NP-completeness of Lambek calculus and multiplicative noncommutative linear logic

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A *formal language* is a set of finite words over a finite alphabet.

Example. Consider the alphabet $\Sigma = \{a, e, v\}$. The set $\{ve, veave, veaveave, veaveaveave, \ldots\}$ is a formal language.

Two important approaches to formal language specification:

- Noam Chomsky (recursion-theoretic approach)
- Jim Lambek (logico-algebraic approach)
  J. Lambek, *The mathematics of sentence structure*, American Mathematical Monthly **65** (1958), no. 3, 154–170.

By $\circ$ we denote the concatenation operator.
Σ^* is the set of all words over the alphabet Σ.
Σ^+ is the set of all non-empty words over the alphabet Σ.

J. Lambek considers three basic operations on languages:

$$\mathcal{A}\cdot\mathcal{B} \risingdotseq \{x \circ y \mid x \in \mathcal{A},\ y \in \mathcal{B}\},$$
$$\mathcal{A}\backslash\mathcal{B} \risingdotseq \{y \in \Sigma^+ \mid \mathcal{A} \cdot \{y\} \subseteq \mathcal{B}\},$$
$$\mathcal{B}/\mathcal{A} \risingdotseq \{x \in \Sigma^+ \mid \{x\} \cdot \mathcal{A} \subseteq \mathcal{B}\}.$$

Example. Let $\mathcal{A} = \{j, m\}$ and $\mathcal{B} = \{je, jrj, jrm, me, mrj, mrm\}$. Then $\mathcal{A}\backslash\mathcal{B} = \{e, rj, rm\}$.

Definition. *Types* are the elements of the minimal set Tp such that

- $\{p_0, p_1, p_2, \dots\} \subset \mathrm{Tp}$
- If $A \in \mathrm{Tp}$ and $B \in \mathrm{Tp}$, then $(A \cdot B) \in \mathrm{Tp}$, $(A\backslash B) \in \mathrm{Tp}$, and $(A/B) \in \mathrm{Tp}$.

Derivable objects of L_H are $A \to B$, where $A \in \mathrm{Tp}$ and $B \in \mathrm{Tp}$.

Axioms and rules of L_H

$$A \to A \qquad (A \cdot B) \cdot C \to A \cdot (B \cdot C) \qquad A \cdot (B \cdot C) \to (A \cdot B) \cdot C$$

$$\frac{A \to B \quad B \to C}{A \to C} \qquad \frac{A \cdot B \to C}{A \to C/B} \qquad \frac{A \cdot B \to C}{B \to A \backslash C}$$

$$\frac{A \to C/B}{A \cdot B \to C} \qquad \frac{B \to A \backslash C}{A \cdot B \to C}$$

We write $L_H \vdash \Gamma \to A$ for "$\Gamma \to A$ is derivable in the calculus L_H".

Example. Let $A, B \in \mathrm{Tp}$. Then $L_H \vdash A \cdot (A \backslash B) \to B$.

$$\frac{A \backslash B \to A \backslash B}{A \cdot (A \backslash B) \to B}$$

Remark. There exist $A, B \in \mathrm{Tp}$ such that $L_H \nvdash B \to A \cdot (A \backslash B)$.

Example. $A \cdot (B/C) \to (A \cdot B)/C$ is derivable in L_H.

$$\dfrac{(A \cdot (B/C)) \cdot C \to A \cdot ((B/C) \cdot C) \qquad \dfrac{\dfrac{\dfrac{B/C \to B/C}{(B/C) \cdot C \to B} \quad \dfrac{A \cdot B \to A \cdot B}{B \to A \backslash (A \cdot B)}}{(B/C) \cdot C \to A \backslash (A \cdot B)}}{A \cdot ((B/C) \cdot C) \to A \cdot B}}{\dfrac{(A \cdot (B/C)) \cdot C \to A \cdot B}{A \cdot (B/C) \to (A \cdot B)/C}}$$

Definition. $A \underset{L_H}{\leftrightarrow} B$ iff $L_H \vdash A \to B$ and $L_H \vdash B \to A$.

Example.

$$(A\backslash B)/C \underset{L_H}{\leftrightarrow} A\backslash(B/C),$$
$$A/(B \cdot C) \underset{L_H}{\leftrightarrow} (A/C)/B,$$
$$A \cdot (A\backslash(A \cdot B)) \underset{L_H}{\leftrightarrow} A \cdot B.$$

Example.

$$L_H \vdash ((B/A)\backslash C)\backslash D \to (B\backslash C)\backslash(A\backslash D),$$
$$L_H \nvdash ((A\backslash B)\backslash C)\backslash D \to C\backslash((B\backslash A)\backslash D).$$

Derivable objects of the calculus L are *sequents* $\Gamma \to A$, where $A \in \mathrm{Tp}$ and $\Gamma \in \mathrm{Tp}^+$.

Axioms and rules of L

$$A \to A \qquad \frac{\Phi \to B \quad \Gamma\, B\, \Delta \to A}{\Gamma\, \Phi\, \Delta \to A}\ (\text{cut})$$

$$\frac{A\, \Pi \to B}{\Pi \to A \backslash B}\ (\to \backslash),\ \text{where } \Pi \neq \Lambda \qquad \frac{\Phi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, \Phi\, (A \backslash B)\, \Delta \to C}\ (\backslash \to)$$

$$\frac{\Pi\, A \to B}{\Pi \to B / A}\ (\to /),\ \text{where } \Pi \neq \Lambda \qquad \frac{\Phi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, (B / A)\, \Phi\, \Delta \to C}\ (/ \to)$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma\, \Delta \to A \cdot B}\ (\to \cdot) \qquad \frac{\Gamma\, A\, B\, \Delta \to C}{\Gamma\, (A \cdot B)\, \Delta \to C}\ (\cdot \to)$$

Here Λ is the empty sequence, $A, B, C \in \mathrm{Tp}$, and $\Gamma, \Delta, \Phi, \Pi \in \mathrm{Tp}^*$.

Theorem 1 (J. Lambek, 1958). $\mathsf{L} \vdash A_1 \ldots A_n \to B$ *if and only if* $\mathsf{L_H} \vdash A_1 \cdot \ldots \cdot A_n \to B$.

Cut-elimination theorem (J. Lambek, 1958). A sequent is derivable in L if and only if it is derivable in L without (cut).

Example. $\mathsf{L} \vdash A \cdot (B/C) \to (A \cdot B)/C$

$$\dfrac{\dfrac{A \to A \qquad \dfrac{C \to C \quad B \to B}{(B/C)\, C \to B}\,(/\to)}{A\,(B/C)\,C \to (A \cdot B)}\,(\to\cdot)}{\dfrac{A\,(B/C) \to (A \cdot B)/C}{A \cdot (B/C) \to (A \cdot B)/C}\,(\cdot\to)}\,(\to/)$$

Remark. $\mathsf{L} \nvdash (A \cdot B)/C \to A \cdot (B/C)$.

Definition. A *Lambek categorial grammar* is a triple $\langle \Sigma, D, f \rangle$ such that $|\Sigma| < \infty$, $D \in \mathrm{Tp}$, $f : \Sigma \to \mathcal{P}(\mathrm{Tp})$, and $|f(t)| < \infty$ for each $t \in \Sigma$.
The grammar *recognizes* the language

$$\mathcal{L}_L(\Sigma, D, f) \rightleftharpoons \{t_1 \ldots t_n \in \Sigma^+ \mid \exists B_1 \in f(t_1) \ldots \exists B_n \in f(t_n) \\ \mathrm{L} \vdash B_1 \ldots B_n \to D\}$$

Example.

$$np = p_1 \quad s = p_2 \quad D = s \quad \Sigma = \{\text{John}, \text{Mary}, \text{works}, \text{recommends}\}$$

$$f(\text{John}) = f(\text{Mary}) = \{np\}$$
$$f(\text{works}) = \{(np \backslash s)\}$$
$$f(\text{recommends}) = \{((np \backslash s)/np)\}$$

$$\dfrac{np \to np \qquad \dfrac{np \to np \quad s \to s}{np\,(np \backslash s) \to s}\,(\backslash \to)}{np \quad ((np \backslash s)/np) \quad np \quad \to s}\,(/ \to)$$

John recommends Mary

B. Carpenter, *Type-Logical Semantics*, MIT Press, Cambridge, MA, 1997.
http://www.colloquial.com/tlg/parser.html

Example.

$$\Sigma = \{\text{Val}, \text{recommends}, \text{he}, \text{she}, \text{him}, \text{her}\}$$

$$\begin{aligned} f(\text{Val}) &= \{np\} \\ f(\text{recommends}) &= \{((np\backslash s)/np)\} \\ f(\text{he}) = f(\text{she}) &= \{(s/(np\backslash s))\} \\ f(\text{him}) = f(\text{her}) &= \{((s/np)\backslash s)\} \end{aligned}$$

$$\dfrac{\dfrac{\dfrac{np \to np \qquad \dfrac{(np\backslash s) \to (np\backslash s) \quad s \to s}{(s/(np\backslash s))\ (np\backslash s) \to s}\,(/\to)}{(s/(np\backslash s))\ ((np\backslash s)/np)\ np \to s}\,(/\to)}{(s/(np\backslash s))\ ((np\backslash s)/np) \to (s/np)}\,(\to /) \qquad s \to s}{\underset{\text{She}}{(s/(np\backslash s))}\ \ \underset{\text{recommends}}{((np\backslash s)/np)}\ \ \underset{\text{him}}{((s/np)\backslash s)} \to s}\,(\backslash\to)$$

Example.

$$\Sigma = \{\text{John}, \text{Val}, \text{succeeds}, \text{exists}, \text{helps}, \text{recommends}, \text{student}, \text{professor}, \text{club}, \text{a}, \text{the}, \text{every}, \text{this}, \text{strange}, \text{whenever}, \text{whom}, \text{relatively}, \text{everywhere}, \text{or}\}$$

John succeeds whenever Val recommends a club or helps the student whom this relatively strange professor recommends.

$$\begin{aligned}
f(\text{Val}) &= \{np\} \\
f(\text{succeeds}) = f(\text{exists}) &= \{(np\backslash s)\} \\
f(\text{helps}) = f(\text{recommends}) &= \{((np\backslash s)/np)\} \\
f(\text{student}) = f(\text{professor}) = f(\text{club}) &= \{n\} \\
f(\text{a}) = f(\text{the}) = f(\text{every}) &= \{(np/n)\} \\
f(\text{this}) &= \{(np/n), np\} \\
f(\text{strange}) &= \{(n/n)\} \\
f(\text{whenever}) &= \{((s\backslash s)/s)\} \\
f(\text{whom}) &= \{((n\backslash n)/(s/np))\} \\
f(\text{relatively}) &= \{((n/n)/(n/n))\} \\
f(\text{everywhere}) &= \{((np\backslash s)\backslash(np\backslash s))\}
\end{aligned}$$

$$f(\text{or}) = \{((np\backslash np)/np), ((s\backslash s)/s), (((np\backslash s)\backslash(np\backslash s))/(np\backslash s))\}$$

Definition. A *context-free grammar* is a 4-tuple $\langle \Sigma, \mathcal{W}, S, \mathcal{R} \rangle$ such that $|\Sigma| < \infty$, $|\mathcal{W}| < \infty$, $\Sigma \cap \mathcal{W} = \varnothing$, $S \in \mathcal{W}$, $\mathcal{R} \subset \{A \mapsto u \mid A \in \mathcal{W} \text{ and } u \in (\Sigma \cup \mathcal{W})^+\}$, and $|\mathcal{R}| < \infty$.
The grammar *recognizes* the language

$$\mathcal{G}(\Sigma, \mathcal{W}, S, \mathcal{R}) \rightleftharpoons \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R}) \cap \Sigma^+.$$

Here $\bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$ is defined inductively.

- $S \in \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$
- If $u_1, u_2, u_3 \in (\Sigma \cup \mathcal{W})^*$, $A \in \mathcal{W}$, $u_1 A u_3 \in \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$, and $A \mapsto u_2 \ \in \mathcal{R}$, then $u_1 u_2 u_3 \in \bar{\mathcal{G}}(\Sigma, \mathcal{W}, S, \mathcal{R})$.

Example.

$$\Sigma = \{\text{John}, \text{Mary}, \text{works}, \text{recommends}\} \qquad \mathcal{W} = \{S, NP, VP, V_t\}$$

$$\mathcal{R} = \{S \mapsto NP \ \ VP, \quad VP \mapsto V_t \ NP, \quad NP \mapsto \text{John}, \\ NP \mapsto \text{Mary}, \quad VP \mapsto \text{works}, \quad V_t \mapsto \text{recommends}\}$$

Theorem 2 (J. M. Cohen, 1967).

$$\forall\langle\Sigma,\mathcal{W},S,\mathcal{R}\rangle\ \exists D\ \exists f\ \textit{such that}\ \mathcal{L}_L(\Sigma,D,f)=\mathcal{G}(\Sigma,\mathcal{W},S,\mathcal{R})$$

Theorem 3 (1992).

$$\forall\langle\Sigma,D,f\rangle\ \exists\mathcal{W}\ \exists S\ \exists\mathcal{R}\ \textit{such that}\ \mathcal{G}(\Sigma,\mathcal{W},S,\mathcal{R})=\mathcal{L}_L(\Sigma,D,f)$$

Definition.

$$\|p_i\| \rightleftharpoons 1,$$
$$\|A \cdot B\| = \|A \backslash B\| = \|A / B\| \rightleftharpoons \|A\| + \|B\|.$$

Proof of Theorem 3.

$$m \rightleftharpoons \max(\|D\|, \max_{t \in \Sigma} \max_{B \in f(t)} \|B\|)$$

$$\mathcal{W} \rightleftharpoons \{A \in \mathrm{Tp} \mid \quad \|A\| \leq m\}$$
$$S \rightleftharpoons D$$
$$\mathcal{R} \rightleftharpoons \{B \mapsto t \mid t \in \Sigma \text{ and } B \in f(t)\} \cup$$
$$\cup \{C \mapsto AB \mid A, B, C \in \mathcal{W} \text{ and } \mathrm{L} \vdash AB \to C\} \cup$$
$$\cup \{D \mapsto A \mid A \in \mathcal{W} \text{ and } \mathrm{L} \vdash A \to D\}$$

□

Example.

$$\Sigma = \{\text{John}, \text{Mary}, \text{recommends}\}$$

$$\begin{aligned}
np &\mapsto \text{John} \in \mathcal{R} \\
np &\mapsto \text{Mary} \in \mathcal{R} \\
((np\backslash s)/np) &\mapsto \text{recommends} \in \mathcal{R} \\
s &\mapsto np \ \ (np\backslash s) \in \mathcal{R} \\
(np\backslash s) &\mapsto ((np\backslash s)/np) \ \ np \in \mathcal{R} \\
&\text{etc.}
\end{aligned}$$

Theorem 3 follows from Lemma 1.

Lemma 1. *If* $\mathrm{L} \vdash B_1 \dots B_n \to D$*, where* $n \geq 2$*,* $\|D\| \leq m$*, and* $\|B_i\| \leq m$ *for each* i*, then* $B_1 \dots B_n \to D$ *follows by means of the cut rule from* $n-1$ *derivable sequents of the form* $A_1 A_2 \to A_3$*, where* $\|A_j\| \leq m$ *for each* j*.*

We construct *links* between primitive type occurrences in a sequent if a derivation of this sequent is given.

- **Axiom:** The two occurrences of the same primitive type are linked to each other.
- **Rule:** Two primitive type occurrences in the conclusion of a rule are connected with a link if and only if they come from the same premise and their ancestors are connected with a link.

Lemma 2. *If* $\Gamma\Phi\Delta \to C$ *has a derivation in* L, *then* $\exists B \in \mathrm{Tp}$ *such that*

(i) $\|B\|$ *is equal to the number of links leading from* Φ *to* $\Gamma\Delta C$,

(ii) $\mathrm{L} \vdash \Phi \to B$,

(iii) $\mathrm{L} \vdash \Gamma B \Delta \to C$.

Lemma 3. *If* $\Gamma\Phi\Delta \to C$ *has a derivation in* $\mathsf{L}(\backslash, /)$, *then* $\exists n\ \ \exists B_1 \in \mathsf{Tp}(\backslash, /) \ \ldots\ \exists B_n \in \mathsf{Tp}(\backslash, /) \quad \exists \Phi_1 \ldots \exists \Phi_n$ *such that*

(i) $\Phi = \Phi_1 \ldots \Phi_n$,

(ii) *there are no links between* Φ_i *and* Φ_k *if* $i \neq k$,

(iii) $\|B_i\|$ *is equal to the number of links leading from* Φ_i *to* $\Gamma\Delta C$,

(iv) $\mathsf{L}(\backslash, /) \vdash \Phi_i \to B_i$ *for each* $i \leq n$,

(v) $\mathsf{L}(\backslash, /) \vdash \Gamma B_1 \ldots B_n \Delta \to C$.

Example.

$$\mathsf{L}(\backslash, /) \vdash \underbrace{p_1 \quad (p_1 \backslash p_2) \quad p_3}_{\Phi} \ \underbrace{(p_3 \backslash (p_2 \backslash p_4))}_{\Delta} \to p_4$$

$$\mathsf{L}(\backslash, /) \vdash \underbrace{p_1 \quad (p_1 \backslash p_2)}_{\Phi_1} \ \underbrace{p_3}_{\Phi_2} \ \underbrace{(p_3 \backslash (p_2 \backslash p_4))}_{\Delta} \to p_4$$

$$B_1 = p_2 \qquad B_2 = p_3$$

Lemma 4.

(i) *If* $L \vdash \Gamma\Phi\Delta \to C$ *and there is a link between* Φ *and* C*, then there is no link between* Γ *and* Δ*.*

(ii) *If* $L \vdash \Gamma\Phi\Delta\Psi \to C$ *and there is a link between* Φ *and* Ψ*, then there is no link between* Γ *and* Δ*.*

Lemma 5. *If* $n \geq 2$ *and* $A_1 \ldots A_n \to A_{n+1}$ *has a derivation in the Lambek calculus, then there exists a number* k *such that* $2 \leq k \leq n$ *and* A_k *is connected by links only with* A_{k-1}, A_k, *and* A_{k+1}.

Proof of Lemma 1. Apply Lemma 5 to $B_1 \dots B_n \to D$.

$$l \rightleftharpoons \text{ the total number of links between } B_{k-1} \text{ and } B_k$$
$$r \rightleftharpoons \text{ the total number of links between } B_k \text{ and } B_{k+1}$$
$$\|B_k\| \geq l + r$$

Case 1: $l \geq r$

$$\underbrace{B_1 \dots B_{k-2}}_{\Gamma} \underbrace{B_{k-1} B_k}_{\Phi} \underbrace{B_{k+1} B_{k+2} \dots B_n}_{\Delta} \to D$$

The number of links from Φ to $\Gamma\Delta D$ does not exceed $(\|B_{k-1}\| - l) + r \leq \|B_{k-1}\| \leq m$.

Case 2: $l < r, \quad k < n$

$$\underbrace{B_1 \dots B_{k-2} B_{k-1}}_{\Gamma} \underbrace{B_k B_{k+1}}_{\Phi} \underbrace{B_{k+2} \dots B_n}_{\Delta} \to D$$

The number of links from Φ to $\Gamma\Delta D$ does not exceed $(\|B_{k+1}\| - r) + l \quad \leq \quad \|B_{k+1}\| \quad \leq \quad m$.

CASE 3: $l < r, \quad k = n$

$$\underbrace{B_1 \ldots B_{n-1}}_{\Phi} \underbrace{B_n}_{\Delta} \to D$$

The number of links from Φ to ΔD does not exceed
$(\|D\| - r) + l \quad \leq \quad \|D\| \quad \leq \quad m.$ □

Definition. A *language model* (*free semigroup model*) is a pair $\langle \Sigma^+, v\rangle$ such that Σ is a finite or countable alphabet and

- $v(p_i) \subseteq \Sigma^+$,
- $v(A \cdot B) = v(A) \circ v(B)$,
- $v(A \backslash B) = v(A) \backslash v(B) = \{y \in \Sigma^+ \mid v(A) \circ \{y\} \subseteq v(B)\}$,
- $v(B / A) = v(B) / v(A) = \{x \in \Sigma^+ \mid \{x\} \circ v(A) \subseteq v(B)\}$.

Remark. L is sound with respect to language models.

Definition. $\mathrm{L}(\backslash, /)$ is the elementary fragment of L without $\cdot$.

Remark. L is conservative over $\mathrm{L}(\backslash, /)$.

Remark (W. Buszkowski, 1982). $\mathrm{L}(\backslash, /)$ is complete with respect to language models.

Proof.

$$\Sigma \rightleftharpoons \mathrm{Tp}$$
$$v(A) \rightleftharpoons \{\Gamma \in \mathrm{Tp}^+ \mid \mathrm{L} \vdash \Gamma \to A\}$$

□

Theorem 4 (1993). *A sequent is derivable in* L *if and only if it is true in every language model.*

Example. Let $p, q \in \mathrm{Pr}$. Then $\mathrm{L} \nvdash p \to p \cdot (q \backslash q)$.

$$\Sigma = \{a_1, a_2\} \qquad v(p) = \{a_1\}$$
$$v(q) = \{a_2\}$$

$$v(q \backslash q) = \varnothing$$
$$v(p \cdot (q \backslash q)) = \varnothing$$

$$v(p) = \{a_1\} \not\subseteq \varnothing = v(p \cdot (q \backslash q))$$

Example. Let $p, q, r \in \mathrm{Pr}$. Then $\mathrm{L} \nvdash (p \cdot q)/r \to p \cdot (q/r)$.

$$\Sigma = \{a_1, a_2, a_3\} \qquad \begin{aligned} v(p) &= \{a_1 a_2\} \\ v(q) &= \{a_3\} \\ v(r) &= \{a_2 a_3\} \end{aligned}$$

$$\begin{aligned} v(p \cdot q) &= \{a_1 a_2 a_3\} \\ v((p \cdot q)/r) &= \{a_1\} \\ v(q/r) &= \varnothing \\ v(p \cdot (q/r)) &= \varnothing \end{aligned}$$

$$v((p \cdot q)/r) = \{a_1\} \not\subseteq \varnothing = v(p \cdot (q/r))$$

Example.

$$\Sigma' = \{b, c\} \qquad \begin{aligned} v'(p) &= \{bcbbccb\} \\ v'(q) &= \{bcccb\} \\ v'(r) &= \{bccbbcccb\} \end{aligned}$$

Corollary 1. *A sequent is derivable in* L *if and only if it is true in every language model over a two-symbol alphabet.*

Proof. Let $\Sigma = \{a_1, a_2, \ldots\}$. Put $\Sigma' = \{b, c\}$.
Map a_i to $b\underbrace{cc\ldots c}_{i}b$. □

Derivable objects of the calculus L^* are *sequents* $\Gamma \to A$, where $A \in \mathrm{Tp}$ and $\Gamma \in \mathrm{Tp}^*$.

Axioms and rules of L^*

$$A \to A \qquad \frac{\Phi \to B \quad \Gamma\, B\, \Delta \to A}{\Gamma\, \Phi\, \Delta \to A}\ (\text{cut})$$

$$\frac{A\, \Pi \to B}{\Pi \to A \backslash B}\ (\to \backslash) \qquad \frac{\Phi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, \Phi\, (A \backslash B)\, \Delta \to C}\ (\backslash \to)$$

$$\frac{\Pi\, A \to B}{\Pi \to B / A}\ (\to /) \qquad \frac{\Phi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, (B / A)\, \Phi\, \Delta \to C}\ (/ \to)$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma\, \Delta \to A \cdot B}\ (\to \cdot) \qquad \frac{\Gamma\, A\, B\, \Delta \to C}{\Gamma\, (A \cdot B)\, \Delta \to C}\ (\cdot \to)$$

Example.

$$\frac{A \to A \qquad \dfrac{B \to B}{\to B \backslash B}\ (\to \backslash)}{A \to A \cdot (B \backslash B)}\ (\to \cdot)$$

Remark. $L^* \vdash A \to A \cdot (B \backslash B)$, but $L \nvdash A \to A \cdot (B \backslash B)$.

Cut-elimination theorem. We may drop (cut).

Definition. A *free monoid model* is a pair $\langle \Sigma^*, v \rangle$ such that Σ is a finite or countable alphabet and

- $v(p_i) \subseteq \Sigma^*$,
- $v(A \cdot B) = v(A) \circ v(B)$,
- $v(A \backslash B) = \{y \in \Sigma^* \mid v(A) \circ \{y\} \subseteq v(B)\}$,
- $v(B / A) = \{x \in \Sigma^* \mid \{x\} \circ v(A) \subseteq v(B)\}$.

Theorem 5 (1996). *A sequent is derivable in* L^* *if and only if it is true in every free monoid model.*

We consider only multiplicative fragments of linear logic calculi.

D. N. Yetter, *Quantales and noncommutative linear logic*, Journal of Symbolic Logic, **55** (1990), no. 1, pp. 41–64.

Definition. Let At $\rightleftharpoons \{p_0, p_1, p_2, \ldots\} \cup \{\overline{p_0}, \overline{p_1}, \overline{p_2}, \ldots\}$. *Linear formulas* are the elements of the minimal set Fm such that

- At $\subset$ Fm,
- if $A \in \mathrm{Fm}$ and $B \in \mathrm{Fm}$, then $(A \otimes B) \in \mathrm{Fm}$ and $(A \parr B) \in \mathrm{Fm}$.

$$(p_i)^{\perp} \rightleftharpoons \overline{p_i} \qquad (\overline{p_i})^{\perp} \rightleftharpoons p_i$$

$$(A \otimes B)^{\perp} \rightleftharpoons (B)^{\perp} \parr (A)^{\perp} \qquad (A \parr B)^{\perp} \rightleftharpoons (B)^{\perp} \otimes (A)^{\perp}$$

Example.
$((\overline{p} \parr ((\overline{r} \parr (\overline{r} \otimes r)) \otimes r)) \otimes q)^{\perp} = (\overline{q} \parr ((\overline{r} \parr ((\overline{r} \parr r) \otimes r)) \otimes p))$.

Definition. The following function $\tau\colon \mathrm{Tp} \to \mathrm{Fm}$ embeds L^* into cyclic linear logic.

$$\begin{aligned} \tau(p_i) &\rightleftharpoons p_i \\ \tau(A \cdot B) &\rightleftharpoons \tau(A) \otimes \tau(B) \\ \tau(A \backslash B) &\rightleftharpoons \tau(A)^{\perp} \wp \tau(B) \\ \tau(A / B) &\rightleftharpoons \tau(A) \wp \tau(B)^{\perp} \end{aligned}$$

Example. $\tau(p_1/(p_2 \cdot p_3)) = p_1 \wp (\overline{p_3} \wp \overline{p_2})$

Derivable objects of cyclic linear logic are *sequents* $\to A_1 \ \ldots \ A_n$, where $A_i \in \mathrm{Tp}$.
The intended meaning of $\to A_1 \ \ldots \ A_n$, is $A_1 \wp \ldots \wp A_n$.

Axioms and rules

$$\to A^{\perp}\, A \qquad \frac{\to \Gamma\, A\, B\, \Delta}{\to \Gamma\, (A \parr B)\, \Delta}\ (\parr) \qquad \frac{\to \Gamma\, A \quad \to B\, \Delta}{\to \Gamma\, (A \otimes B)\, \Delta}\ (\otimes)$$

$$\frac{\to \Gamma\, \Delta}{\to \Delta\, \Gamma}\ (\text{rotate}) \qquad \frac{\to \Gamma\, A \quad \to A^{\perp}\, \Delta}{\to \Gamma\, \Delta}\ (\text{cut})$$

Cut-elimination theorem. We may drop (cut).

Another calculus for the same logic.
Axioms and rules of MCLL

$$\to \overline{p_i}\, p_i \qquad \to p_i\, \overline{p_i}$$

$$\frac{\to \Gamma\, A\, B\, \Delta}{\to \Gamma\, (A \parr B)\, \Delta} \qquad \frac{\to \Gamma\, A \quad \to \Phi\, B\, \Delta}{\to \Phi\, \Gamma\, (A \otimes B)\, \Delta} \qquad \frac{\to \Gamma\, A\, \Pi \quad \to B\, \Delta}{\to \Gamma\, (A \otimes B)\, \Delta\, \Pi}$$

Example. MCLL $\vdash\ \to (\overline{p} \otimes q)\,(\overline{q} \otimes r)\,(\overline{r} \mathbin{⅋} p)$.

$$\dfrac{\dfrac{\dfrac{\to \overline{p}\,p \quad \to q\,\overline{q}}{\to (\overline{p} \otimes q)\,\overline{q}\,p} \quad \to r\,\overline{r}}{\to (\overline{p} \otimes q)\,(\overline{q} \otimes r)\,\overline{r}\,p}}{\to (\overline{p} \otimes q)\,(\overline{q} \otimes r)\,(\overline{r} \mathbin{⅋} p)}$$

Example. MCLL $\vdash\ \to (\overline{r} \otimes r)\,(\overline{r} \otimes r)\,(\overline{r} \mathbin{⅋} r)$

Remark. $L^* \vdash A_1 \ \ldots\ A_n \to B$ if and only if MCLL $\vdash\ \to \tau(A_n)^{\perp}\ \ldots\ \tau(A_1)^{\perp}\,\tau(B)$.

Example. $L^* \vdash ((q \backslash r) \cdot s) \to (q \backslash (r \cdot s))$ and MCLL $\vdash\ \to (\overline{s} \mathbin{⅋} (\overline{r} \otimes q))\,(\overline{q} \mathbin{⅋} (r \otimes s))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\to \overline{r}\,r \quad \to \overline{s}\,s}{\to \overline{s}\,\overline{r}\,(r \otimes s)} \quad \to q\,\overline{q}}{\to \overline{s}\,(\overline{r} \otimes q)\,\overline{q}\,(r \otimes s)}}{\to \overline{s}\,(\overline{r} \otimes q)\,(\overline{q} \mathbin{⅋} (r \otimes s))}}{\to (\overline{s} \mathbin{⅋} (\overline{r} \otimes q))\,(\overline{q} \mathbin{⅋} (r \otimes s))}$$

M. Pentus, *Lambek calculus is* NP-*complete*, CUNY Ph.D. Program in Computer Science Technical Report TR-2003005, CUNY Graduate Center, New York, May 2003.
http://www.cs.gc.cuny.edu/tr/techreport.php?id=79

Remark. The derivability problem for MCLL is in NP.

Theorem 6 (2003). *The derivability problem for* MCLL *is* NP-*complete.*

We shall reformulate the well-known NP-complete problem *SAT* (satisfiability in the classical propositional logic) in terms of electrical circuits.
Let $c_1 \wedge \ldots \wedge c_m$ be a Boolean formula in conjunctive normal form with clauses $c_1, \ldots, c_m$ and variables $x_1, \ldots, x_n$.
We construct a frame (with m lamps and n sockets) and a set of $2n$ blocks (each of which fits into one socket only) so that the formula $c_1 \wedge \ldots \wedge c_m$ is satisfiable if and only if there is a way to plug n blocks into the sockets so that no lamp will be switched on.
Each block (and each socket) has $2m$ contacts.

Example. $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_3)$.

To model the circuits in MCLL we shall construct (in polynomial time) formulas G, $E_i(0)$, $E_i(1)$, F_i (where $1 \leq i \leq n$) such that

- $c_1 \wedge \ldots \wedge c_m$ is satisfiable if and only if MCLL $\vdash\ \rightarrow E_1(t_1)\ \ldots\ E_n(t_n)\ G$ for some $t_1, \ldots, t_n \in \{0, 1\}$,
- MCLL $\vdash\ \rightarrow F_1\ \ldots\ F_n\ G$ is satisfiable if and only if MCLL $\vdash\ \rightarrow E_1(t_1)\ \ldots\ E_n(t_n)\ G$ for some $t_1, \ldots, t_n \in \{0, 1\}$.

We shall denote p_{n+1} by r.
In the following definitions $1 \leq j < m$, $1 \leq i \leq n$ and $t \in \{0, 1\}$.

$$
\begin{aligned}
G^0 &\rightleftharpoons (\overline{r} \parr r),\\
G^j &\rightleftharpoons ((\overline{r} \parr G^{j-1}) \otimes r),\\
G &\rightleftharpoons ((\overline{p_n} \parr G^{m-1}) \otimes p_0),\\
H^0 &\rightleftharpoons (\overline{r} \otimes r),\\
H^j &\rightleftharpoons ((\overline{r} \parr H^{j-1}) \otimes r),\\
H_i &\rightleftharpoons ((\overline{p_{i-1}} \parr H^{m-1}) \otimes p_i),\\
E_i^0(t) &\rightleftharpoons (\overline{r} \otimes r),\\
E_i^j(t) &\rightleftharpoons \begin{cases} (\overline{r} \parr (E_i^{j-1}(t) \otimes r)) & \text{if } [\![x_i]\!] = t \rightarrow [\![c_j]\!] = 1,\\ ((\overline{r} \parr E_i^{j-1}(t)) \otimes r) & \text{otherwise,} \end{cases}\\
E_i(t) &\rightleftharpoons \begin{cases} (\overline{p_{i-1}} \parr (E_i^{m-1}(t) \otimes p_i)) & \text{if } [\![x_i]\!] = t \rightarrow [\![c_m]\!] = 1,\\ ((\overline{p_{i-1}} \parr E_i^{m-1}(t)) \otimes p_i) & \text{otherwise,} \end{cases}\\
F_i &\rightleftharpoons ((E_i(0) \otimes H_i^{\perp}) \parr H_i \parr (H_i^{\perp} \otimes E_i(1))).
\end{aligned}
$$

Lemma 6. $\mathrm{MCLL} \vdash\ \to E_i(t)\, H_i^{\perp}$ *for each* $1 \leq i \leq n$ *and* $t \in \{0, 1\}$.

Lemma 7. $\mathrm{MCLL} \vdash\ \to F_i\, E_i(t)^{\perp}$ *for each* $1 \leq i \leq n$ *and* $t \in \{0, 1\}$.

Lemma 8. *If* $\mathrm{MCLL} \vdash\ \to \Gamma\, A^{\perp}$ *and* $\mathrm{MCLL} \vdash\ \to \Phi\, A\, \Delta$, *then* $\mathrm{MCLL} \vdash\ \to \Phi\, \Gamma\, \Delta$.

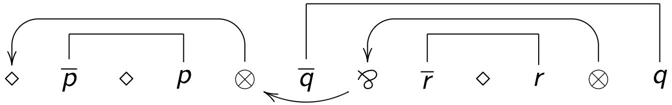
Theorem 7 (2003). *The derivability problems for* L^* *and* L *are* NP*-complete.*

Remark. It is unknown whether the same holds for $\mathrm{L}(\backslash, /)^*$ and $\mathrm{L}(\backslash, /)$.

Example. The derivation

$$\dfrac{\rightarrow \overline{p}\, p \qquad \dfrac{\rightarrow \overline{r}\, r \qquad \rightarrow \overline{q}\, q}{\rightarrow \overline{q}\, \overline{r}\, (r \otimes q)}}{\rightarrow \overline{p}\, (p \otimes (\overline{q} \parr \overline{r}))\, (r \otimes q)}$$

corresponds to the following proof net.



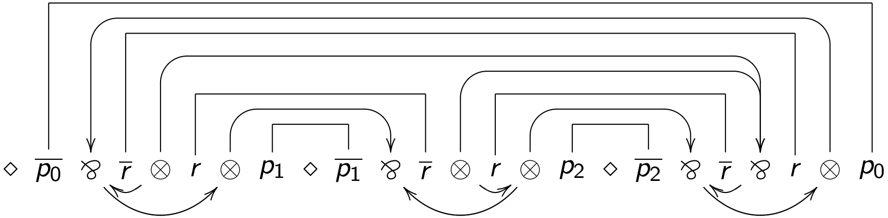
A *proof net* for Γ must satisfy the following conditions.

- $|\Gamma|_{\parr} + |\Gamma|_{\diamond} = |\Gamma|_{\otimes} + 2.$
- No intersections.
- Acyclic.

Example. Let

$$\Gamma = ((\overline{p_0} \invamp (\overline{r} \otimes r)) \otimes p_1)\,(\overline{p_1} \invamp ((\overline{r} \otimes r) \otimes p_2))\,((\overline{p_2} \invamp (\overline{r} \invamp r)) \otimes p_0).$$

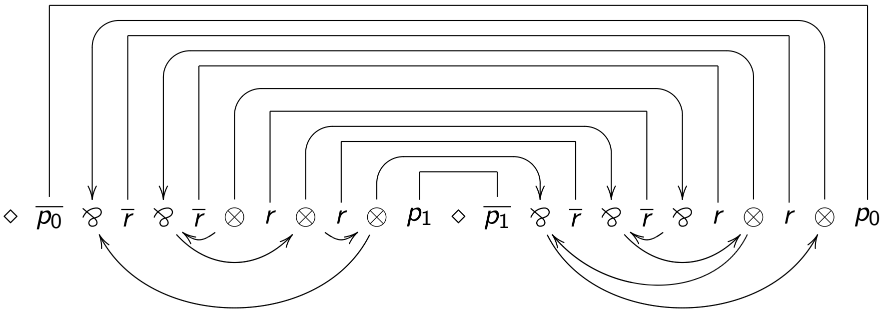
The following figure shows a proof net for Γ.



Example. Let

$$\Gamma = (\overline{p_0} \mathbin{⅋} (((\bar r \mathbin{⅋} (\bar r \otimes r)) \otimes r) \otimes p_1))\,((\overline{p_1} \mathbin{⅋} ((\bar r \mathbin{⅋} (\bar r \mathbin{⅋} r)) \otimes r)) \otimes p_0).$$

The following is not a valid proof net for $\to \Gamma$ (it contains a cycle).



Definition. $\| \cdot \| \colon \mathsf{Fm} \to \mathbb{Z}$

$$\|p_i\| = \|\overline{p_i}\| \rightleftharpoons 2,$$
$$\|A \otimes B\| = \|A \parr B\| \rightleftharpoons \|A\| + \|B\|,$$
$$\|A_1 \dots A_n\| \rightleftharpoons \|A_1\| + \dots + \|A_n\|.$$

Definition. $\mathsf{Occ} \rightleftharpoons \mathsf{Fm} \times \mathbb{Z}$.

Definition. $c \colon \mathsf{Occ} \to \mathbb{Z}$

$$c(p_i) = c(\overline{p_i}) \rightleftharpoons 1,$$
$$c(A \otimes B) = c(A \parr B) \rightleftharpoons \|A\|.$$

Definition. $\prec$ is the following binary relation on Occ.

$$\langle A, k - \|A\| + c(A)\rangle \prec \langle (A \lambda B), k\rangle,$$
$$\langle B, k + c(B)\rangle \prec \langle (A \lambda B), k\rangle,$$
if $\langle A, i\rangle \prec \langle B, j\rangle$ and $\langle B, j\rangle \prec \langle C, k\rangle$, then $\langle A, i\rangle \prec \langle C, k\rangle$.

Here $\lambda \in \{\otimes, \parr\}$.

Definition. Let $\diamond \notin \mathrm{Fm}$. Let $\Gamma = A_1 \dots A_n$. Then $\boldsymbol{\Omega}_\Gamma \rightleftharpoons \langle \Omega_\Gamma, \prec_\Gamma, <_\Gamma \rangle$, where

$$\Omega_\Gamma \rightleftharpoons \{\langle B, k + \|A_1 \dots A_{i-1}\| \rangle \mid 1 \le i \le n \text{ and } \langle B, k \rangle \preceq \langle A_i, c(A_i) \rangle \} \\ \cup \{\langle \diamond, \|A_1 \dots A_{i-1}\| \rangle \mid 1 \le i \le n\},$$

$\langle A, k \rangle \prec_\Gamma \langle B, l \rangle$ iff $A \neq \diamond$, $B \neq \diamond$, and $\langle A, k \rangle \prec_\Gamma \langle B, l \rangle$,

$\langle A, k \rangle <_\Gamma \langle B, l \rangle$ iff $k < l$.

Definition.

$$\Omega_\Gamma^\diamond \rightleftharpoons \{\langle C, k \rangle \in \Omega_\Gamma \mid C = \diamond\},$$
$$\Omega_\Gamma^{\mathrm{At}} \rightleftharpoons \{\langle C, k \rangle \in \Omega_\Gamma \mid C \in \mathrm{At}\},$$
$$\Omega_\Gamma^\otimes \rightleftharpoons \{\langle C, k \rangle \in \Omega_\Gamma \mid C = A \otimes B \text{ for some } A \text{ and } B\},$$
$$\Omega_\Gamma^\parr \rightleftharpoons \{\langle C, k \rangle \in \Omega_\Gamma \mid C = A \parr B \text{ for some } A \text{ and } B\}.$$

Definition. A *proof net for* Γ is a relational structure $\langle \mathbf{\Omega}_\Gamma, \mathcal{A}, \mathcal{E} \rangle$, where

- $\flat(\Omega_\Gamma^{\wp}) + \flat(\Omega_\Gamma^{\diamond}) - \flat(\Omega_\Gamma^{\otimes}) = 2$,
- $\mathcal{A}$ is a map from $\Omega_\Gamma^{\otimes}$ to $\Omega_\Gamma^{\wp} \cup \Omega_\Gamma^{\diamond}$,
- $\mathcal{E}$ is a map from $\Omega_\Gamma^{\mathsf{At}}$ to $\Omega_\Gamma^{\mathsf{At}}$,
- if $\langle \alpha, \beta \rangle \in \mathcal{E}$, then $\langle \beta, \alpha \rangle \in \mathcal{E}$,
- if $\langle \langle A, i \rangle, \langle B, j \rangle \rangle \in \mathcal{E}$, then $A = B^{\perp}$,
- the edges of the graph $\langle \Omega_\Gamma, \mathcal{A} \cup \mathcal{E} \rangle$ can be drawn without intersections on a semiplane while the vertices of the graph are ordered according to $<_\Gamma$ on the border of the semiplane,
- the graph $\langle \Omega_\Gamma, \prec_\Gamma \cup \mathcal{A} \rangle$ is acyclic.

Theorem 8 (1998). MCLL $\vdash\ \rightarrow \Gamma$ *if and only if there exists a proof net for* Γ*.*

Definition. $\mathrm{MCLL} \vdash A \to B$ iff $\mathrm{MCLL} \vdash\ \to A^\perp\, B$.

Definition. $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$ iff $\mathrm{MCLL} \vdash A \to B$ and $\mathrm{MCLL} \vdash B \to A$.

Lemma 9.
- *$A \underset{\mathrm{MCLL}}{\leftrightarrow} A$.*
- *If $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$, then $B \underset{\mathrm{MCLL}}{\leftrightarrow} A$.*
- *If $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$ and $B \underset{\mathrm{MCLL}}{\leftrightarrow} C$, then $A \underset{\mathrm{MCLL}}{\leftrightarrow} C$.*
- *If $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$ and $C \underset{\mathrm{MCLL}}{\leftrightarrow} D$, then $A \otimes C \underset{\mathrm{MCLL}}{\leftrightarrow} B \otimes D$.*
- *If $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$ and $C \underset{\mathrm{MCLL}}{\leftrightarrow} D$, then $A \parr C \underset{\mathrm{MCLL}}{\leftrightarrow} B \parr D$.*
- *If $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$, then $A^\perp \underset{\mathrm{MCLL}}{\leftrightarrow} B^\perp$.*

Definition. $\sharp\colon \mathrm{Fm} \to \mathbb{Z}$

$$\sharp(p_i) = \sharp(\overline{p_i}) \rightleftharpoons 0,$$
$$\sharp(A \parr B) \rightleftharpoons \sharp A + \sharp B + 1,$$
$$\sharp(A \otimes B) \rightleftharpoons \sharp A + \sharp B - 1.$$

Lemma 10. *If* MCLL $\vdash A \to B$, *then* $\sharp A = \sharp B$.

Definition. $\mathrm{at}_0\colon \mathrm{Fm} \to \mathcal{P}(\mathrm{At})$ and $\mathrm{at}_1\colon \mathrm{Fm} \to \mathcal{P}(\mathrm{At})$:

$$\mathrm{at}_0(C) \rightleftharpoons \{C\} \text{ if } C \in \mathrm{At},$$
$$\mathrm{at}_1(C) \rightleftharpoons \{C^{\perp}\} \text{ if } C \in \mathrm{At},$$
$$\mathrm{at}_k(A \parr B) = \mathrm{at}_k(A \otimes B) \rightleftharpoons \mathrm{at}_k(A) \cup \mathrm{at}_{(k+1+\sharp A \bmod 2)}(B).$$

Lemma 11. *If* $A \underset{\mathrm{MCLL}}{\leftrightarrow} B$, *then* $\mathrm{at}_0(A) = \mathrm{at}_0(B)$.

Theorem 9 (2002). *$A \underset{\mathsf{MCLL}}{\leftrightarrow} p_i$ if and only if $\mathrm{at}_0(A) = \{p_i\}$, $\sharp A = 0$, and $\sharp C \in \{-1, 0, 1\}$ whenever C is a subformula of A.*

Corollary 2. *There is a deterministic polynomial time algorithm for the special equivalence problem: given $A \in \mathrm{Tp}$ and p_i, to decide whether $A \underset{\mathsf{MCLL}}{\leftrightarrow} p_i$.*

Remark. It is unknown whether the same holds for the problem $A \underset{\mathsf{MCLL}}{\leftrightarrow} B$.

V. M. Abrusci. *Phase semantics and sequent calculus for pure noncommutative classical linear propositional logic*, Journal of Symbolic Logic **56** (1991), no. 4, pp. 1403–1451.

Definition. *Formulas of* PNCL are the elements of the minimal set $\mathrm{Fm}_{\mathrm{PNCL}}$ such that

- $\mathbf{1} \in \mathrm{Fm}_{\mathrm{PNCL}}$ and $\bot \in \mathrm{Fm}_{\mathrm{PNCL}}$
- $\{p_i \mid i > 0\} \subset \mathrm{Fm}_{\mathrm{PNCL}}$
- $\{p_i^{\overbrace{\bot\ldots\bot}^{n}} \mid i > 0 \text{ and } n > 0\} \subset \mathrm{Fm}_{\mathrm{PNCL}}$
- $\{{}^{\overbrace{\bot\ldots\bot}^{n}}p_i \mid i > 0 \text{ and } n > 0\} \subset \mathrm{Fm}_{\mathrm{PNCL}}$
- If $A \in \mathrm{Fm}_{\mathrm{PNCL}}$ and $B \in \mathrm{Fm}_{\mathrm{PNCL}}$, then $(A \otimes B) \in \mathrm{Fm}_{\mathrm{PNCL}}$ and $(A \parr B) \in \mathrm{Fm}_{\mathrm{PNCL}}$.

$$(A \otimes B)^\perp \rightleftharpoons B^\perp \parr A^\perp \qquad {}^\perp(A \otimes B) \rightleftharpoons {}^\perp B \parr {}^\perp A$$

$$(A \parr B)^\perp \rightleftharpoons B^\perp \otimes A^\perp \qquad {}^\perp(A \parr B) \rightleftharpoons {}^\perp B \otimes {}^\perp A$$

$$\mathbf{1}^\perp \rightleftharpoons \perp \qquad {}^\perp\mathbf{1} \rightleftharpoons \perp$$

$$\perp^\perp \rightleftharpoons \mathbf{1} \qquad {}^\perp\perp \rightleftharpoons \mathbf{1}$$

$$(p_i^{\overbrace{\perp\cdots\perp}^{n}})^\perp \rightleftharpoons p_i^{\overbrace{\perp\cdots\perp}^{n+1}} \qquad {}^\perp(p_i^{\overbrace{\perp\cdots\perp}^{n}}) \rightleftharpoons p_i^{\overbrace{\perp\cdots\perp}^{n-1}}$$

$$({}^{\overbrace{\perp\cdots\perp}^{n}}p_i)^\perp \rightleftharpoons {}^{\overbrace{\perp\cdots\perp}^{n-1}}p_i \qquad {}^\perp({}^{\overbrace{\perp\cdots\perp}^{n}}p_i) \rightleftharpoons {}^{\overbrace{\perp\cdots\perp}^{n+1}}p_i$$

$$\tau(p_i) \rightleftharpoons p_i$$

$$\tau(A \bullet B) \rightleftharpoons \tau(A) \otimes \tau(B)$$

$$\tau(A \backslash B) \rightleftharpoons \tau(A)^\perp \parr \tau(B)$$

$$\tau(A / B) \rightleftharpoons \tau(A) \parr {}^\perp\tau(B)$$

Axioms and rules of PNCL

$$\to (A^{\perp})\,A \qquad\qquad \to \mathbf{1} \qquad\qquad \frac{\to \Gamma\,\Delta}{\to \Gamma\,\perp\,\Delta}\ (\perp)$$

$$\frac{\to \Gamma\,A\,B\,\Delta}{\to \Gamma\,(A \parr B)\,\Delta}\ (\parr) \qquad\qquad \frac{\to \Gamma\,A \quad \to B\,\Delta}{\to \Gamma\,(A \otimes B)\,\Delta}\ (\otimes)$$

$$\frac{\to \Gamma\,\Delta}{\to (\Delta^{\perp\perp})\,\Gamma}\ (\text{rotate}) \qquad\qquad \frac{\to \Gamma\,A \quad \to A^{\perp}\,\Delta}{\to \Gamma\,\Delta}\ (\text{cut})$$

Cut-elimination theorem. A sequent is derivable in PNCL if and only if it is derivable in PNCL without (cut).

Remark. $L^* \vdash A_1\ \ldots\ A_n \to B$ if and only if PNCL $\vdash\ \to \tau(A_n)^{\perp}\ \ldots\ \tau(A_1)^{\perp}\,\tau(B)$.