Additive Combinatorics and Discrete Logarithm Based Range Protocols

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Outline I

1. Motivation
   - Zero-Knowledge Proofs
   - Additive Combinatorics

Zero-Knowledge Proofs

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- We actually are interested in Σ-protocols (see the paper)

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- Example: to prove that $x \in [0, 2^\ell - 1]$, commit to bits x_i, then ZK-prove that $x_i \in [0, 1]$, then compute $Com(x) = \prod Com(x_i)^{2^i} = Com(\sum x_i 2^i)$

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- Additive combinatorics is the sexy subject that studies the properties of sumsets
- Nobel price winners Terry Tao, Tim Gowers work on additive combinatorics, and recently Luca Trevisan and others have tried to apply additive combinatorics in theoretical computer science

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  - **efficient ZK-proofs** that $x_i \in S_i$ — small/structured sets S_i

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- Needed in e-voting, e-auctions and many other applications

Range Proofs: Previous Work

- Folklore: to prove $x \in [0, H]$, prove that $x \in [0, 2^\ell] \wedge x \in [H - 2^\ell, H]$ for $H \leq 2^\ell < 2H$
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- Answer 2: we give a semi-closed form for G_i

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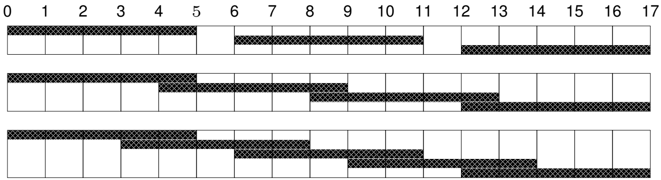

$[0, 17] = 6 * [0, 2] + [0, 5] = 4 * [0, 3] + [0, 5] = 3 * [0, 4] + [0, 5]$

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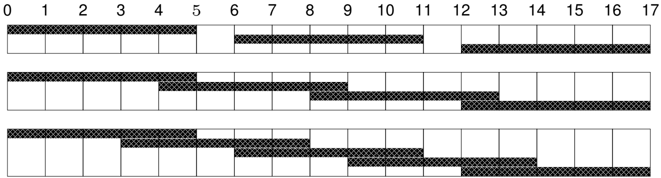

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- Optimal solution to $[0, H_0] = G_0 * [0, u-1] + H_1$

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- Set $H' := H_\ell = H - \lfloor H/(u - 1) \rfloor \cdot (u - 1)$

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$[0, H] = \sum_{i=0}^{\ell} G_i * [0, u-1] + [0, H']$ *with* $\ell \leq \log_u(H+1)$, G_i *given by recursive formulas, and* H' *as in the last slide*

Optimal case: $u \approx \log_2 H / \log_2 \log_2 H$, then the range proof has length $\Theta(\log H / \log H \log H)$

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Theorem

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[LAN02] result follows: there $u = 2$, thus $anything \equiv 0 \mod u - 1$

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- Range proof twice more efficient than [CCS08] for general H

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- Question: Can you use existing techniques from AC?
- Open question: devise an “efficient” sumset-representation for a large family of sets A