfrac{
    \dfrac{
      \dfrac{
        (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0
        \qquad \vdots
      }{
        (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot
      }\to_L
    }{
      \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot
    }\to_{R2}
  }{
    \color{blue}{\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot}
  }\to_{R1}
}<
$$

(I) $\langle \{\}, \{p_1, ((\top \prec p_0) \to \bot) \to \bot\} \rangle$

Restart Example

$$
\dfrac{\dfrac{\vdots \qquad \dfrac{\dfrac{\dfrac{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0 \qquad \vdots}{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot} \to_L}{p_0 \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0} \to_{R2}}{} }{\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0} \prec_R
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0}{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0} < \qquad \vdots}{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot} \to_L}{\color{blue}{\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot}} \to_{R2}}{\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot} \to_{R1}
$$

(I) $\langle \{\}, \{p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot\} \rangle$

Restart Example

$$\dfrac{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0), p_0, (\top -\!\!< p_0) \to \bot \vdash \bot, \top -\!\!< p_0 \qquad \vdots}{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0), p_0, (\top -\!\!< p_0) \to \bot \vdash \bot} \; \to_L$$

$$\dfrac{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0), p_0, (\top -\!\!< p_0) \to \bot \vdash \bot}{p_0 \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0} \; \to_{R2}$$

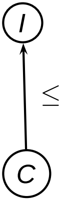
$$\dfrac{\vdots \qquad p_0 \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0}{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0} \; -\!\!<_R$$

$$\dfrac{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0}{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot, \top -\!\!< p_0} \; <$$

$$\dfrac{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot, \top -\!\!< p_0 \qquad \vdots}{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot} \; \to_L$$

$$\dfrac{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot}{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot} \; \to_{R2}$$

$$\dfrac{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot}{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot} \; \to_{R1}$$



$$I: \quad \langle \{(\top -\!\!< p_0) \to \bot\}, \{\bot\} \rangle$$

$$C: \quad \langle \{\}, \{p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot\} \rangle$$

Restart Example

$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0 \qquad \vdots$$

$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot \quad \to_L$$

$$\vdots \qquad p_0 \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0 \quad \to_{R2}$$

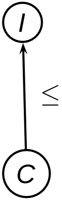
$$\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0 \quad \prec_R$$

$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0 \quad < \qquad \vdots$$

$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot \quad \to_L$$

$$\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot \quad \to_{R2}$$

$$\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot \quad \to_{R1}$$



$$I: \langle \{ (\top \prec p_0) \to \bot \}, \{ \bot, \top \prec p_0 \} \rangle$$

$$C: \langle \{\}, \{ p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot \} \rangle$$

Restart Example

$$\dfrac{\dfrac{\vdots \qquad \dfrac{\dfrac{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0 \qquad \vdots}{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot} \to_L}{p_0 \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0} \to_{R2}}{\color{blue}{\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0}} \prec_R}{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0} <$$

$$\dfrac{\dfrac{\dfrac{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0 \qquad \vdots}{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot} \to_L}{\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot} \to_{R2}}{\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot} \to_{R1}$$

(I) $\langle \{\}, \{p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0\} \rangle$

Restart Example

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          (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0
          \qquad \vdots
        }{
          (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot
        } \to_L
      }{
        {\color{blue} p_0 \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0}
      } \to_{R2}
    }{
      \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0
    } \prec_R
  }{
    (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0
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      \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot
    } \to_{R2}
  }{
    \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot
  } \to_{R1}
} \to_L
$$

I $\quad \langle \{p_0\}, \{p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0\} \rangle$

Restart Example

$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0 \qquad \vdots$$

$$\rule{20em}{0.4pt} \; \to_L$$

$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot$$

$$\rule{20em}{0.4pt} \; \to_{R2}$$

$$\vdots \qquad p_0 \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0$$

$$\rule{20em}{0.4pt} \; \prec_R$$

$$\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0$$

$$\rule{20em}{0.4pt} \; <$$

$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0 \qquad \vdots$$

$$\rule{20em}{0.4pt} \; \to_L$$

$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot$$

$$\rule{20em}{0.4pt} \; \to_{R2}$$

$$\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot$$

$$\rule{20em}{0.4pt} \; \to_{R1}$$

$$\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot$$

I: $\langle \{p_0, (\top \prec p_0) \to \bot\}, \{\bot\} \rangle$

$\uparrow \; \leq$

C: $\langle \{p_0\}, \{p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0\} \rangle$

Restart Example

$$\vdots$$

$$(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0), p_0, (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot, \top \mathrel{-\!\!<} p_0 \quad \to_L$$

$$(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0), p_0, (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot \quad \to_{R2}$$

$$p_0 \vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0$$

$$\vdots \qquad \mathrel{-\!\!<}_R$$

$$\vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0 \quad <$$

$$(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot), (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot, \top \mathrel{-\!\!<} p_0 \qquad \vdots \quad \to_L$$

$$(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot), (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot \quad \to_{R2}$$

$$\vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot \quad \to_{R1}$$

$$\vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot$$

I: $\langle \{p_0, (\top \mathrel{-\!\!<} p_0) \to \bot\}, \{\bot, \top \mathrel{-\!\!<} p_0\} \rangle$

$\leq$ ($C \to I$)

C: $\langle \{p_0\}, \{p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0\} \rangle$

Restart Example

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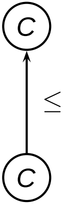
        (< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0), p_0, (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot, \top \mathrel{-\!\!<} p_0 \qquad \vdots
      }{
        (< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0), p_0, (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot
      } \to_L
    }{
      p_0 \vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0
    } \to_{R2}
  }{
    \vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0
  } \mathrel{-\!\!<}_R
}{
  \dfrac{
    \dfrac{
      \dfrac{
        (< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot), (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot, \top \mathrel{-\!\!<} p_0 \qquad \vdots
      }{
        (< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot), (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot
      } \to_L
    }{
      \vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot
    } \to_{R2}
  }{
    \vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot
  } \to_{R1}
} <
$$



$$\langle \{p_0, (\top \mathrel{-\!\!<} p_0) \to \bot\}, \{\bot, \top \mathrel{-\!\!<} p_0\} \rangle$$

$$\langle \{p_0\}, \{p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0\} \rangle$$

Conclusions and Further Work

- Bi-Intuitionistic logic presents challenges for proof search
- Nested sequents give:
  - a cut-free sequent calculus LBiInt
  - a complete, terminating proof search strategy
- Further work:
  - Generalise LBiInt to other similar logics (KtS4, S5, bi-Lambek, $\cdots$)
  - Explore connection with type theory