On two translation-invariant pseudo-distances between infinite words and their applications to cellular automata

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Overview

- In a sense, the product topology on bi-infinite words cannot be induced by a "good" distance.
- The Besicovitch and Weyl pseudo-distances define new quotient spaces where the shift "behaves well".
- Cellular automata (CA) also behave "well" with respect to them.

The space of bi-infinite words

Let S be an alphabet—finite, $|S| \geq 2$.
The product topology on $\mathcal{C} = S^{\mathbb{Z}}$ is induced by the distance

$$d(c_1, c_2) = 2^{-r} \text{ if } r = \min\{|x| \mid c_1(x) \neq c_2(x)\}$$

The shift map $\sigma \in S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is defined by

$$\sigma(c)(x) = c(x+1)$$

The temporally periodic points for σ
are precisely the spatially periodic words.

"Its's not a bug, it's a feature"

1. σ is transitive, *i.e.*,
   $\forall U, V \subseteq S^{\mathbb{Z}}$ open $\exists n \mid \sigma^{-n}(U) \cap V \neq \emptyset$.
   - True for the cylinders $C_r(c) = \{c' \in S^{\mathbb{Z}} \mid d(c, c') < r\}$
2. σ has a dense set of periodic points
   - If $c_r = (c[-r : r])^{\mathbb{Z}}$ then $d(c, c_r) \leq 2^{-r}$.
3. σ is sensitive to initial conditions, *i.e.*,
   $\exists \delta > 0 \mid \forall c \in S^{\mathbb{Z}}, r > 0 \exists c' \in S^{\mathbb{Z}}, n \in \mathbb{N} \mid$
   $d(c_1, c_2) < r, d(\sigma^n(c), \sigma^n(c')) > \delta$
   - Put $\delta = 1/2$. Choose $n > -\log_2 r$, $d(c, c') = 2^{-n}$.

This is Devaney's definition of chaos! so that:

- the shift is a chaotic map—**WHAT!??**
- no translation-invariant distance induces the product topology

How to Solve It[1]

Blanchard, Formenti, and Kůrka. (1999)

1. Construct a pseudo-distance d on $S^{\mathbb{Z}}$.
2. Consider the relation $c_1 \sim_d c_2$ iff $d(c_1, c_2) = 0$.
3. Work in the quotient space $S^{\mathbb{Z}} / \sim_d$.

[1] Apologies to G. Polya.

The Besicovitch pseudo-distance

Consider the sets of the form $B_n = \{-n, \ldots, n\}$.
Put

$$d_{\mathcal{B}}(c_1, c_2) = \limsup_{n \to \infty} \frac{|\{x \in B_n \mid c_1(x) \neq c_2(x)\}|}{2n+1}$$

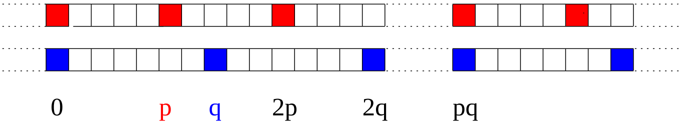
$d_{\mathcal{B}}$ is translation-invariant.

- Call $H_n(c_1, c_2) = |\{x \in B_n \mid c_1(x) \neq c_2(x)\}|$.
- Then $|H_n(c_1, c_2) - H_n(\sigma(c_1), \sigma(c_2))| \leq 2$.

$d_{\mathcal{B}}(c_1, c_2) = 0$ iff $\{x \mid c_1(x) \neq c_2(x)\}$ is sparse.

Example

Let p, q be primes.
Let $c_1(x)$ be 1 if $x \in p\mathbb{Z}$, 0 otherwise.
Let $c_2(x)$ be 1 if $x \in q\mathbb{Z}$, 0 otherwise.



Then

$$d_{\mathcal{B}}(c_1, c_2) = \frac{1}{p} + \frac{1}{q} - \frac{1}{pq}$$

The Weyl pseudo-distance

Consider the sets of the form $W_n = \{0 \ldots, n-1\}$.
Put

$$d_{\mathcal{W}}(c_1, c_2) = \limsup_{n \to \infty} \sup_{z \in \mathbb{Z}} \frac{|\{x \in W_n \mid c_1(z+x) \neq c_2(z+x)\}|}{n}$$

$d_{\mathcal{W}}(c_1, c_2) = 0$ iff $\{x \mid c_1(x) \neq c_2(x)\}$ is uniformly sparse.

- $d_{\mathcal{W}}$ is actually a limit because of Fekete's lemma.
- $d_{\mathcal{B}}$ is usually not a limit.

Fekete's lemma

Let $f : \{1, 2, \ldots\} \to [0, \infty)$ satisfy

$$f(m+n) \leq f(m) + f(n) \ \forall m, n \geq 1 .$$

Then

$$\lim_{n \to \infty} \frac{f(n)}{n}$$

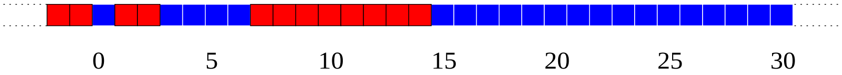
exists, and equals

$$\inf_{n \geq 1} \frac{f(n)}{n}$$

Example

Let $a_n = \sum_{i=1}^{n} 2^i$. Let $c_0(x) = 0$ for all x and

$$c(x) = \begin{cases} 1 & \text{if } x \in \{a_{2k-1}+1, \ldots, a_{2k}\}, \\ 0 & \text{if } x \in \{a_{2k}+1, \ldots, a_{2k+1}\}, \\ c(-x) & \text{if } x < 0\,. \end{cases}$$



- $d_{\mathcal{W}}(c_0, c) = \lim_{n\to\infty} \sup_{z\in\mathbb{Z}} \dfrac{|\{x \in W_n \mid c(x) \neq 0\}|}{n} = 1.$
- $d_{\mathcal{B}}(c_0, c) = \dfrac{2}{3}$ but $\liminf_n \dfrac{|\{x \in \{-n, \ldots, n\} \mid c(x) \neq 0\}|}{2n+1} = \dfrac{1}{3}$.

Completeness

$\mathcal{C}$ is complete.

- Being Cauchy means being ultimately equal on any finite set.

$\mathcal{B}$ is complete.

- A Cauchy sequence has a subsequence that "stays tight".
- $\{B_n\}$ has a subsequence that "grows fast".
- Join to find a (unique) limit point.

$\mathcal{W}$ is not complete.

- Non-trivial.

Completeness is the reason why $d_{\mathcal{B}}$ has been preferred to $d_{\mathcal{W}}$.

Compactness

$\mathcal{C}$ is compact.

- By Tychonoff's theorem.

$\mathcal{B}$ is not compact.

- Non-trivial; proof based on Sturmian sequences.

$\mathcal{W}$ is not compact.

- Compact metric spaces are complete.

Connectedness

$\mathcal{C}$ is totally disconnected.

- $\mathcal{C}$ is product of t.d. spaces.

$\mathcal{B}$ is arcwise connected.

- Construction based on Toeplitz sequences.

$\mathcal{W}$ is arcwise connected.

- Same as above.

Dense subsets

Let $\mathcal{P}$ be the set of periodic configurations.

$\mathcal{P}$ is dense in $\mathcal{C}$.

- Let $c_n = c_{[-n \dots n]}$.
- Then $\lim_{n\to\infty} c_n = c$ in the product topology.

$\mathcal{P}$ is not dense in $\mathcal{B}$.

- Let $c(x) = 1$ iff $x \geq 0$.
- Then $d_{\mathcal{B}}(c, c') \geq \dfrac{1}{2}$ for any periodic c'.

$\mathcal{P}$ is not dense in $\mathcal{W}$.

- Follows from previous and $d_{\mathcal{W}} \geq d_{\mathcal{B}}$.

Note that: distinct elements of $\mathcal{P}$ have $d_{\mathcal{W}} \geq d_{\mathcal{B}} > 0$.

Overview

	$\mathcal{C}$	$\mathcal{B}$	$\mathcal{W}$
complete	yes	yes	no
compact	yes	no	no
connected	no	yes	yes
$\mathcal{P}$ dense	yes	no	no

$\mathcal{P}$: set of periodic configurations.

Cellular automata

A cellular automaton (CA) is a triple $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ where

- S is an alphabet
- $\mathcal{N} = \{n_1, \ldots, n_k\}$ is a finite subset of $\mathbb{Z}$—neighborhood
- $f : S^k \to S$ is a function—local map

The local map induces a global map $F_{\mathcal{C}} : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ by

$$F_{\mathcal{C}}(c)(x) = f(c(x + n_1), \ldots, c(x + n_k))$$

CA are well-defined on Besicovitch and Weyl classes

- Let $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ be a CA.
- Then the value of c at a point affects the values of $F_{\mathcal{C}}(c)$ at no more than $|\mathcal{N}|$ points.
- Consequently,

$$d_{\mathcal{B}}(F_{\mathcal{C}}(c_1), F_{\mathcal{C}}(c_2)) \leq |\mathcal{N}| \cdot d_{\mathcal{B}}(c_1, c_2)$$

and

$$d_{\mathcal{W}}(F_{\mathcal{C}}(c_1), F_{\mathcal{C}}(c_2)) \leq |\mathcal{N}| \cdot d_{\mathcal{W}}(c_1, c_2)$$

so that the following are well defined:

$$F_{\mathcal{B}}([c]_{\mathcal{B}}) = [F(c)]_{\mathcal{B}} \ ; \ F_{\mathcal{W}}([c]_{\mathcal{W}}) = [F(c)]_{\mathcal{W}}$$

Surjectivity

Theorem (well-known)
Let $\mathcal{A} = \langle S, \{-r, \ldots, r\}, f \rangle$ be a CA. The following are equivalent.

1. $F_{\mathcal{C}} : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is surjective.
2. For every $p : \{0, \ldots, n-1\} \to S$ there exists $\pi : \{-r, \ldots, n+r-1\} \to S$ s.t.

$$f(\pi(i-r), \ldots, \pi(i+r)) = p(i) \; \forall i \in \{0, \ldots, n-1\}$$

Reason why: the product space is compact.

Example

The AND CA on two neighbors

- $S = \{0, 1\}$
- $\mathcal{N} = \{0, 1\}$
- $f(a, b) = a \text{ AND } b$

is not surjective because the pattern 101 is "orphan":

$$
\begin{array}{lcccccc}
c_{t+1} & \cdots & 1 & 0 & 1 & \cdot & \cdots \\
c_t & \cdots & 1 & 1 & 1 & 1 & \cdots \\
 & \cdots & \cdot & \uparrow & \cdot & \cdot & \cdots
\end{array}
$$

Surjectivity of 1D CA in Besicovitch and Weyl spaces

Theorem (Blanchard, Cervelle, Formenti 2005)
Let $\mathcal{A}$ be a CA. The following are equivalent.

- $F_{\mathcal{C}}$ is surjective.
- $F_{\mathcal{B}}$ is surjective.
- $F_{\mathcal{W}}$ is surjective.

Reason why:

- $F_{\mathcal{B}}$ is surjective iff $\forall c \exists c' \mid d_{\mathcal{B}}(c, F_{\mathcal{C}}(c')) = 0$. Similar for $F_{\mathcal{W}}$.
- If $F_{\mathcal{B}}$ or $F_{\mathcal{W}}$ is surjective, then $F_{\mathcal{C}}$ is surjective on $\mathcal{P}$.

Other links with CA properties

Theorem (Blanchard, Formenti, Kůrka 1999)

- If $F_{\mathcal{C}}$ is equicontinuous,
  then $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$ are equicontinuous.
- If $F_{\mathcal{C}}$ has equicontinuity points,
  then $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$ have equicontinuity points.
- If $F_{\mathcal{B}}$ or $F_{\mathcal{W}}$ is transitive,
  then $F_{\mathcal{C}}$ is transitive.
- If $F_{\mathcal{B}}$ or $F_{\mathcal{W}}$ is sensitive,
  then $F_{\mathcal{C}}$ is sensitive.

A starting point for generalization

Consider $B_n = \{-n, \ldots, n\}$ and $W_n = \{0, \ldots, n = 1\}$ as windows.

- Both $d_{\mathcal{B}}$ and $d_{\mathcal{W}}$ are upper limits of relative densities under some window.
- For $d_{\mathcal{B}}$ the window is kept in place and progressively enlarged.
- For $d_{\mathcal{W}}$ the window is moved all around between enlargement.
- In fact,

$$d_{\mathcal{W}}(c_1, c_2) = \limsup_{n \to \infty} \sup_{z \in \mathbb{Z}} \frac{|\{x \in B_n \mid c_1(x+z) \neq c_2(x+z)\}|}{|2n+1|}$$

i.e., $d_{\mathcal{W}}$ can be found via the B_n's.

The key is thus to find a good sequence of windows.

Exhaustive sequences

The sequence $B_n = \{-n, \ldots, n\}$ satisfies the following properties:

1. $|B_n| < \infty$ for every n.
2. $B_n \subseteq B_{n+1}$ for every n.
3. $\bigcup_{n \in \mathbb{N}} B_n = \mathbb{Z}$.

A sequence $\mathcal{X} = \{X_n\}$ such that:

1. $|X_n| < \infty$ for every n,
2. $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$,
3. $\bigcup_{n \in \mathbb{N}} X_n = \mathbb{Z}^d$

shall be said to be exhaustive.

Generalized Besicovitch and Weyl distances

Let $\mathcal{X} = \{X_n\}_{n \in \mathbb{N}}$ be an exhaustive sequence for $\mathbb{Z}^d$. Then

$$d_{\mathcal{B},\mathcal{X}}(c_1, c_2) = \limsup_{n \to \infty} \frac{|\{x \in X_n \mid c_1(x) \neq c_2(x)\}|}{|X_n|}$$

and

$$d_{\mathcal{W},\mathcal{X}}(c_1, c_2) = \limsup_{n \to \infty} \sup_{z \in \mathbb{Z}^d} \frac{|\{x \in X_n \mid c_1(x+z) \neq c_2(x+z)\}|}{|X_n|}$$

generalize $d_{\mathcal{B}}$ and $d_{\mathcal{W}}$ from the one-dimensional case.
Call $\mathcal{B}_{\mathcal{X}}$ and $\mathcal{W}_{\mathcal{X}}$ the corresponding quotient spaces.

Families of disks

It is of interest when $\mathcal{X}$ is the family of disks w.r.t.

- the von Neumann neighborhood

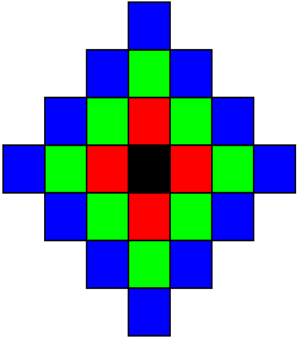
$$\mathrm{vN} = \{x \mid \sum_{i=1}^{d} |x_i| \leq 1\}$$

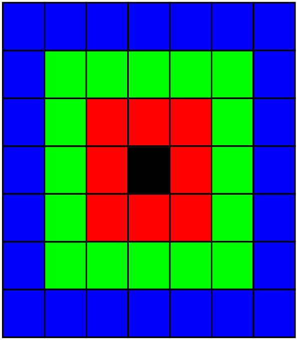
- the Moore neighborhood

$$\mathrm{M} = \{x \mid \max_{i=1}^{d} |x_i| \leq 1\}$$

- in general, any set of generators T for $\mathbb{Z}^d$ over $\mathbb{Z}$.

von Neumann and Moore disks for radius 1, 2, 3





Equivalence does not depend on generators

Theorem (Capobianco, 2009)
Let T be a set of generators for $\mathbb{Z}^d$.
Let $\mathcal{X}$ be the family of disks w.r.t. T.

1. As T varies, the associate Besicovitch pseudo-distances $d_{\mathcal{B},\mathcal{X}}$ are pairwise metrically equivalent.
2. In particular, if $d_{B,T}(c_1, c_2) = 0$ for any T, then $d_{B,T}(c_1, c_2) = 0$ for every T.
3. The above also hold for the Weyl pseudo-distances $d_{\mathcal{W},\mathcal{X}}$.

Reason why: the "rate of polynomial growth" is defined in a precise sense.

Topological properties in higher dimension

Theorem (Capobianco, to appear)
Consider $S^{\mathbb{Z}^d}$ with $\mathcal{X}$ the Moore disks sequence.

- $\mathcal{B}_{\mathcal{X}}$ is complete.
- $\mathcal{B}_{\mathcal{X}}$ and $\mathcal{W}_{\mathcal{X}}$ are arcwise connected.
- If $c_1, c_2 \in \mathcal{P}$, $c_1 \neq c_2$ then $d_{\mathcal{W},\mathcal{X}}(c_1, c_2) \geq d_{\mathcal{B},\mathcal{X}}(c_1, c_2) > 0$.
- $\mathcal{P}$ is not dense in either $\mathcal{B}_{\mathcal{X}}$ or $\mathcal{W}_{\mathcal{X}}$.

Overview for higher dimension

	$\mathcal{C}$	$\mathcal{B}$	$\mathcal{W}$
complete	yes	yes	??
compact	yes	??	??
connected	no	yes	yes
$\mathcal{P}$ dense	yes	no	no

$\mathcal{P}$: set of periodic configurations.

Surjectivity in higher dimension

Theorem (Capobianco, 2009)
Let $\langle S, \mathcal{N}, f \rangle$ a d-dimensional CA. The following are equivalent.

1. $F_{\mathcal{C}}$ is surjective.
2. $F_{\mathcal{B}}$ is surjective.
3. $F_{\mathcal{W}}$ is surjective.

Reason why: an "orphan" pattern can be use to construct a c such that $d_{\mathcal{B},\mathcal{X}}(c, F_{\mathcal{C}}(c')) \geq \delta > 0$ for all c'.

Surjectivity in local terms for $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$?

- Surjective CA on $\mathcal{C}$ are characterized in local terms.
- Is there anything like that for $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$?
- Problem: In $\mathcal{B}$ and $\mathcal{W}$ single occurrences don't count.
- Idea: But the set of all occurrences might!

Measuring sets

Given $\mathcal{X}$, for $U \subseteq \mathbb{Z}^d$ put

$$\mathrm{dens\,sup}_{\mathcal{B},\mathcal{X}} U = \limsup_{n \to \infty} \frac{|U \cap X_n|}{|X_n|}$$

and

$$\mathrm{dens\,sup}_{\mathcal{W},\mathcal{X}} U = \limsup_{n \to \infty} \sup_{z \in \mathbb{Z}^d} \frac{|(z + U) \cap X_n|}{|X_n|}$$

$\mathrm{dens\,inf}_{\mathcal{B},\mathcal{X}}(U)$ and $\mathrm{dens\,inf}_{\mathcal{W},\mathcal{X}}(U)$ are defined similarly.

Measuring pattern occurrences

Theorem (Capobianco, to appear)
Suppose $d_{\mathcal{B},\mathcal{X}}(c_1, c_2) = 0$. Then for any pattern p

$$\mathrm{dens\,sup}_{\mathcal{B},\mathcal{X}}\mathrm{occ}(p, c_1) = \mathrm{dens\,sup}_{\mathcal{B},\mathcal{X}}\mathrm{occ}(p, c_2)$$

and

$$\mathrm{dens\,inf}_{\mathcal{B},\mathcal{X}}\mathrm{occ}(p, c_1) = \mathrm{dens\,inf}_{\mathcal{B},\mathcal{X}}\mathrm{occ}(p, c_2)$$

A similar result holds for $d_{\mathcal{W},\mathcal{X}}$, $\mathrm{dens\,sup}_{\mathcal{W},\mathcal{X}}(U)$ and $\mathrm{dens\,inf}_{\mathcal{W},\mathcal{X}}(U)$.
Reason why: a counting argument.

Surjectivity in local terms for $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$!

Theorem (Capobianco, to appear)
Suppose $\mathcal{X}$ is a family of disks.
Let F be the global function of a CA.
The following are equivalent.

1. For every c there exists c' s.t. $d_{\mathcal{B},\mathcal{X}}(c, F(c')) = 0$.
2. For every p there exists c' s.t. $\mathrm{dens\,sup}_{\mathcal{B},\mathcal{X}} \mathrm{occ}(p, F(c')) > 0$.

A similar result holds for $d_{\mathcal{W},\mathcal{X}}$ and $\mathrm{dens\,sup}_{\mathcal{W},\mathcal{X}}$.
Reason why: invariance of upper density.

Configurations and CA on finitely generated groups

Suppose every element of a group G "is" a finite word on a finite $T \subseteq G$.

- We can again define configurations.
- We can define translation by $g \in G$ as

$$c^g(h) = c(gh) \quad \forall h \in G$$

- We can define CA on G via

$$F(c)(g) = f\left(c^g|_{\mathcal{N}}\right)$$

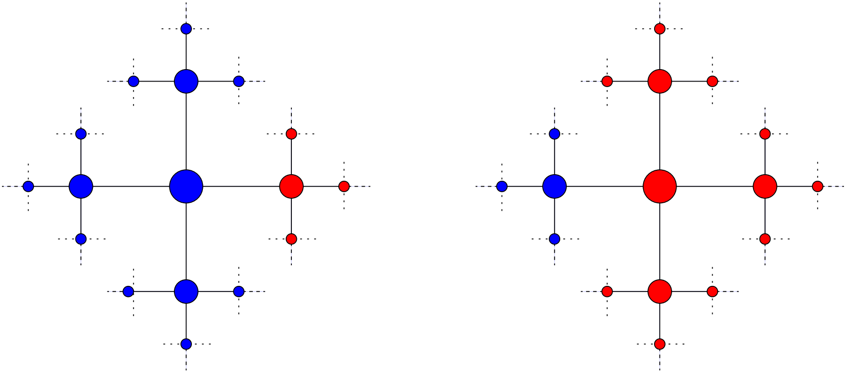
- We can define Besicovitch and Weyl spaces on S^G via $\mathcal{X}$.
- And we run risks!

$d_{\mathcal{B}}$ might not be translation-invariant!

- Let G be the free group on two generators a, b.
- Let X_n be the set of reduced words on $\{a, b, a^{-1}, b^{-1}\}$ of length $\leq n$.
- Let $c_1 = 0 \, \forall g \in G$.
- Let $c_2(g) = 1$ if $g \in aG$, 0 otherwise.

Then $d_{\mathcal{B},\mathcal{X}}(c_1, c_2) = \dfrac{1}{4}$ but $d_{\mathcal{B},\mathcal{X}}(c_1^a, c_2^a) = \dfrac{3}{4}$.

c_2 and c_2^a from previous example



The case of subexponential growth

Theorem (Capobianco, 2009 and to appear)
Let G be a group of subexponential growth.
Let $\mathcal{X}$ be the sequence of disks w.r.t. a set of generators T.

- CAare well-defined on $\mathcal{B}_{\mathcal{X}}$ and $\mathcal{W}_{\mathcal{X}}$.
- For arbitrary CA, the following are equivalent.
  - $F_{\mathcal{C}}$ is surjective.
  - $F_{\mathcal{B}}$ is surjective.
  - $F_{\mathcal{W}}$ is surjective.
  - For every p there exists c' s.t. $\operatorname{dens\,sup}_{\mathcal{B},\mathcal{X}} \operatorname{occ}(p, F(c')) > 0$.
  - For every p there exists c' s.t. $\operatorname{dens\,sup}_{\mathcal{W},\mathcal{X}} \operatorname{occ}(p, F(c')) > 0$.

Moreover, if G is of polynomial growth then:

- $d_{\mathcal{B},\mathcal{X}}$ is invariant by translations.
- The $d_{\mathcal{B},\mathcal{X}}$'s are pairwise metrically equivalent as T varies.
- Same for the $d_{\mathcal{W},\mathcal{X}}$.
- In particular, $\mathcal{B}$ and $\mathcal{W}$ do not depend on the choice of T.

Conclusions and further research

Conclusions

- The Besicovitch and Weyl pseudo-distance determine a structure of the space much different than the product topology.
- Nonetheless, they are interesting items *per se*.
- And they may provide insights on CA behavior.

Further research

- Determine the topological properties of $\mathcal{B}$ and $\mathcal{W}$ in more general setting.
- Find "good" dense sets.
- Find links between properties of $F_{\mathcal{C}}$, $F_{\mathcal{B}}$ and $F_{\mathcal{W}}$.
- Refine existing results.

Suggested readings

- F. Blanchard, E. Formenti, P. Kůrka. (1999) Cellular automata in the Cantor, Besicovitch and Weyl topological spaces. *Complex Systems* **11(2)**, 107–123.
- F. Blanchard, J. Cervelle, E. Formenti. (2005) Some results about the chaotic behavior of cellular automata. *Theor. Comp. Sci.* **349**, 318–336.
- S. Capobianco. (2009) Surjunctivity for cellular automata in Besicovitch spaces. *Journal of Cellular Automata* **4(2)**, 89–98.

Thank you for attention!

Any questions?