Lower Bounds on Circuit Complexity

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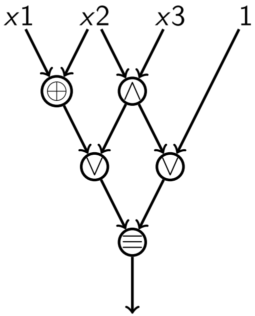
Boolean Circuits

- inputs: propositional variables $x_1, x_2, \ldots, x_n$ and constants 0, 1
- gates: binary functions
- fan-out of a gate is unbounded



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$$\left(16(t+n+2)^2\right)^t .$$
Each of t gates is assigned one of 16 possible binary Boolean functions that acts on two previous nodes, and each previous node can be either a previous gate ($\leq t$ choices) or a variables or a constant ($\leq n+2$ choices).

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Each of t gates is assigned one of 16 possible binary Boolean functions that acts on two previous nodes, and each previous node can be either a previous gate ($\leq t$ choices) or a variables or a constant ($\leq n+2$ choices).
- For $t = 2^n/(10n)$, $F(n,t)$ is approximately $2^{2^n/5}$, which is $\ll 2^{2^n}$.
- Thus, the circuit complexity of almost all Boolean functions on n variables is exponential in n. Still, we do not know any explicit function with super-linear circuit complexity.

Known Lower Bounds

	circuit size	formula size
full binary basis B_2	$3n - o(n)$ [Blum]	$n^{2-o(1)}$ [Nechiporuk]
basis $U_2 = B_2 \setminus \{\oplus, \equiv\}$	$5n - o(n)$ [Iwama et al.]	$n^{3-o(1)}$ [Hastad]
monotone basis $M_2 = \{\vee, \wedge\}$	exponential [Razborov; Alon, Boppana; Andreev; Karchmer, Wigderson]	

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- To avoid tricks like this one, we say that a function f is explicitly defined if $f^{-1}(1)$ is in NP.
- Usually, under a Boolean function f we actually understand an infinite sequence $\{f_n \mid n = 1, 2, \dots\}$.

Known Lower Bounds for Circuits over B_2

Known Lower Bounds

$2n - c$	[Schnorr, 74]
$2.5n - o(n)$	[Paul, 77]
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Gate Elimination

All the proofs are based on the so-called gate elimination method. This is essentially the only known method for proving lower bounds on circuit complexity.

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Remark

This method is very unlikely to produce non-linear lower bounds.

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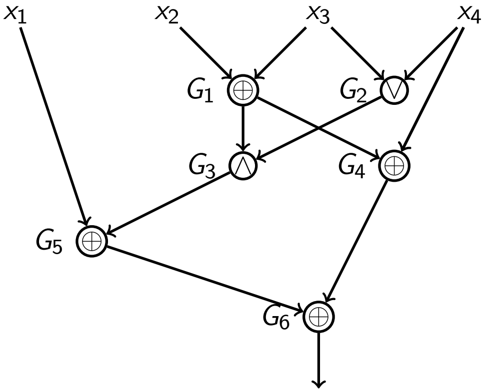

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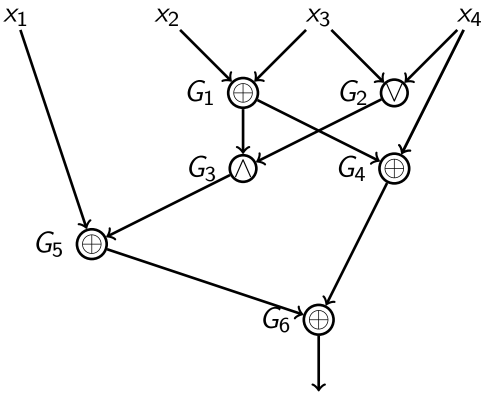

assign $x_1 = 1$

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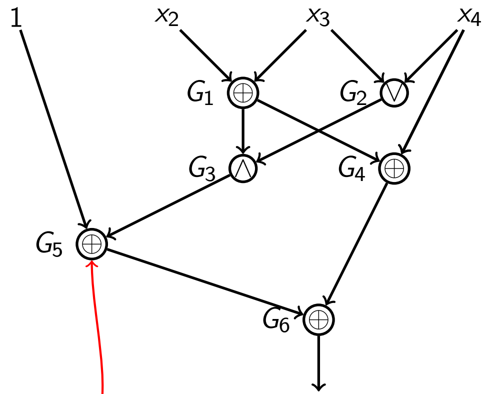

G_5 now computes $G_3 \oplus 1 = \neg G_3$

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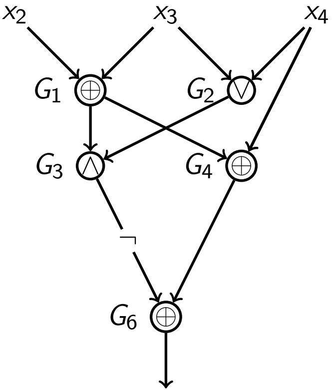

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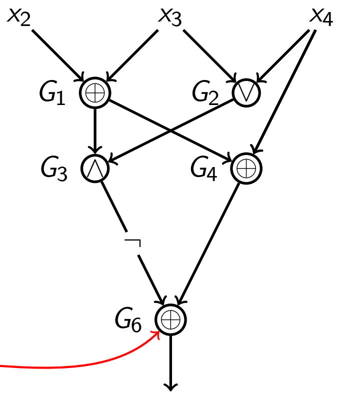

now we can change the binary function assigned to G_6

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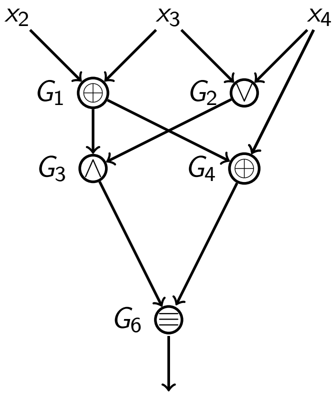

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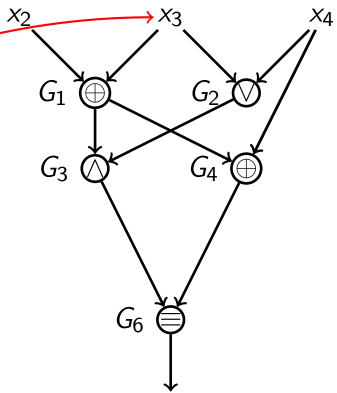
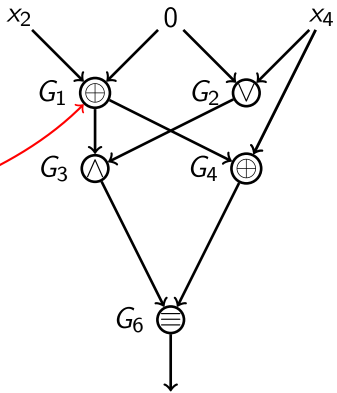


now assign $x_3 = 0$

Example



G_1 then is equal to x_2

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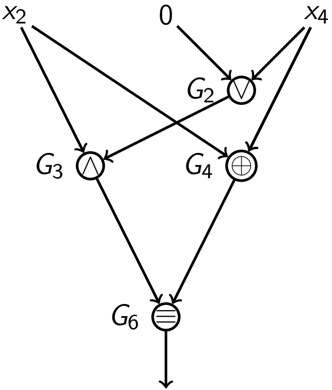

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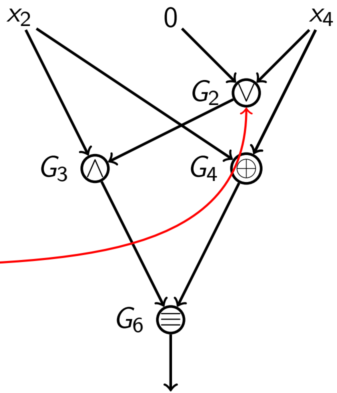

$G_2 = x_4$

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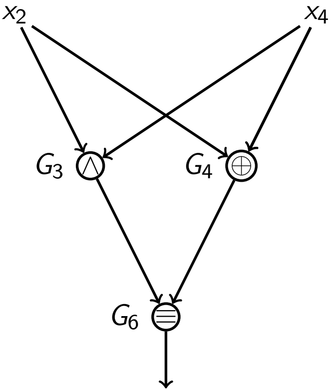

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- Then $\mathrm{MOD}_{3,r}^n, \mathrm{MOD}_{4,r}^n \in Q_{2,3}^n$, but $\mathrm{MOD}_{2,r}^n \notin Q_{2,3}^n$.

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- Thus, either x_i or x_j fans out to another gate P.
- By assigning this variable, we eliminate at least two gates and get a subfunction from $Q^{n-1}_{2,3}$. $\square$

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Remark

Optimal circuits contain AND- and XOR-type gates only, as constant and degenerate gates can be easily eliminated.

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- That is why, in particular, the current record bounds for circuits over $U_2 = B_2 \setminus \{\oplus, \equiv\}$ are stronger than the bounds over B_2.
- Usually, the main bottleneck of a proof based on gate elimination is a circuit whose top contains many XOR-type gates.

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Lemma (Degree lower bound)

Any circuit computing f contains at least $\deg(\tau(f)) - 1$ AND-type gates.

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Definition

For a circuit C, let $A(C)$ and $X(C)$ denote the number of AND- and XOR-type gates in C, respectively. Let also $\mu(C) = 3X(C) + 2A(C)$.

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Lemma

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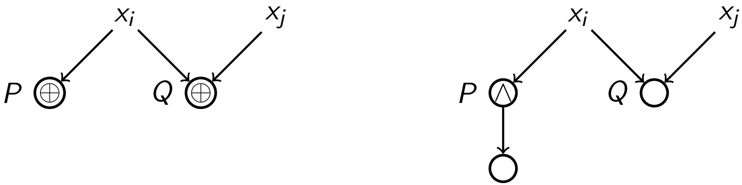

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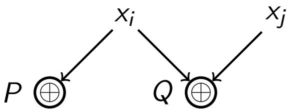

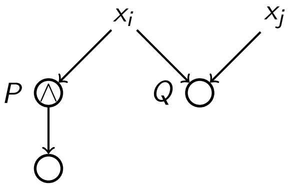

- In both cases, we can assign x_i a constant such that μ is reduced at least by 6. □

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$$\begin{array}{r}
3X(C)+2A(C) \geq 6n - 24 \\
A(C) \geq n - c - 1 \\
\hline
C(f) = 3X(C)+3A(C) \geq 7n - 25 - c
\end{array}$$

$\square$

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  - No, there is a circuit computing MOD_3^n of size $3n$ containing exactly n AND-type gates.
  - Moreover, any symmetric function can be computed using only n AND-type gates.
  - No lower bound better than $n-1$ is known! Though the multiplicative complexity of almost all functions is exponential.

MOD_3 vs MOD_4

MOD_3 is not simpler than MOD_4

For circuits and formulas over B_2 and U_2, it is known that MOD_3 is not simpler than MOD_4. The exact complexity of MOD_4 is known for some of these models: $C_{B_2}(\mathrm{MOD}_4^n) = 2.5n - c$, $L_{B_2}(\mathrm{MOD}_4^n) = \Theta(n \log n)$. The exact complexity of MOD_3 is known for none of these models.

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- MOD_3 survives under substitutions like $x_i = x_j$.
- $C_{B_2}(\mathrm{MOD}_3^n)$ for $n \leq 5$ "grows like" $3n$.

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2. Prove a cn lower bound (for a constant $c > 1$) on the multiplicative complexity of an explicit Boolean function.

Thank you for your attention!