Using Widenings/Narrowings in Data Flow Analyses

An introduction

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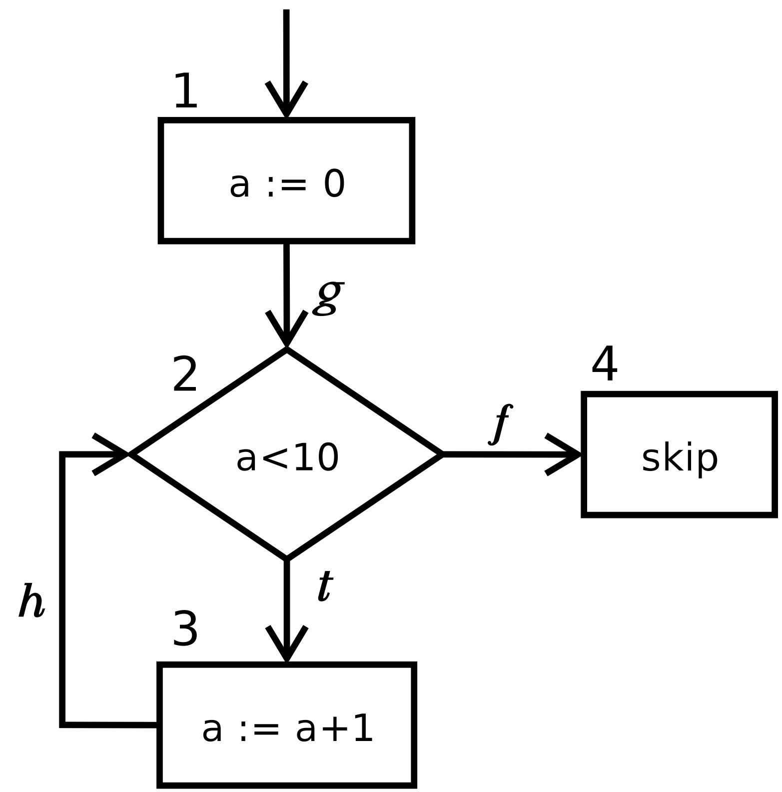
Overview

- Introduction $\longleftarrow$ you are here!
- Galois Connections
- From MOP to MFP
- Insufficiency of Galois connections
- Widenings/Narrowings
- Examples

Interval Analysis

- The goal is to do an interval analysis on the following simple program.

```
a := 0;
while a < 10 do
  a := a + 1;
```



Standard Semantics

- The standard semantics of a programming language defines how expression modify the state.
- The semantics is embodied in the transfer functions of the graph.
- The analysis must take into account
  - all possible input states
  - all possible executions

All possible inputs

Sets of States Semantics

- To analyse to program for all possible input states, we have to use the Set of States semantics.
- It is naturally derrived from the Standard semantics.
- Most precise domain, but impossible to use.
- Approximation is necessary.

The Lattice ordering

- Interesting properties are undecidable.
- Analyses must err on the safe side.
- The Lattice ordering: $x \sqsubseteq y$ means:
  - The set of programs satisfying x is included in the set of programs satisfying y.
  - The state y is a correct approximation of x.
  - The state x is more precise than y.

Galois connections

- Galois connection (more precisely adjunctions):

$$\alpha\colon X \to Y$$
$$\gamma\colon Y \to X$$

total functions that satisfy

$$\alpha(x) \sqsubseteq y \iff x \sqsubseteq \gamma(y)$$

Meaning of Galois connections

$$\alpha(x) \sqsubseteq y \iff x \sqsubseteq \gamma(y)$$

- $\alpha(x)$ is the most precise approximation of x.
- $\gamma(y)$ is the most general element, which can be soundly approximated by y.

Moving to the interval domain

- We can now abstract
  - Sets of State Semantics (visualize as points in space)
  - Collecting Semantics (abstract with projection, concretize as grid)
  - Interval domain (project to interval, concreetizes to rectangle)
- The transfer functions are induced by our abstraction:

$$f' = \alpha \circ f \circ \gamma$$

All possible executions

Notation

- The Control Flow Graph $G = (N, E, s)$, where
  - N is the set of nodes.
  - $E = N \times N$
  - s is the initial node.
- The analysis is an annotation $N \rightarrow D$.
- The transfer functions $tf : E \rightarrow (D \rightarrow D)$
- ι is the initial state.

Merge Over all Paths

Definition 1. The path semantics $[\![\pi]\!]_{tf}$ is simply the composition of the transfer functions along that path

$$[\![\varepsilon]\!]_{tf} = id_{D \to D}$$
$$[\![e_1, \ldots, e_n]\!]_{tf} = [\![e_2, \ldots, e_n]\!]_{tf} \circ tf(e_1)$$

Definition 2. The MOP solution to the data flow problem is defi ned at each node by

$$\mathsf{MOP}(n) = \bigsqcup \left\{ [\![\pi]\!]_{tf}\,(\iota) \mid \pi \text{ is a path from } s \text{ to } n \right\}$$

Least Fixed-point

MOP is not calculable.

Definition 3. The least (minimal) fixed point $\mathsf{MFP}(n)$ is the least solution to the following system:

$$\mathsf{MFP}(n) = \begin{cases} \iota & \text{if } n = s \\ \bigsqcup \{ tf(e)(\mathsf{MFP}(n')) \mid e = (n', n) \in E \} & \text{otherwise} \end{cases}$$

This is correct with respect to MOP, i.e. $\forall n$: $\mathsf{MOP}(n) \sqsubseteq \mathsf{MFP}(n)$.

PAG

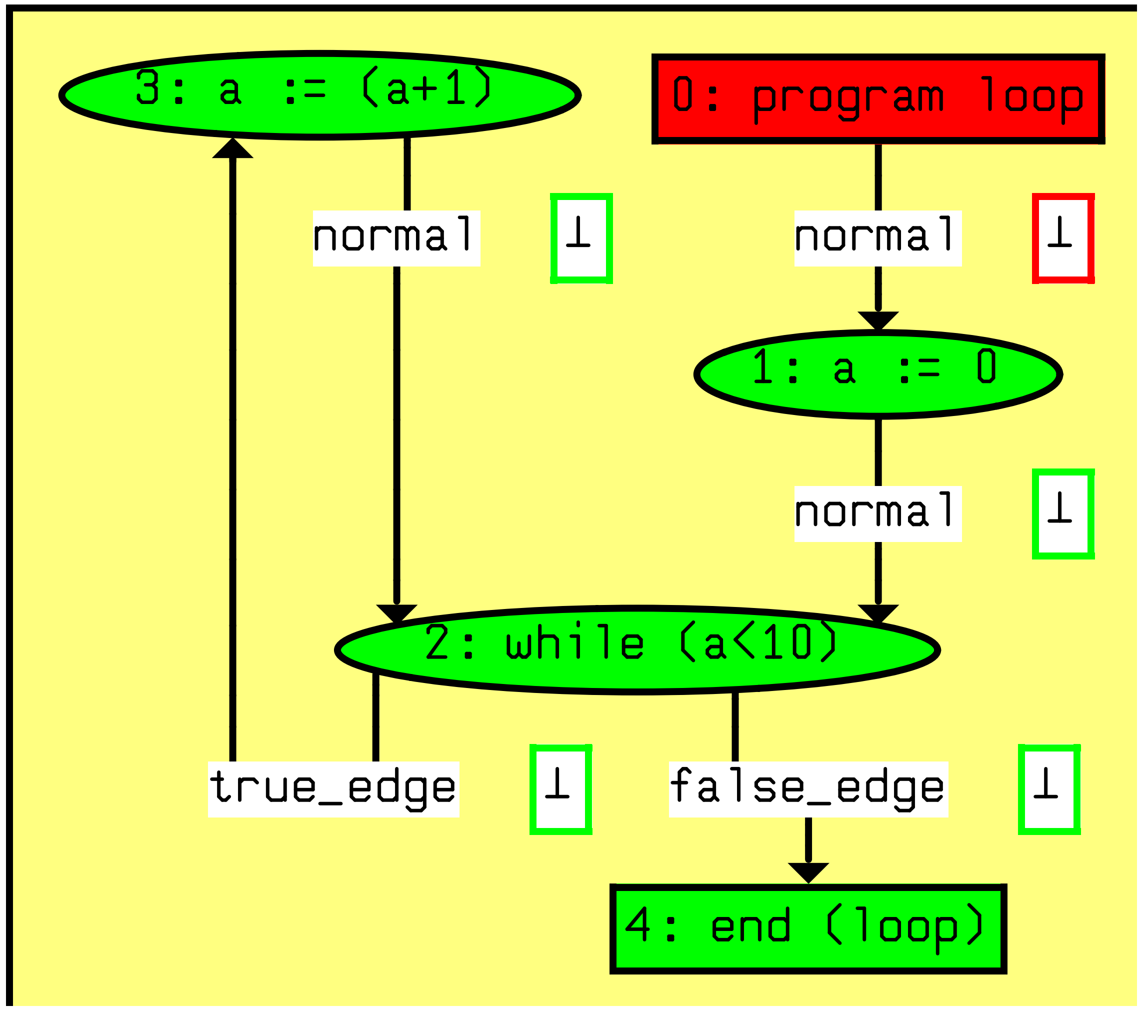
- The analysis is implemented in PAG – a Program Analysis Generator. See www.absint.com for further information.
- The output represents functions.

$$f(x) = \begin{cases} (0, 10) & \text{if } x = \texttt{a} \\ (-\infty, \infty) & \text{otherwise} \end{cases}$$

is written as

$$D \rightarrow (\perp, \top)$$
$$0 \rightarrow (0, 10)$$

Testing the Analysis



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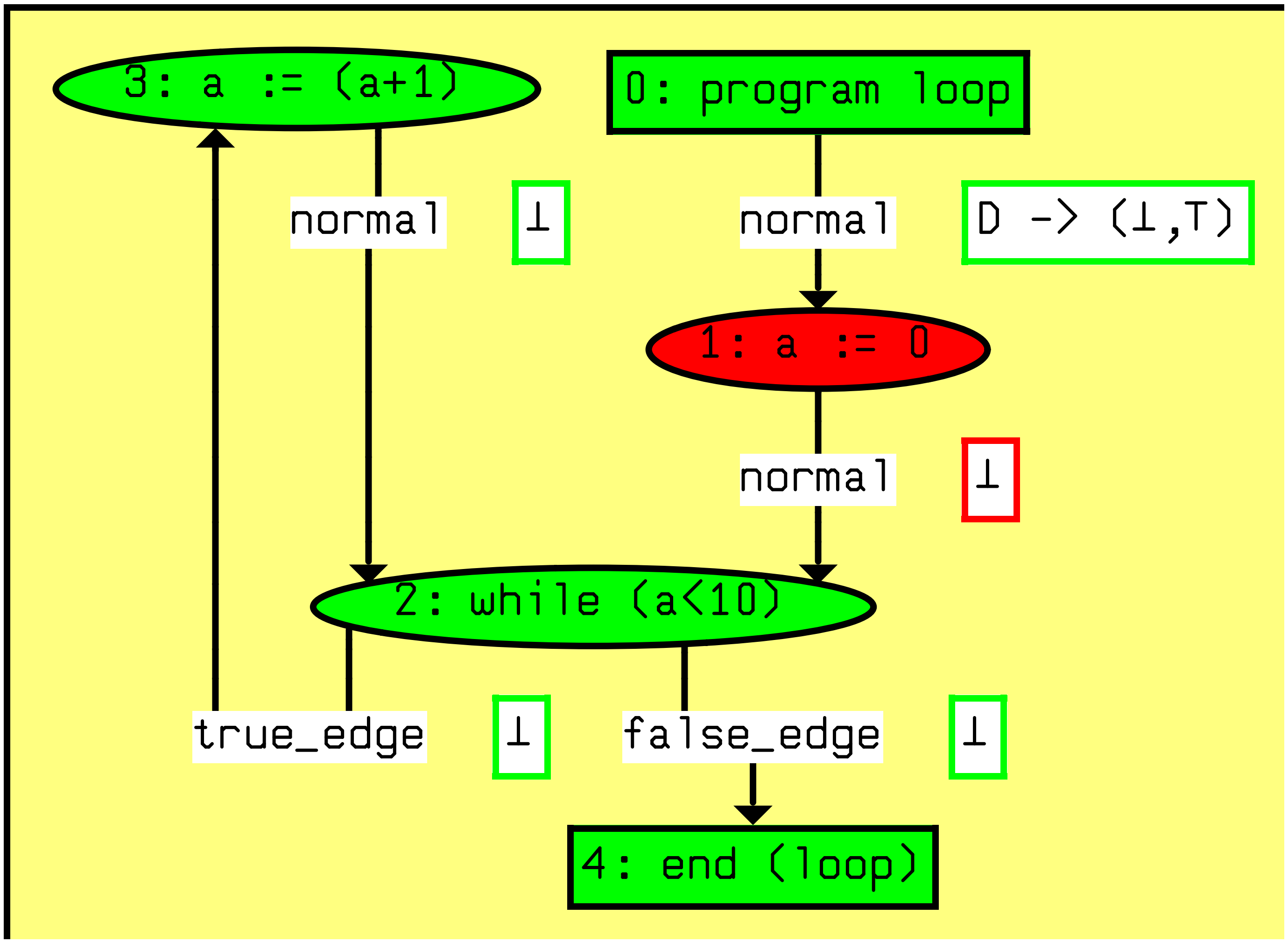

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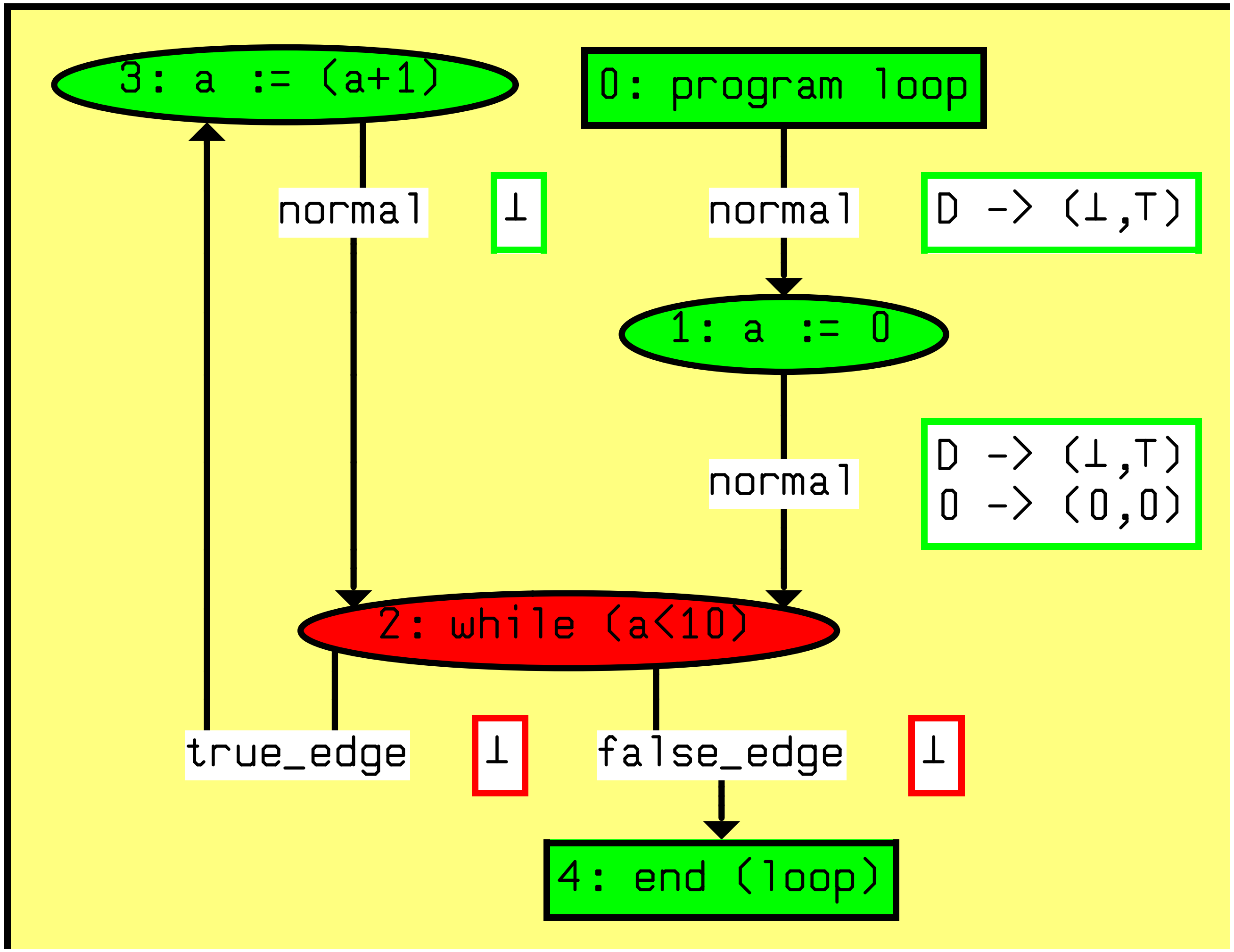

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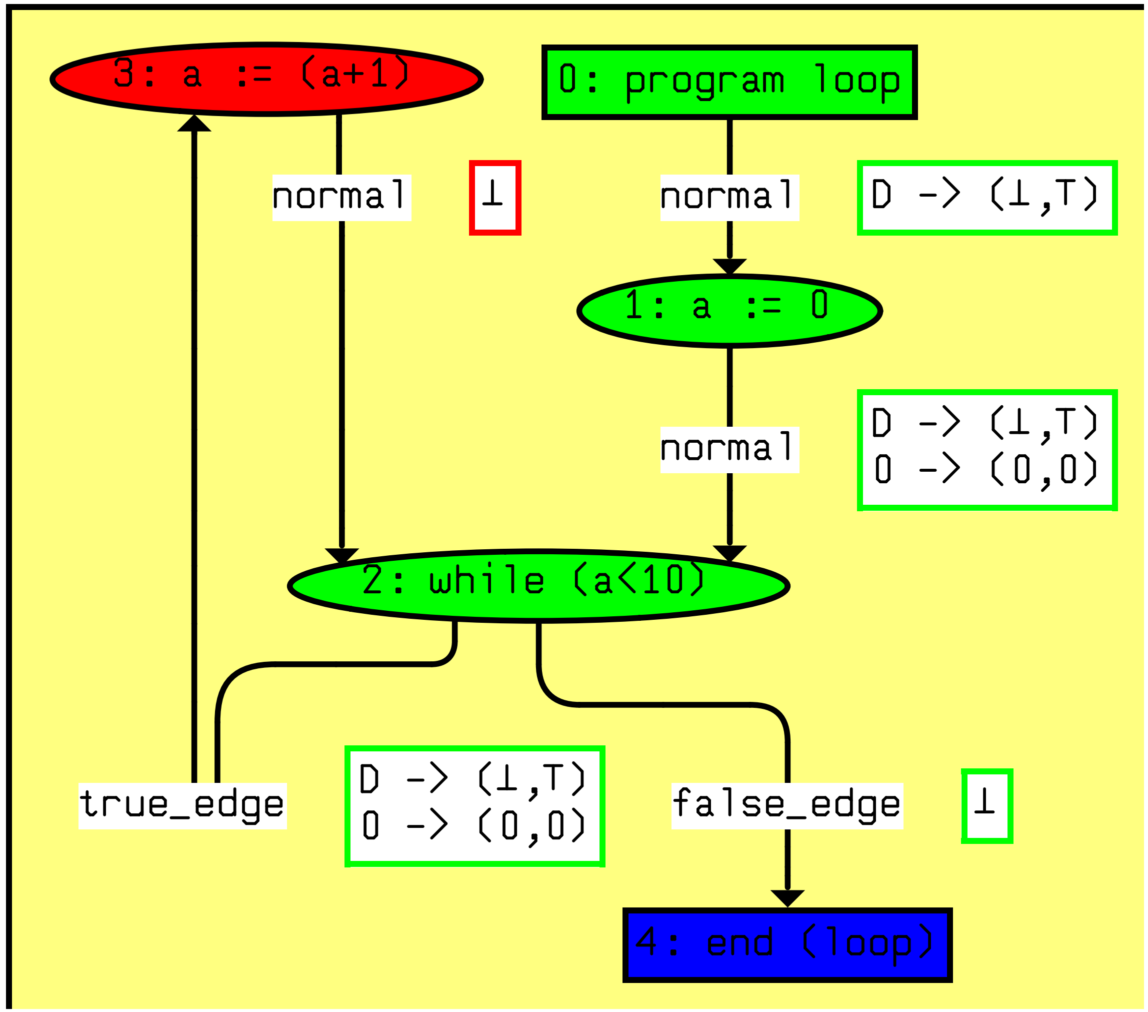

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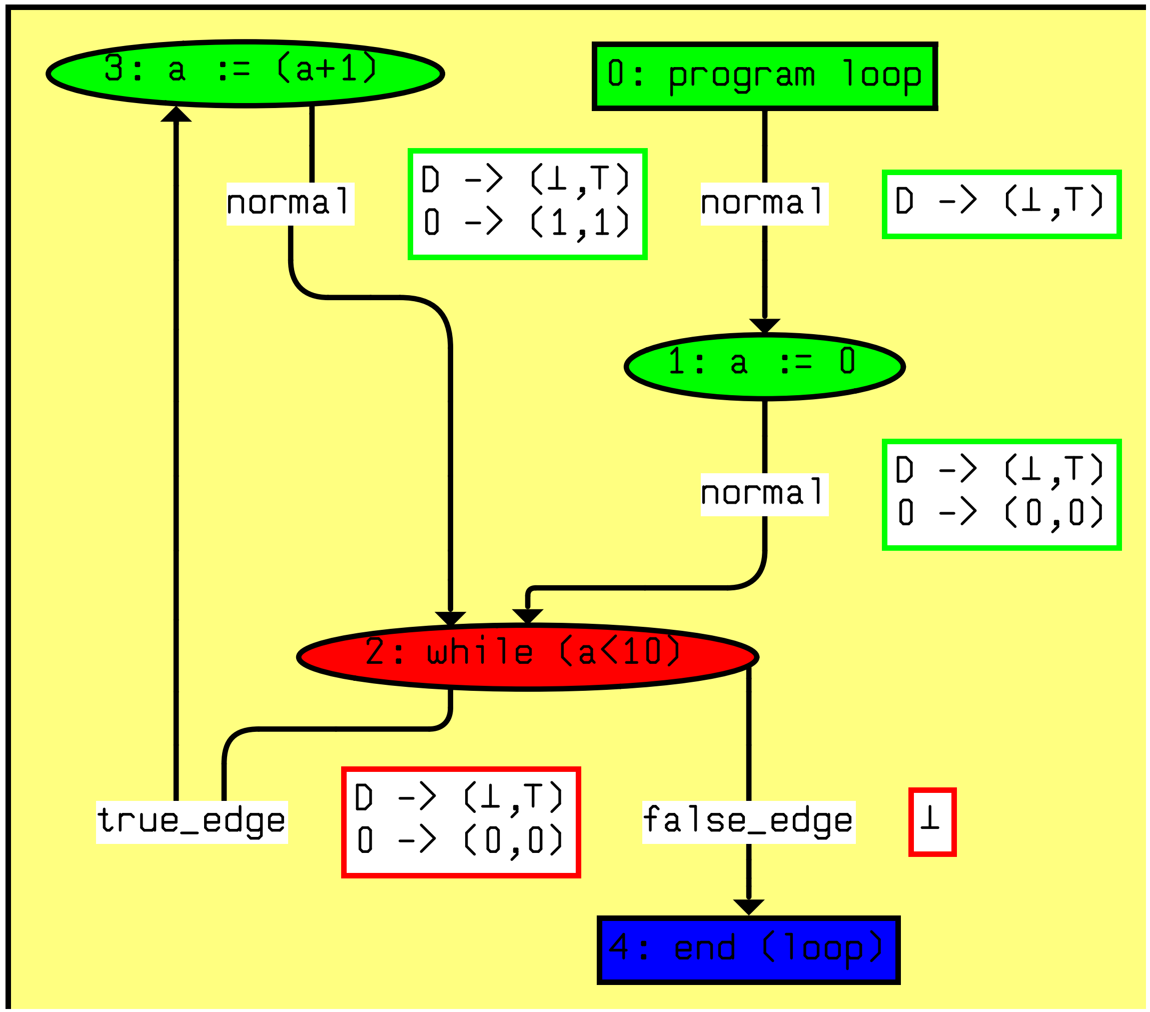

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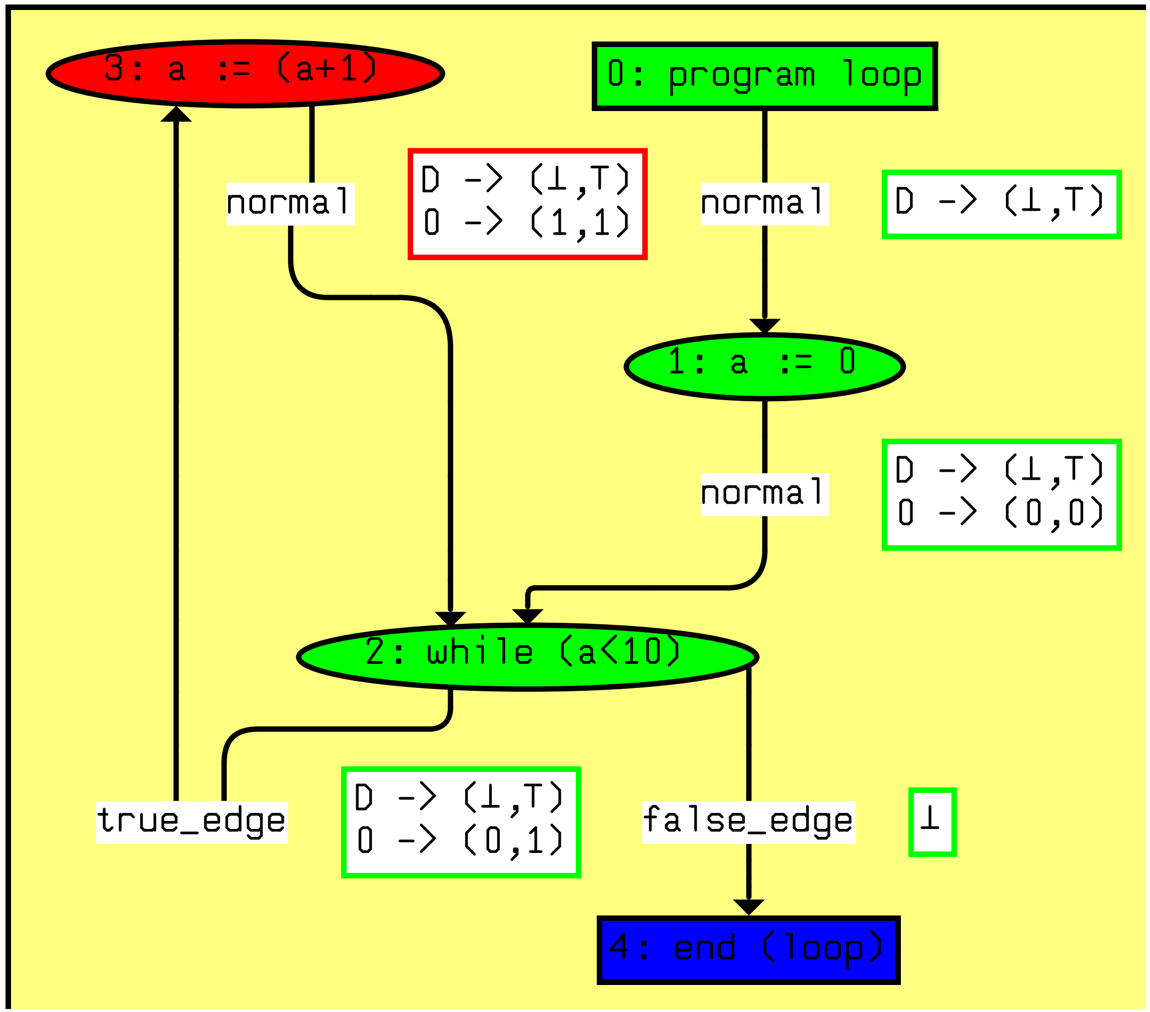

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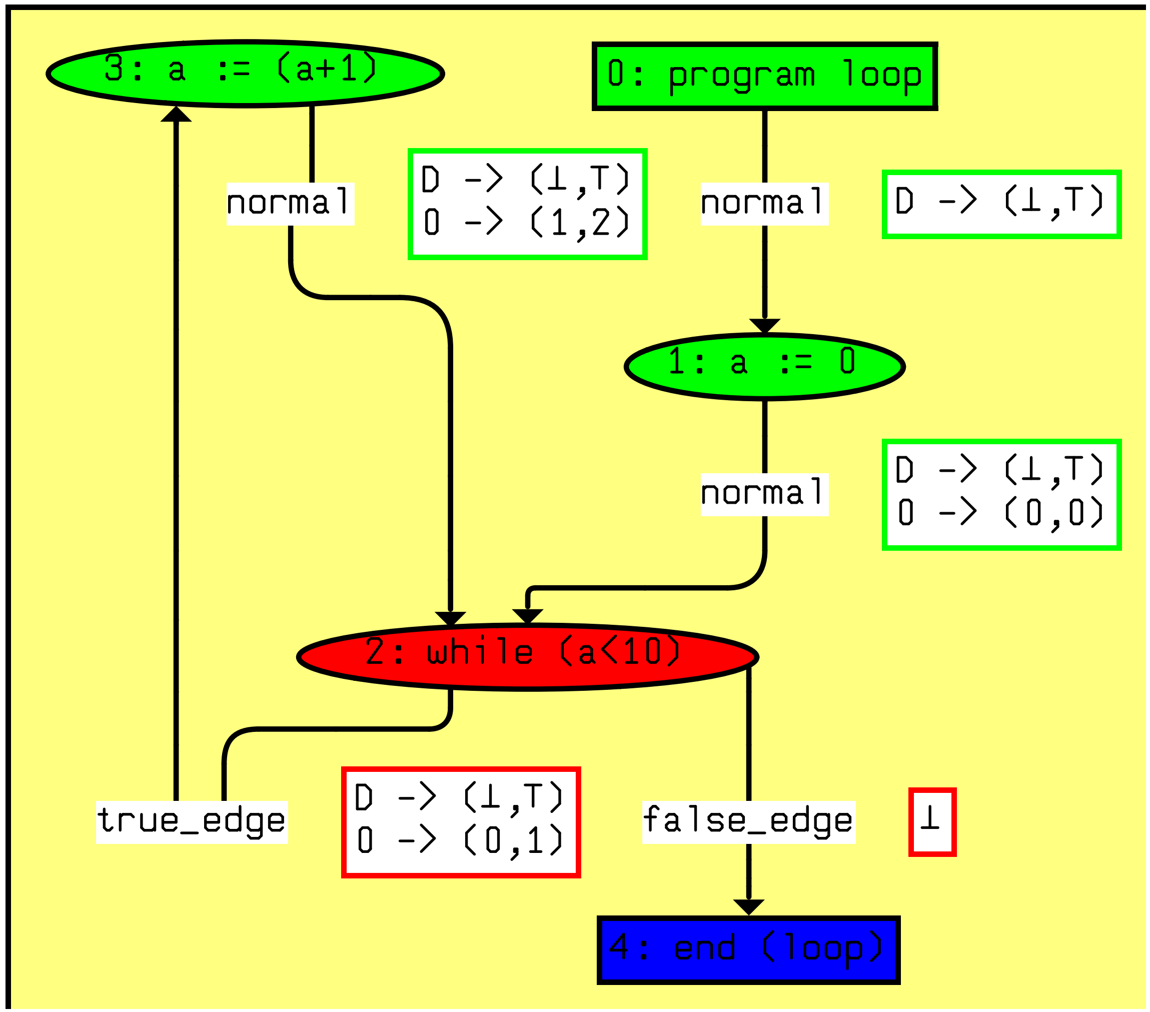

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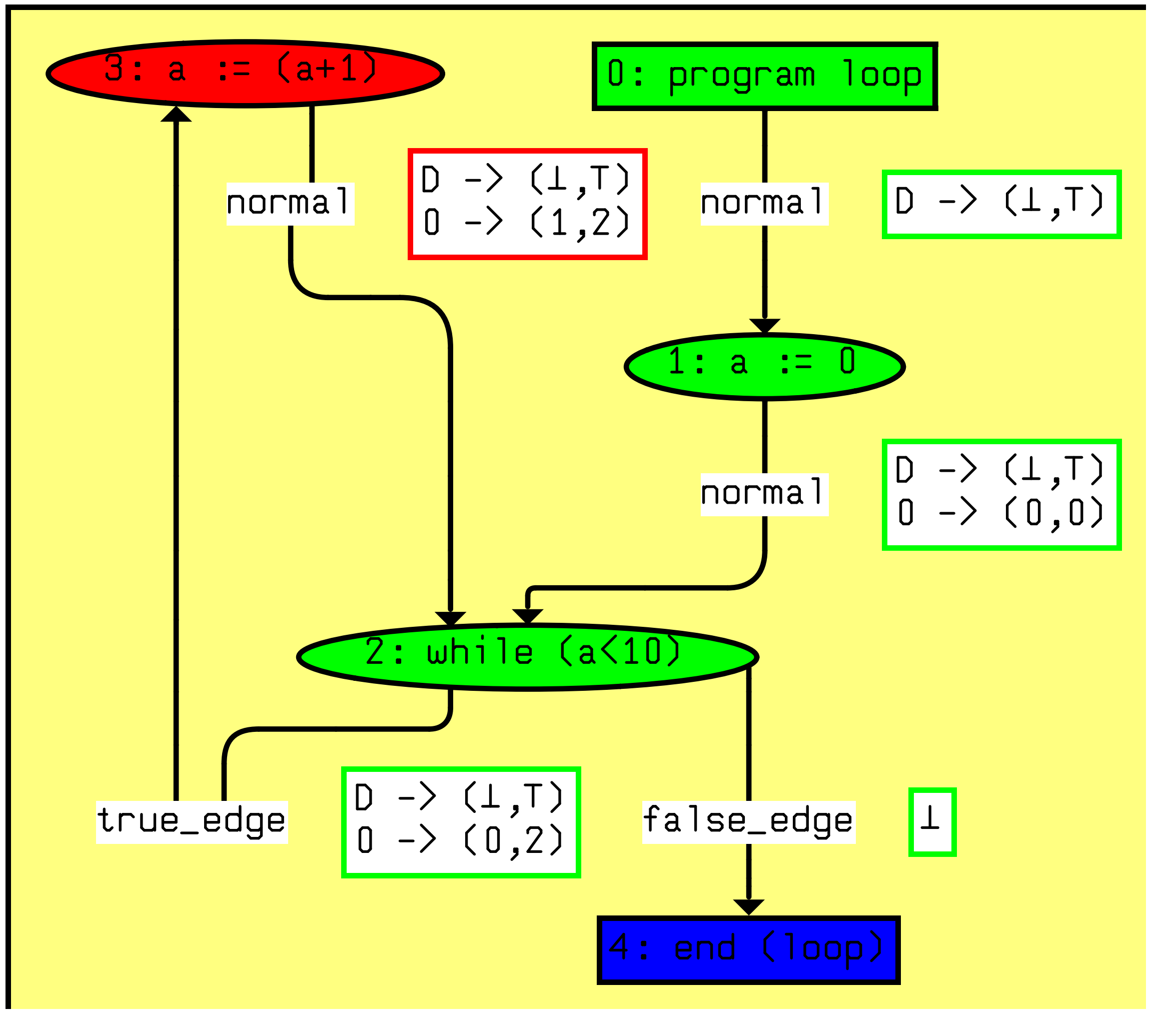

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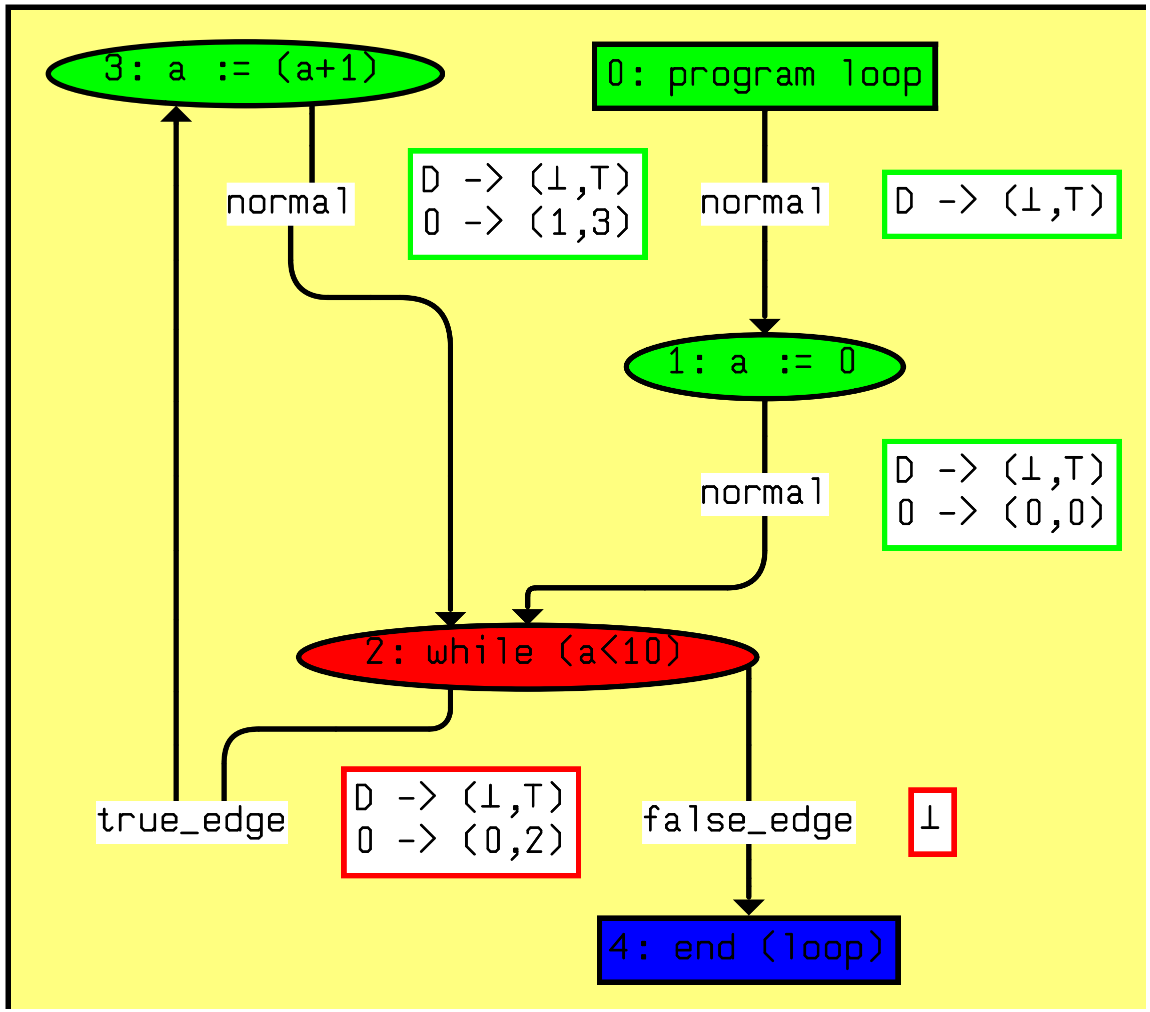

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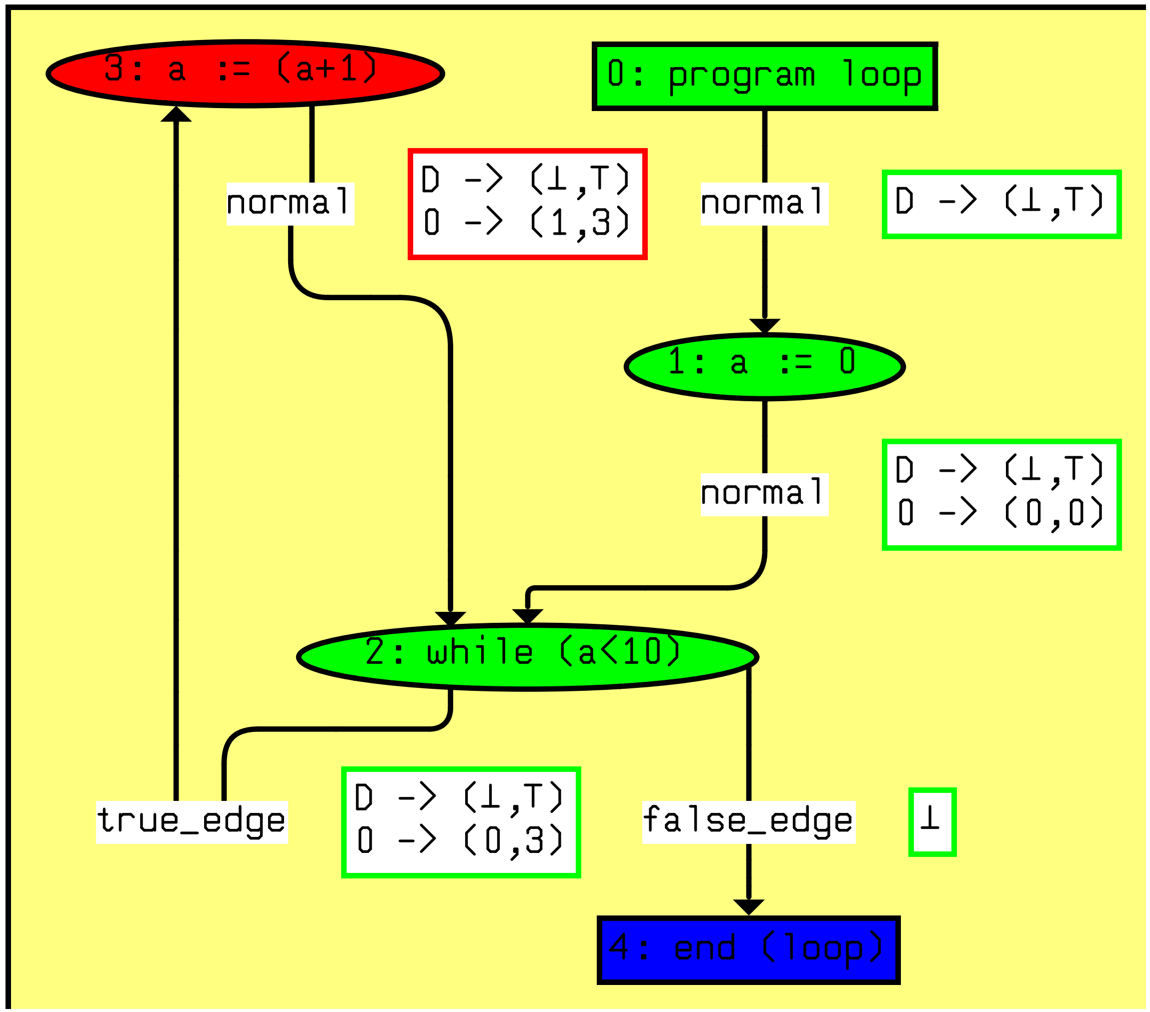

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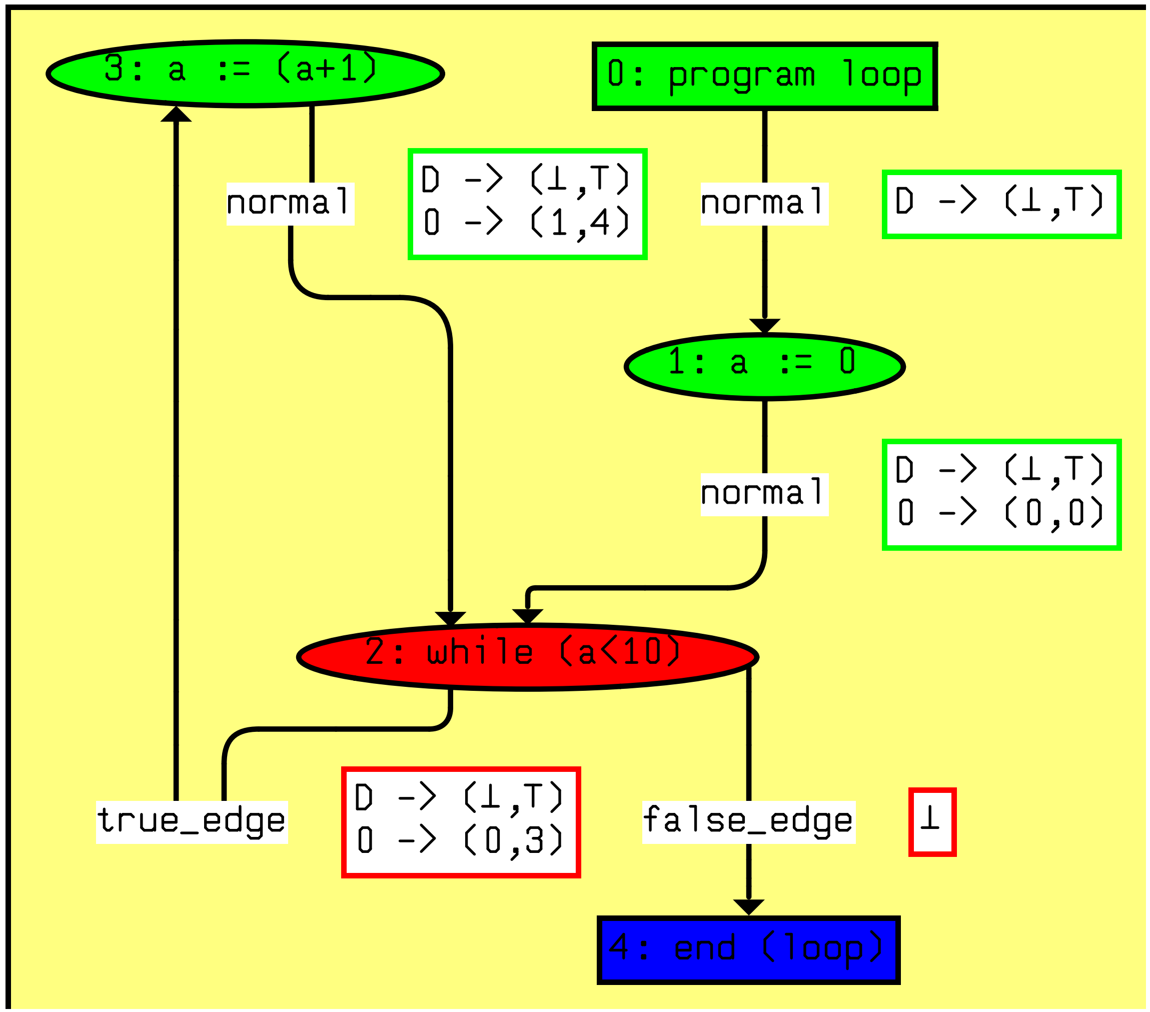

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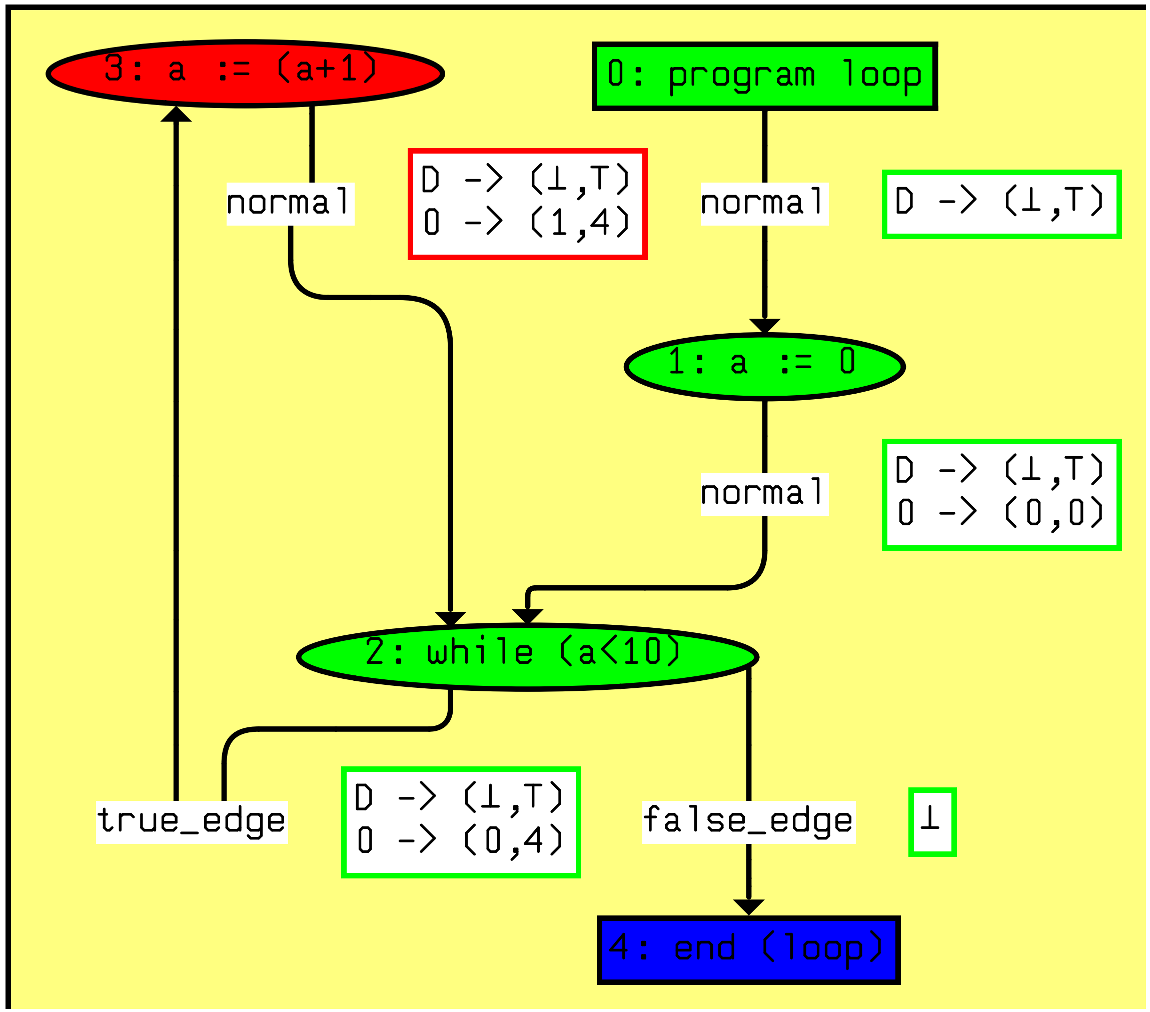

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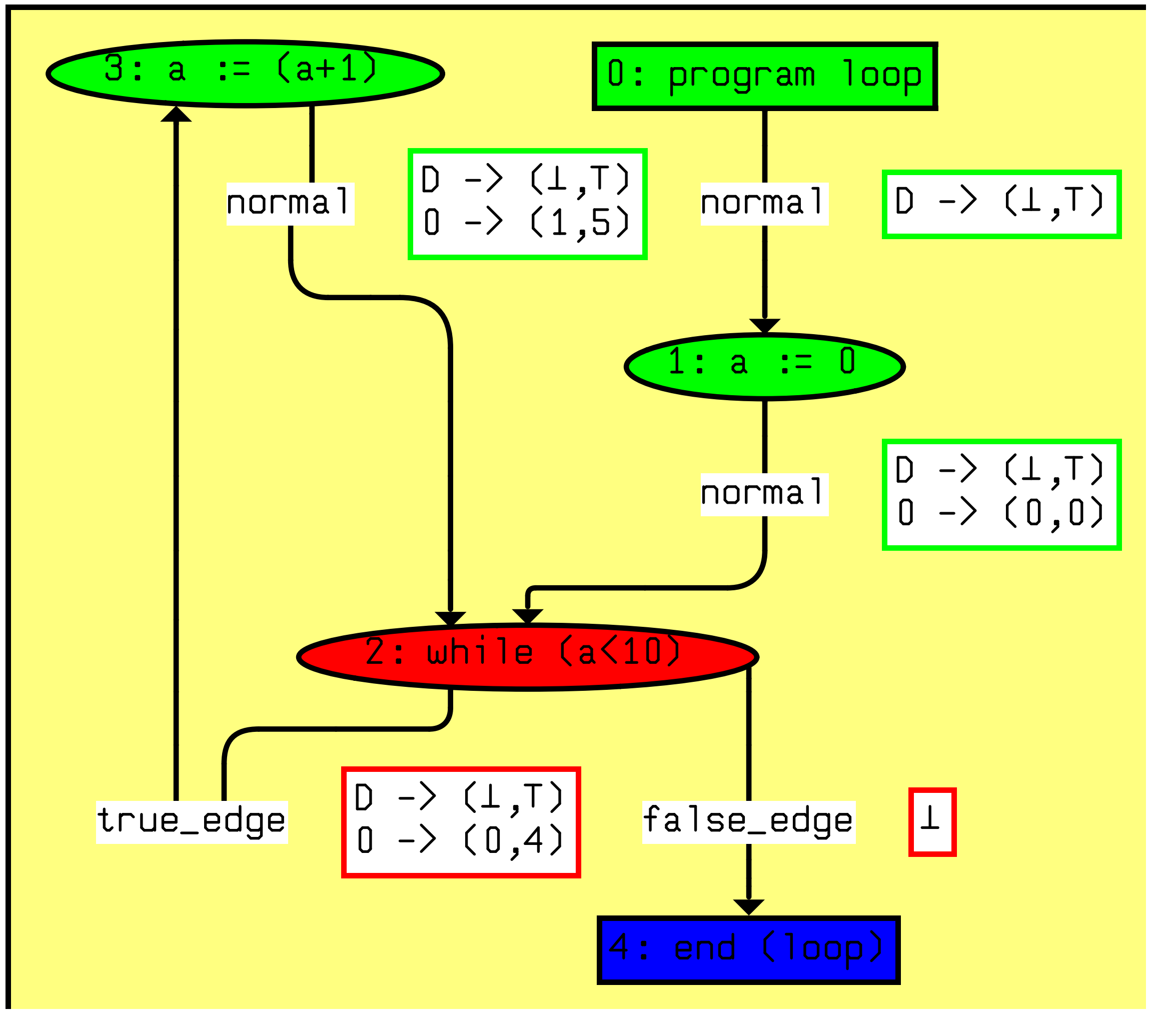

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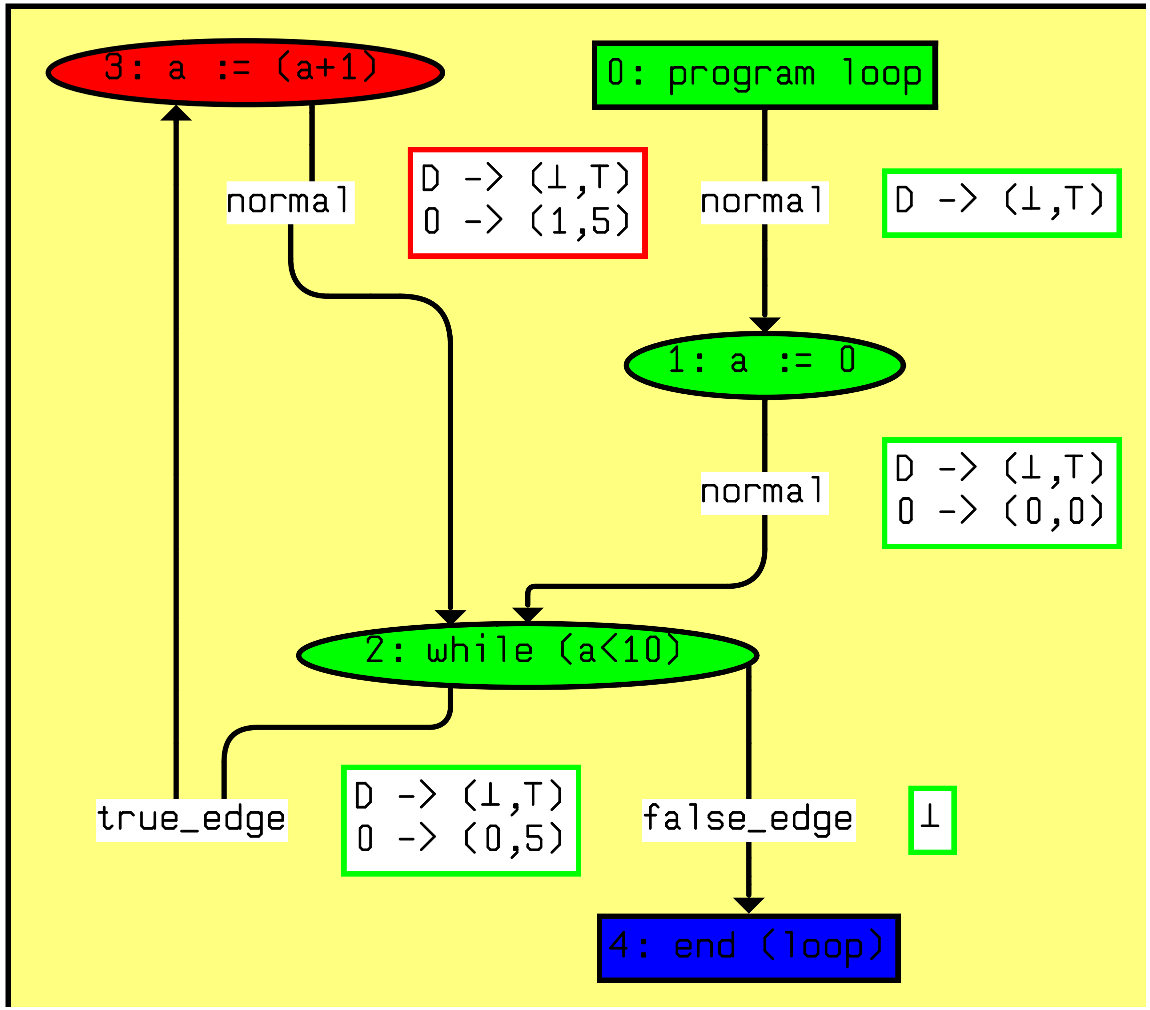

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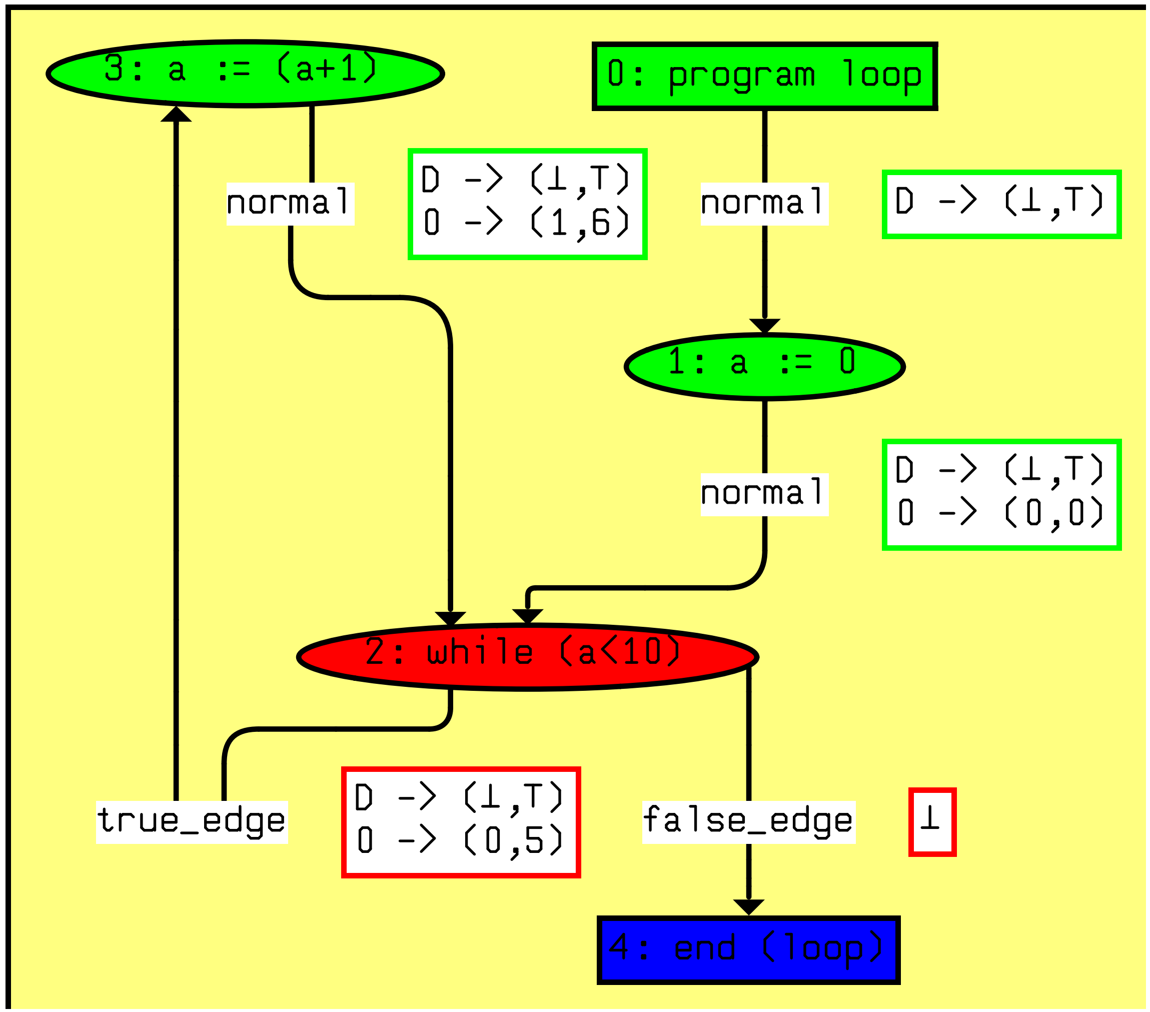

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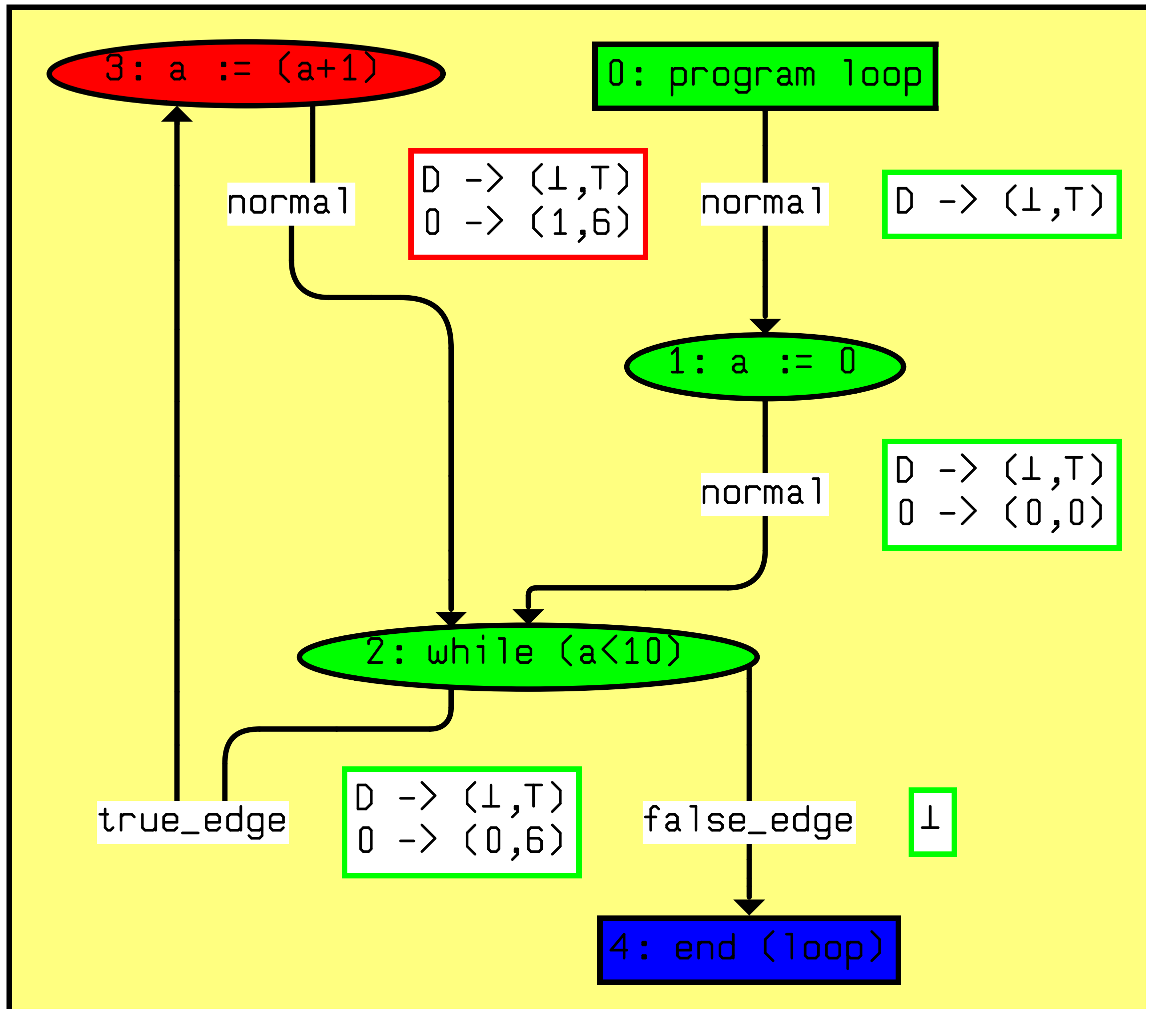

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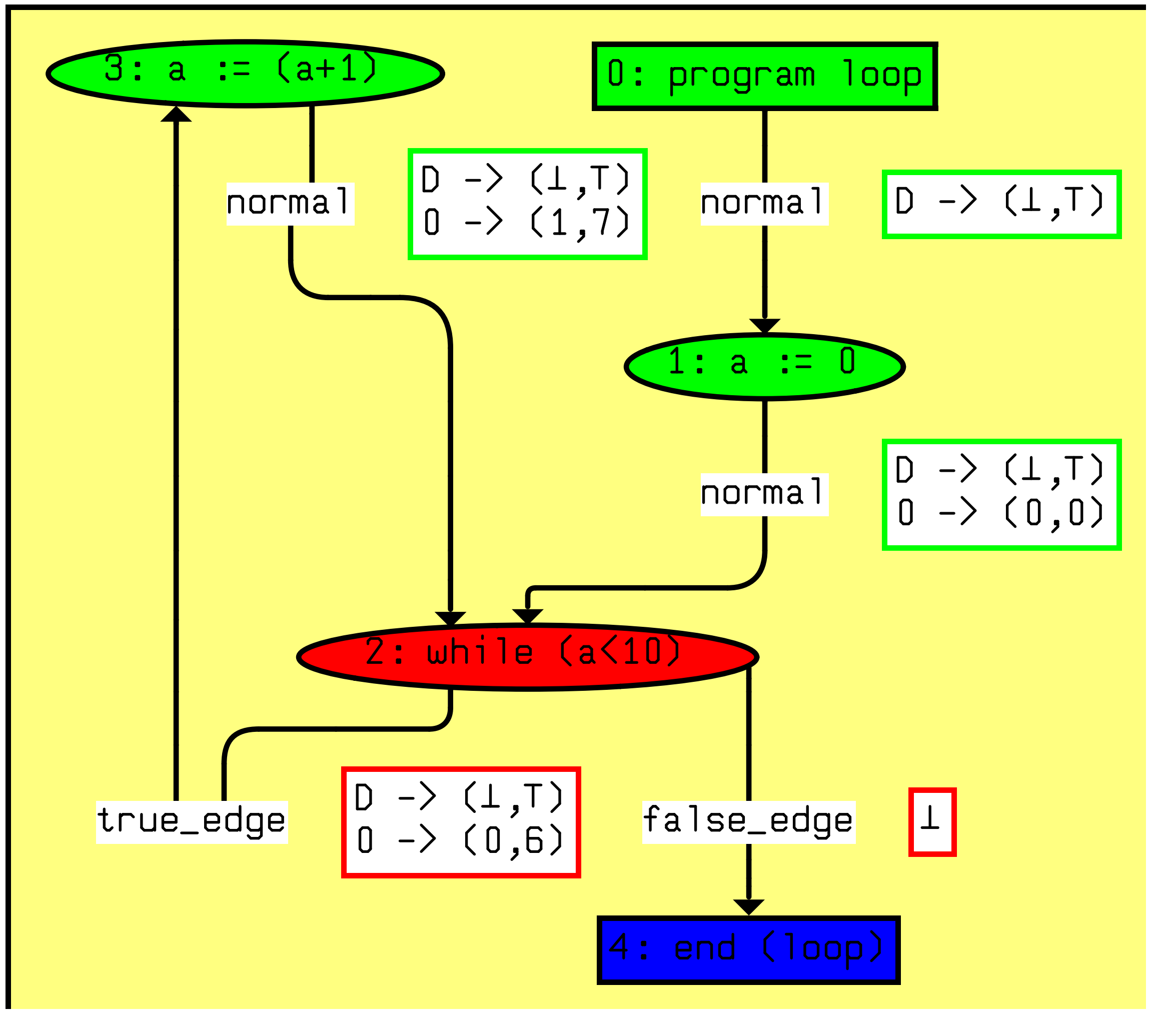

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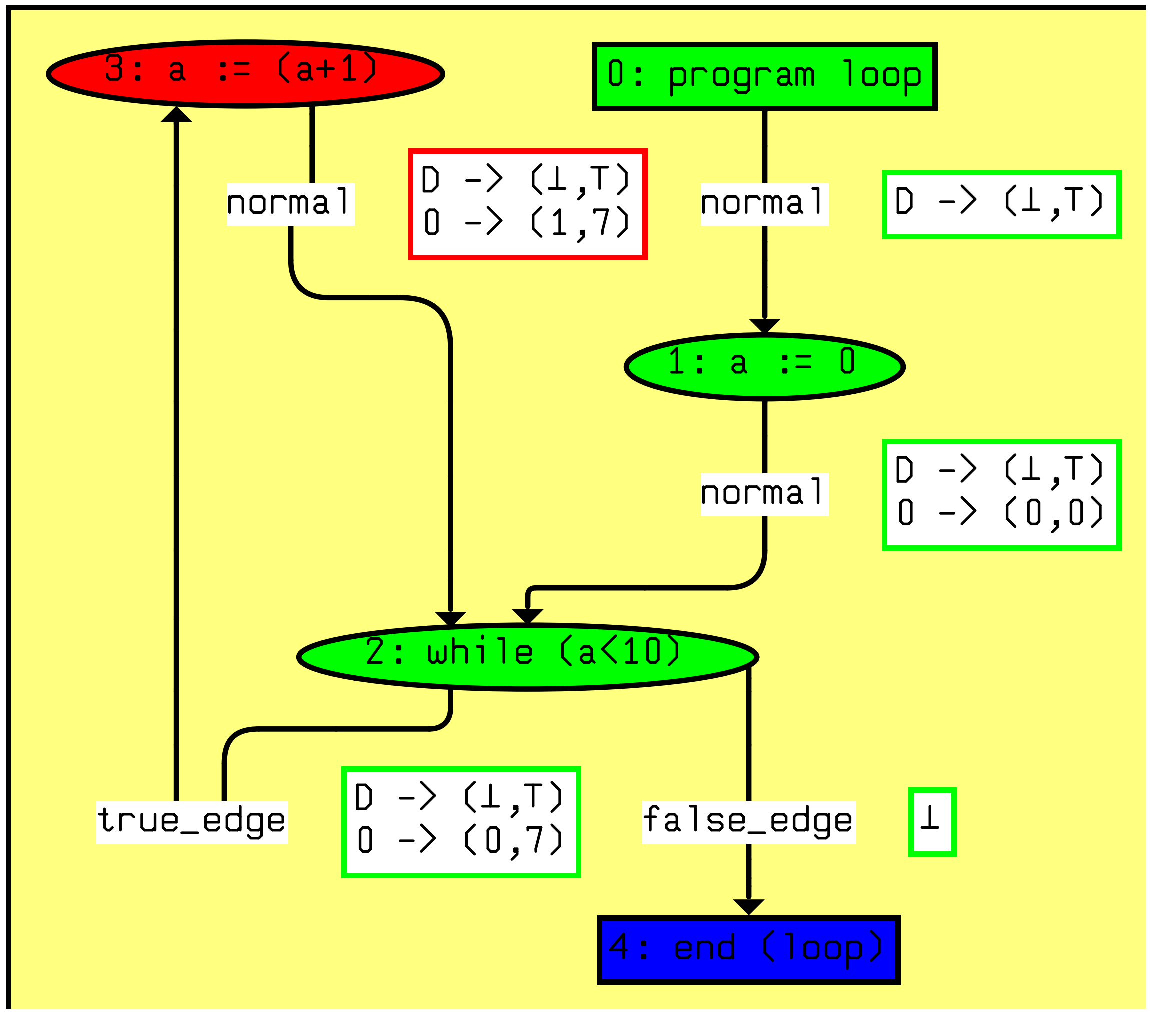

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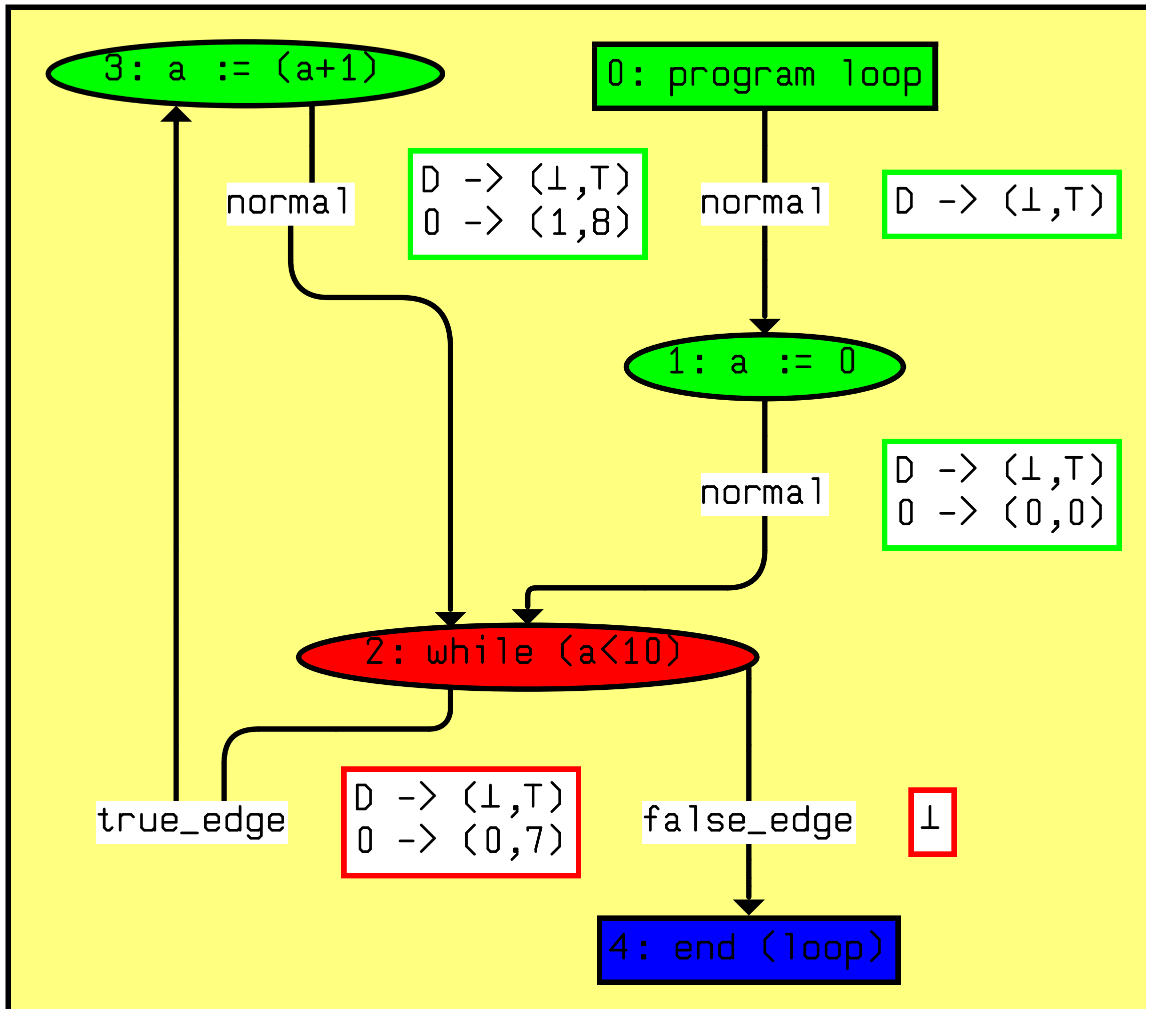

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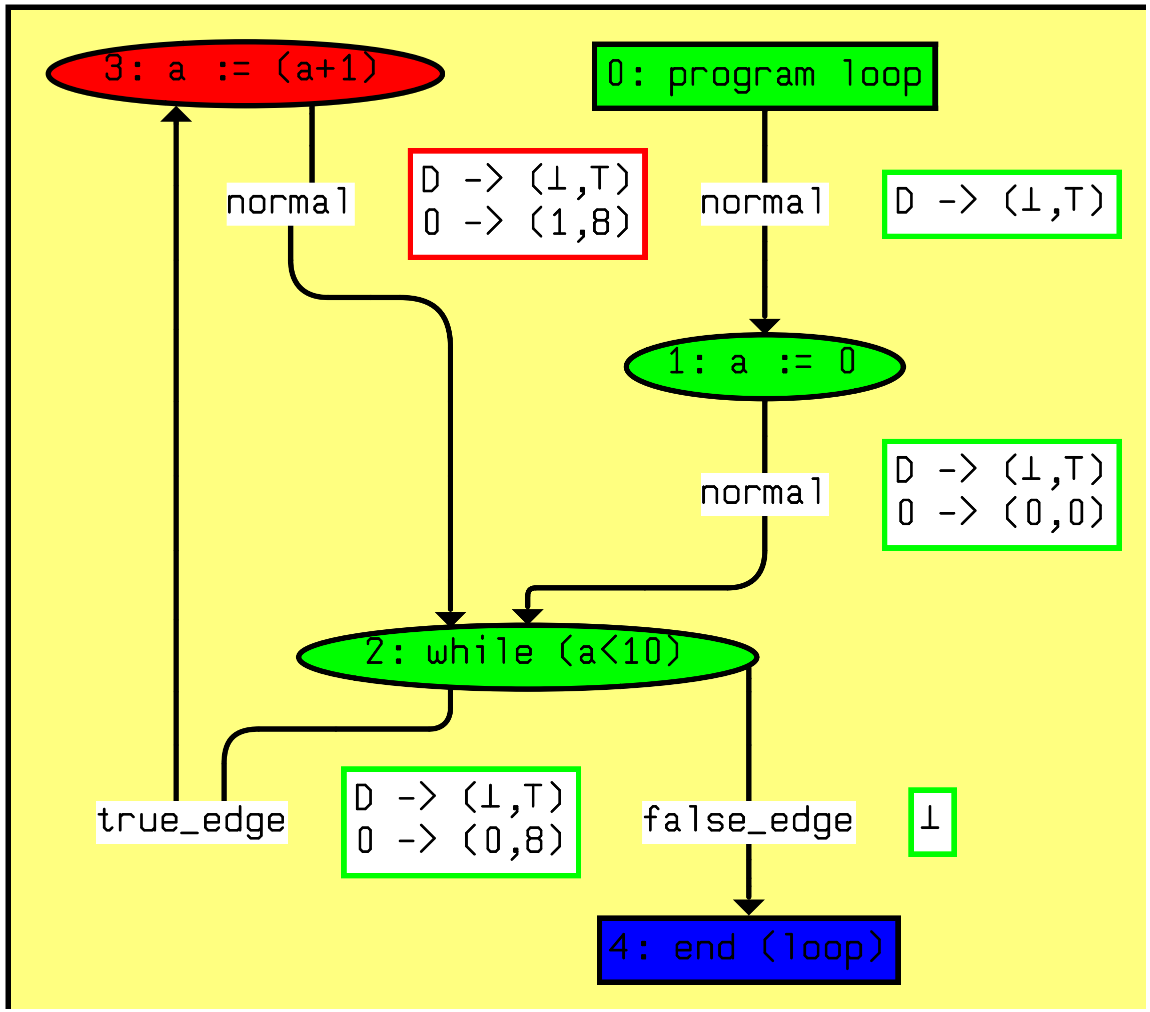

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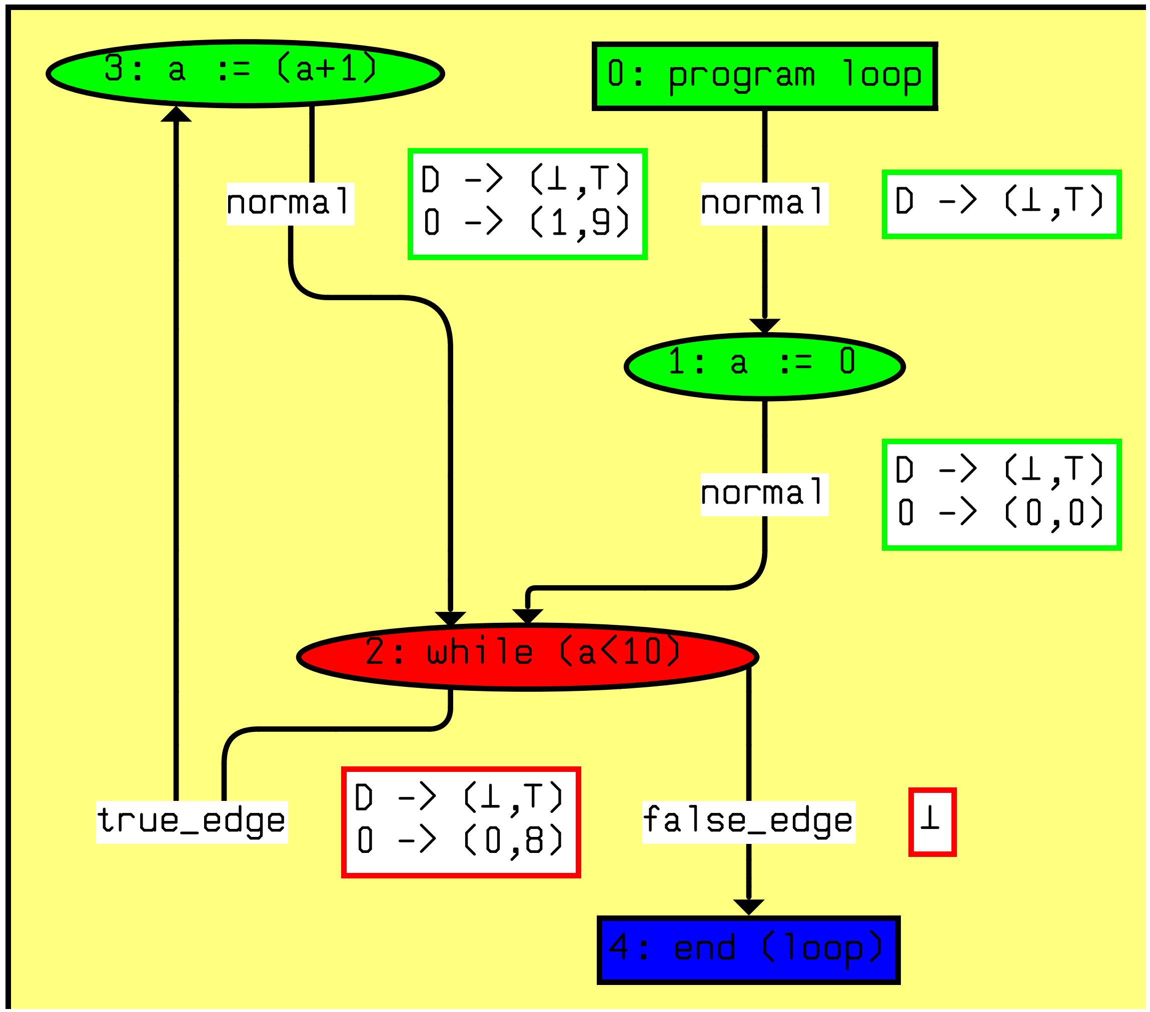

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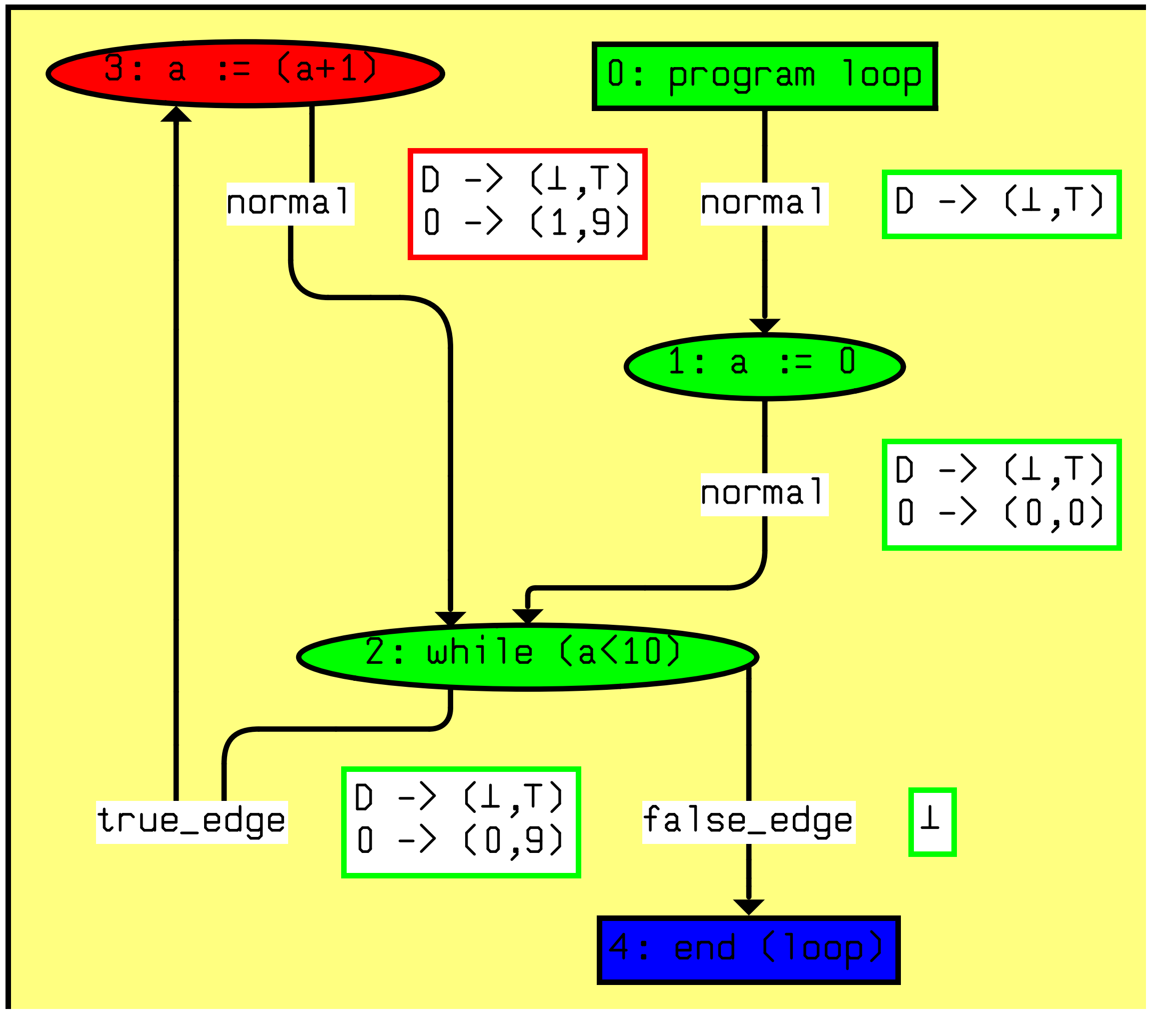

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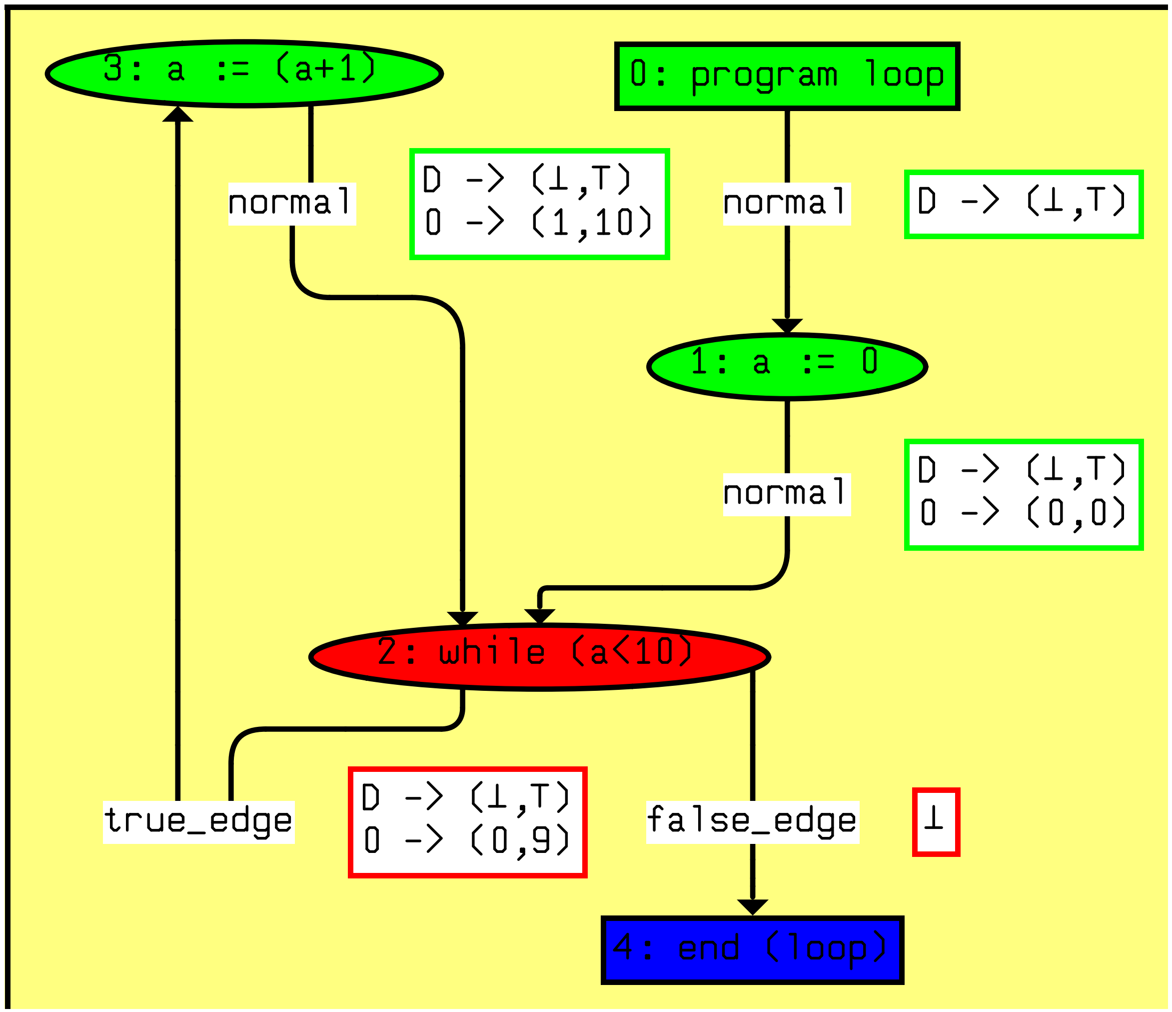

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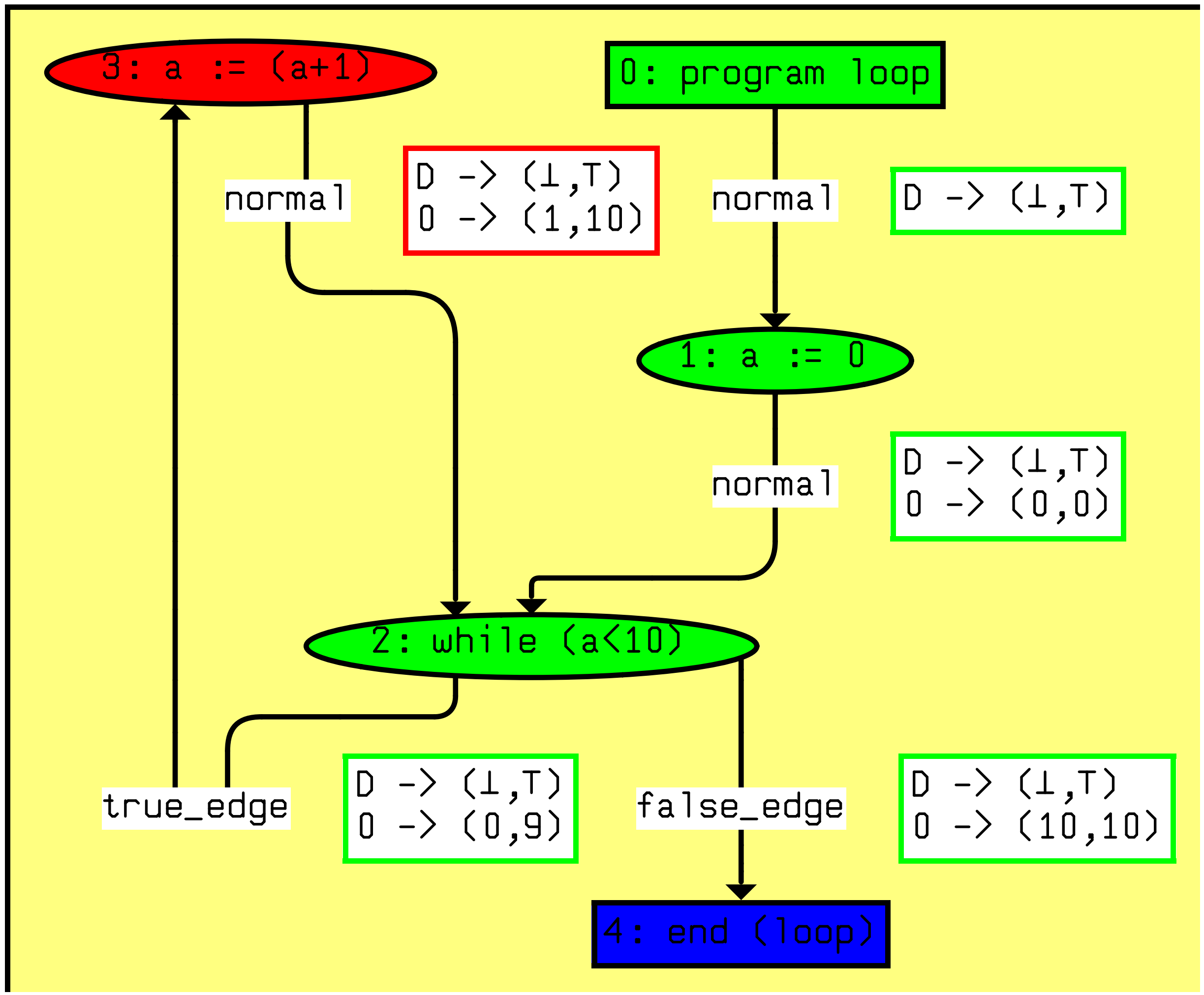

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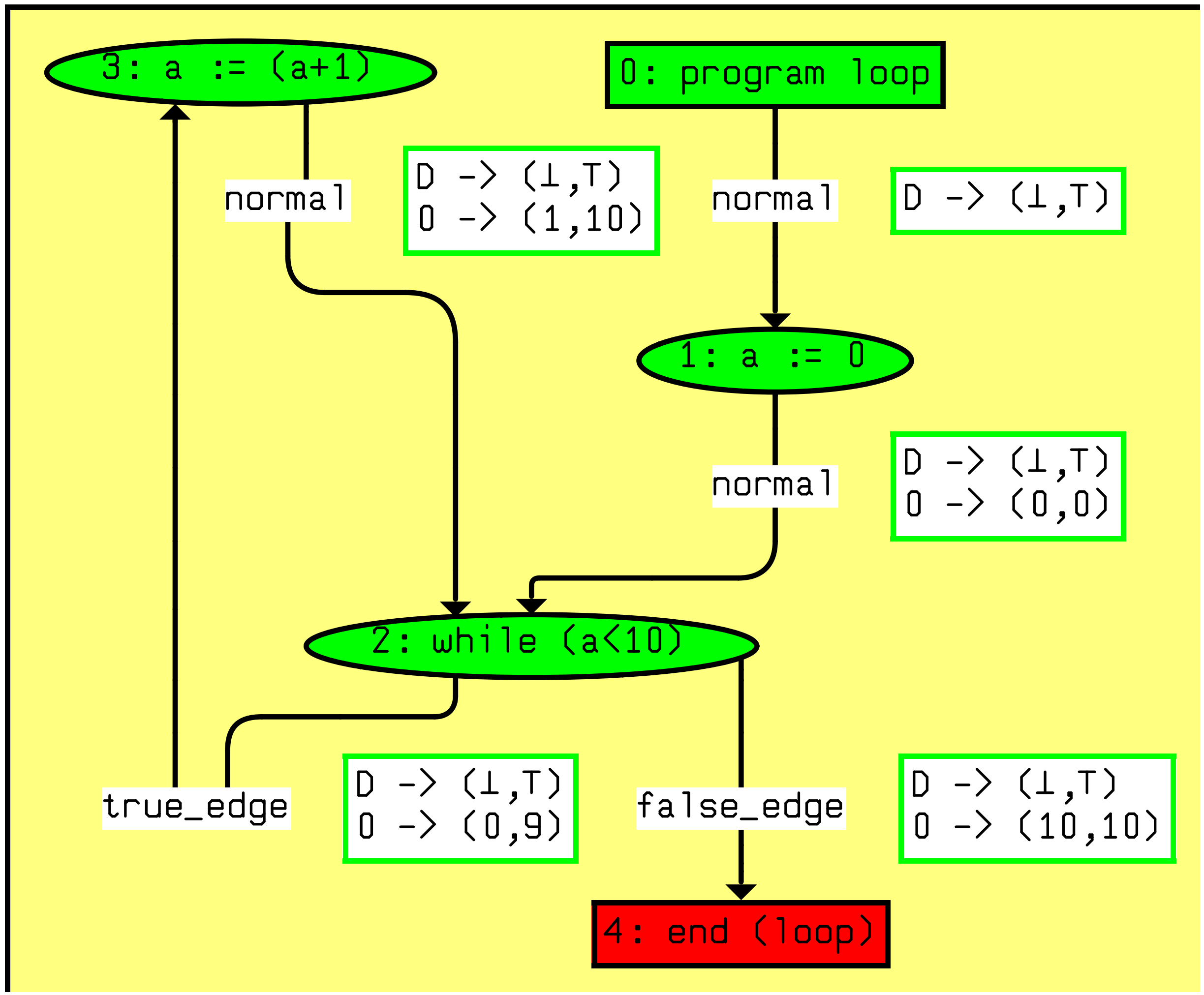


Insufficiency

This will not do

- You might as well run the program. We were lucky the analysis terminated.
- We need further approximation. . . But that's not possible!
  - For each program there is a finite domain.
  - But no single finite domain will do for all programs.
  - Can we find the suitable domain by some form of textual analysis?

Textual analysis?

- Cousot brings the following examples
  - McCarty's 91-function (Bourdoncle)
  - Rational congruence analysis (Granger)
  - Linear inequality analysis (Cousot/Halbwachs)
- Here's the 91-function

$$f(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{otherwise} \end{cases}$$

How did they do it?

Widenings

- A *widening* is a lattice operator $\nabla : L \times L \to L$ such that
  - It's an upper bound operator
  - For all increasing chains $x_0 \sqsubseteq x_1 \sqsubseteq \cdots$, the increasing chain defined by

$$\begin{aligned} y_0 &= x_0 \\ y_{i+1} &= y_i \nabla x_{i+1} \end{aligned}$$

is not strictly increasing.

Iterating with widenings

- The upward iteration sequence with widening

$$X_0 = \bot$$

$$X_{i+1} = \begin{cases} X_i & \text{if } F(X_i) \sqsubseteq X_i \\ X_i \nabla F(X_i) & \text{otherwise} \end{cases}$$

stabilizes to a safe approximation of the fix-point of F.

- F is *reductive* at that point.

Why is that?

- If we reach a point in $\mathrm{Red}(F)$, then we're ok.

$$X_0 = \bot$$

$$X_{i+1} = \begin{cases} X_i & \text{if } F(X_i) \sqsubseteq X_i \\ X_i \nabla F(X_i) & \text{otherwise} \end{cases}$$

- The sequence is clearly ascending.
- If $\exists i : F(X_i) \sqsubseteq X_i$, then
  - The sequence will immediately stabilize.
  - Since $X_i \in \mathrm{Red}(F)$, we have $\mathrm{lfp}(F) \sqsubseteq X_i$.

Will we get there?

- Assume $\forall i : X_i \sqsubset F(X_i)$, then

$$X_0 = \bot$$
$$X_{i+1} = X_i \nabla F(X_i)$$

- This is just the widening of the following ascending chain

$$Y_0 = \bot$$
$$Y_{i+1} = F(X_i)$$

QED! Really!

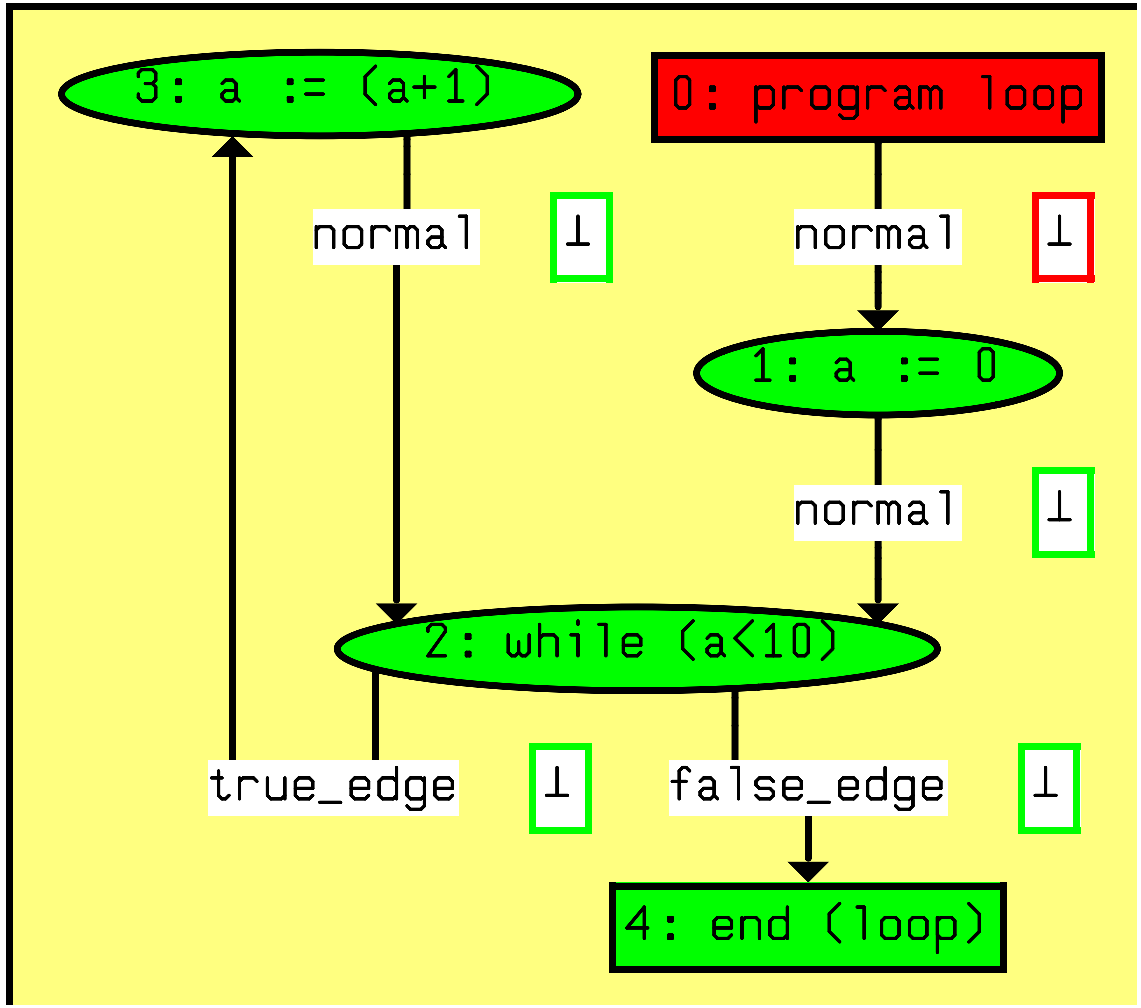
Widening for Upper bounds

- The upper bound of an interval, can be widened by

$$x \nabla y = \begin{cases} \infty & \text{if } x < y \\ x & \text{otherwise} \end{cases}$$

- It's a widening! Any chain is stablized immediately.

Approximating the Fixed-Point



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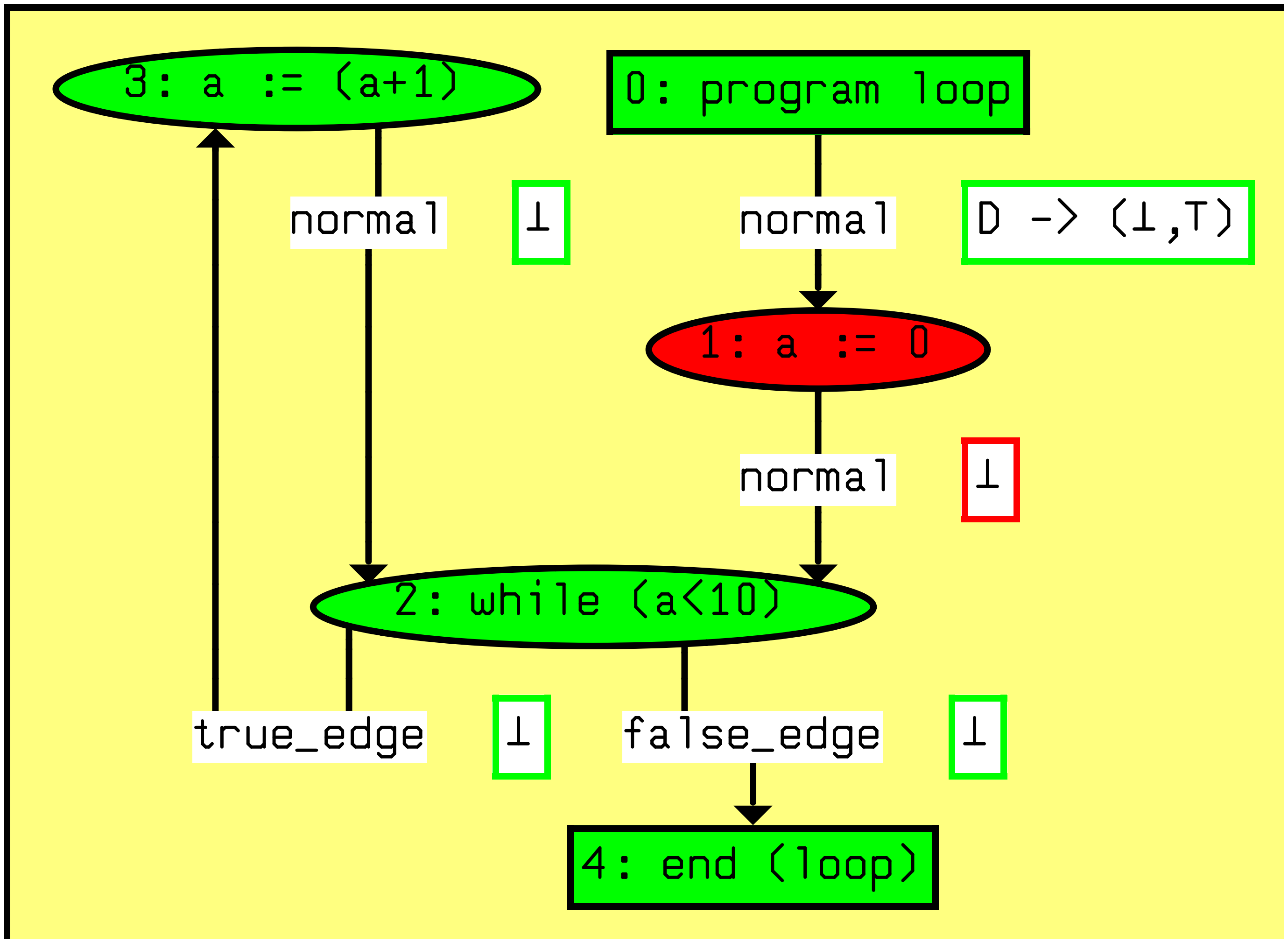

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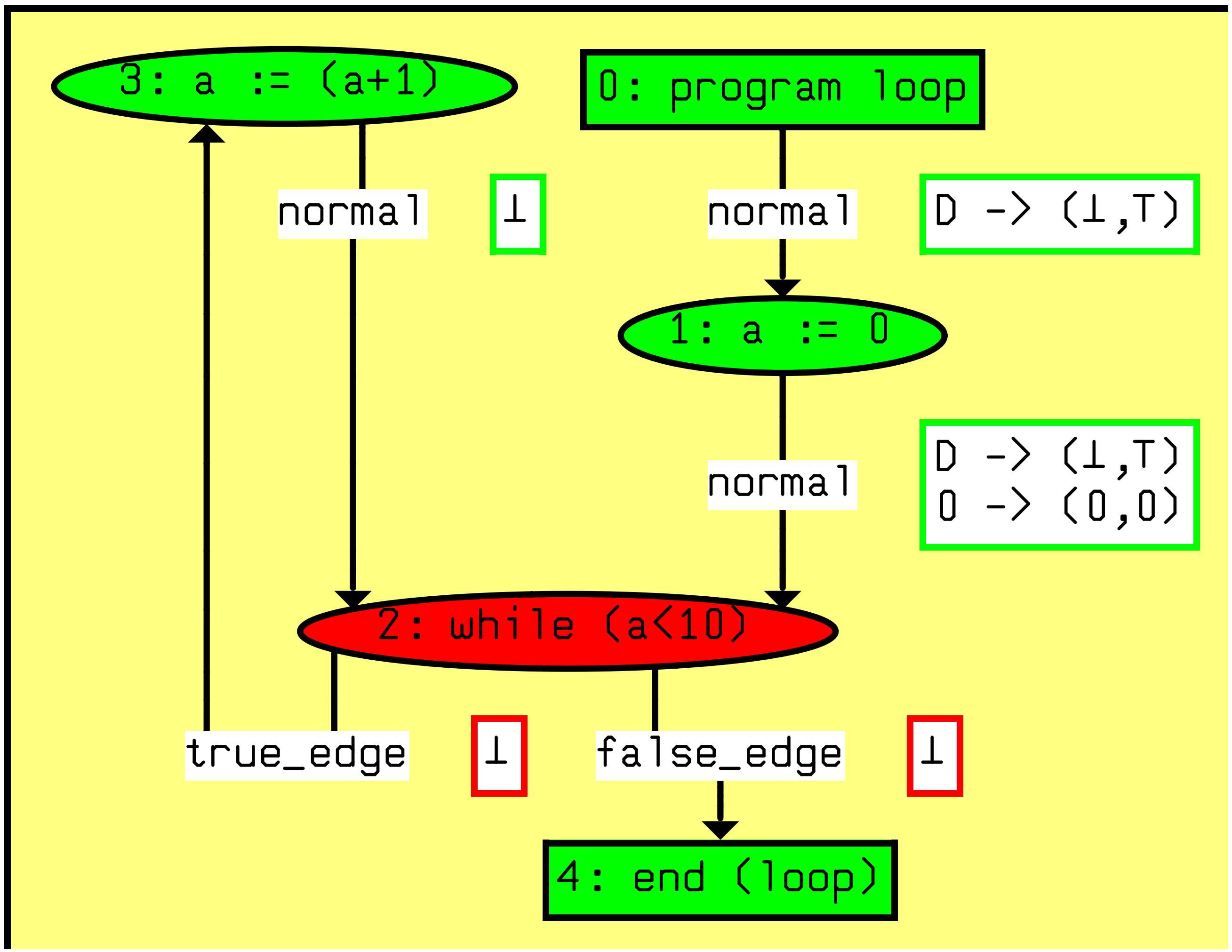

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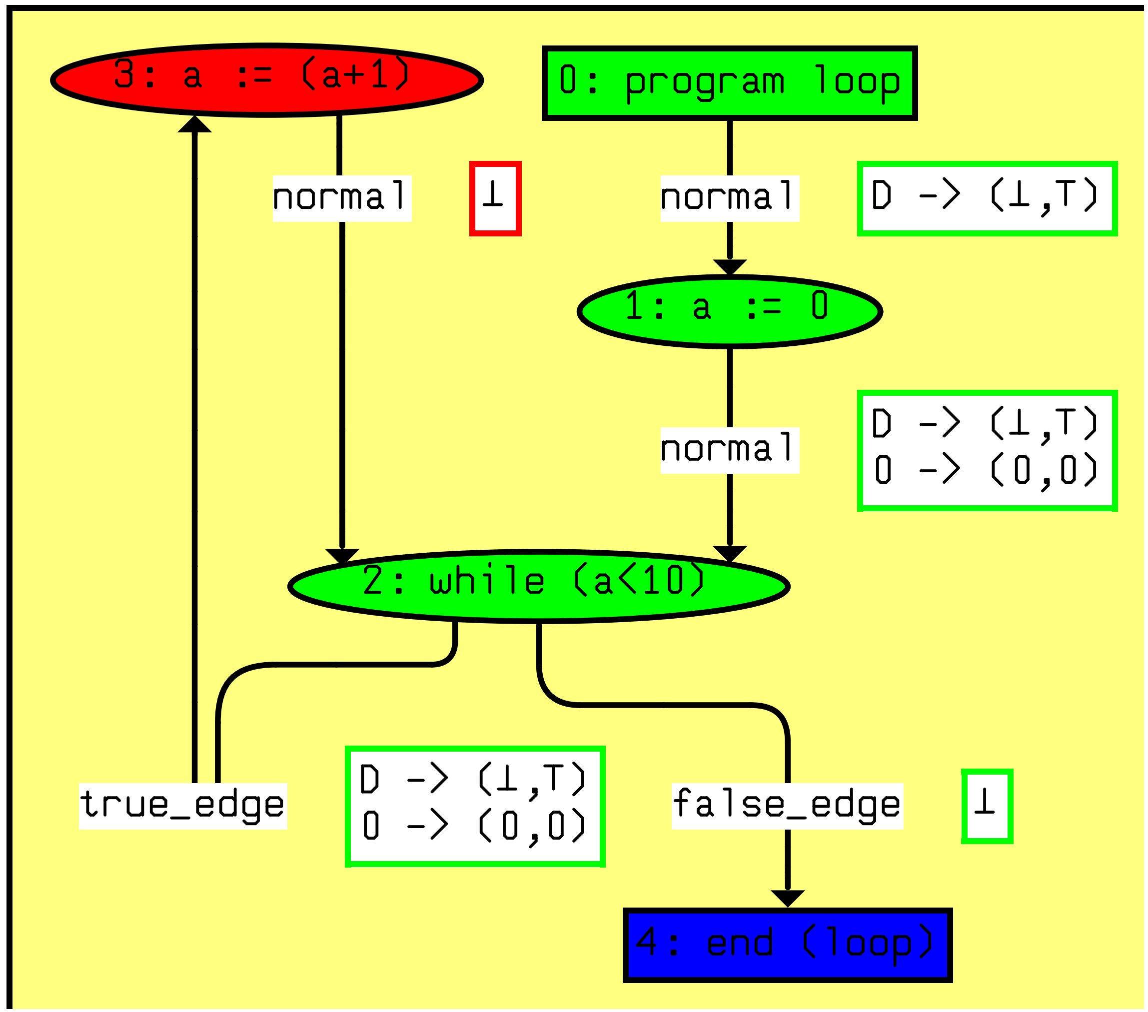

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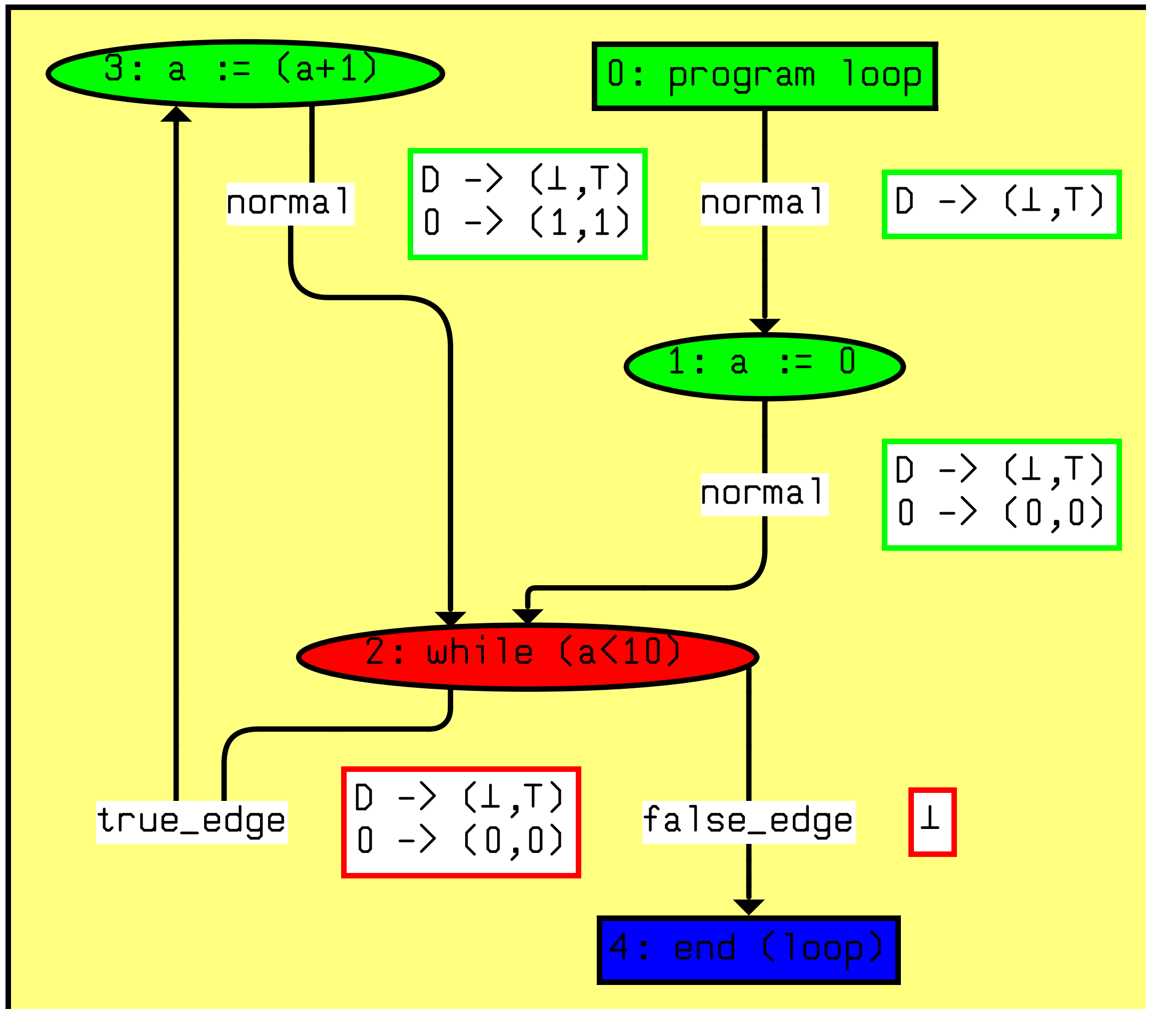
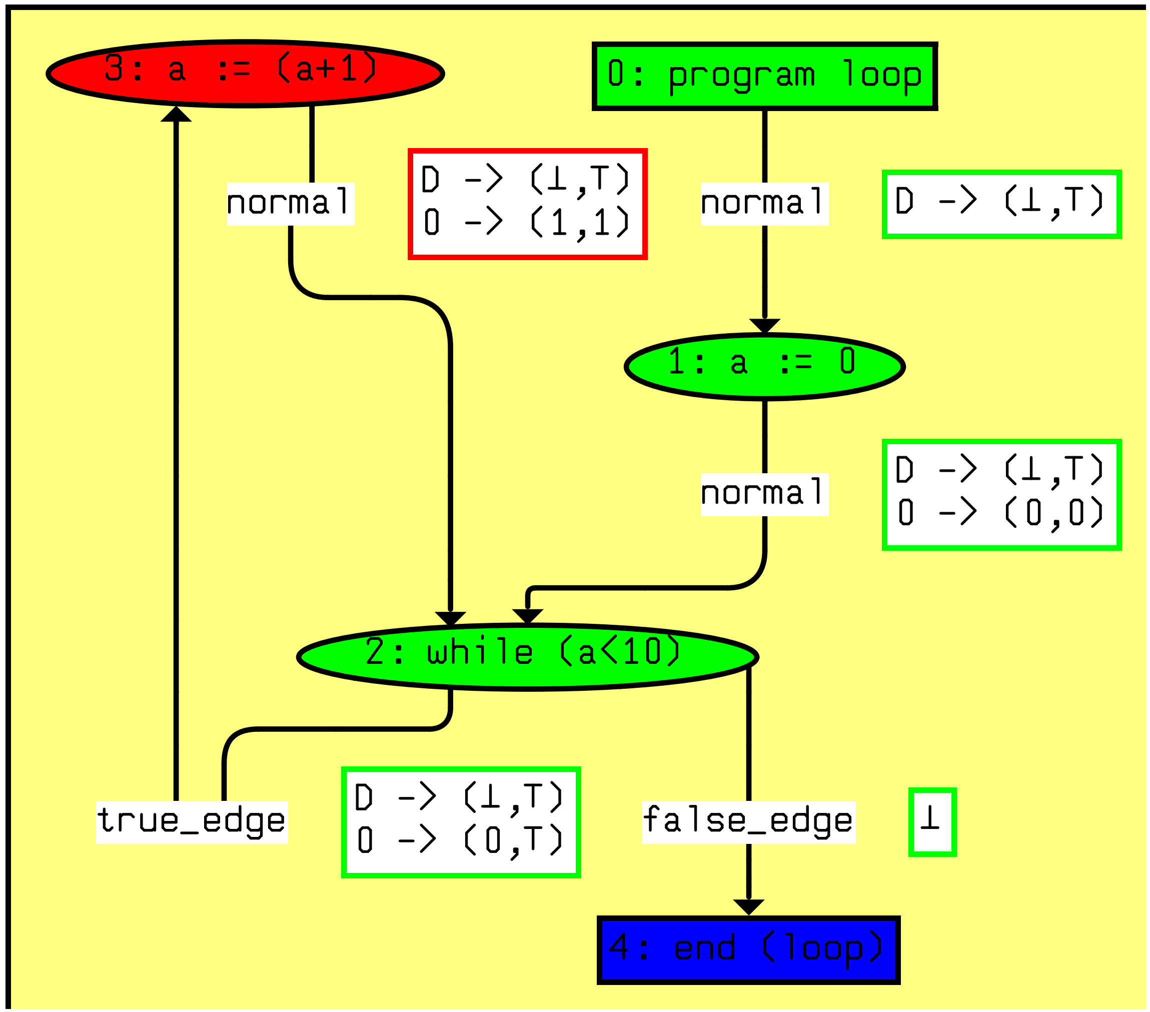


widening about to happen!

Approximating the Fixed-Point



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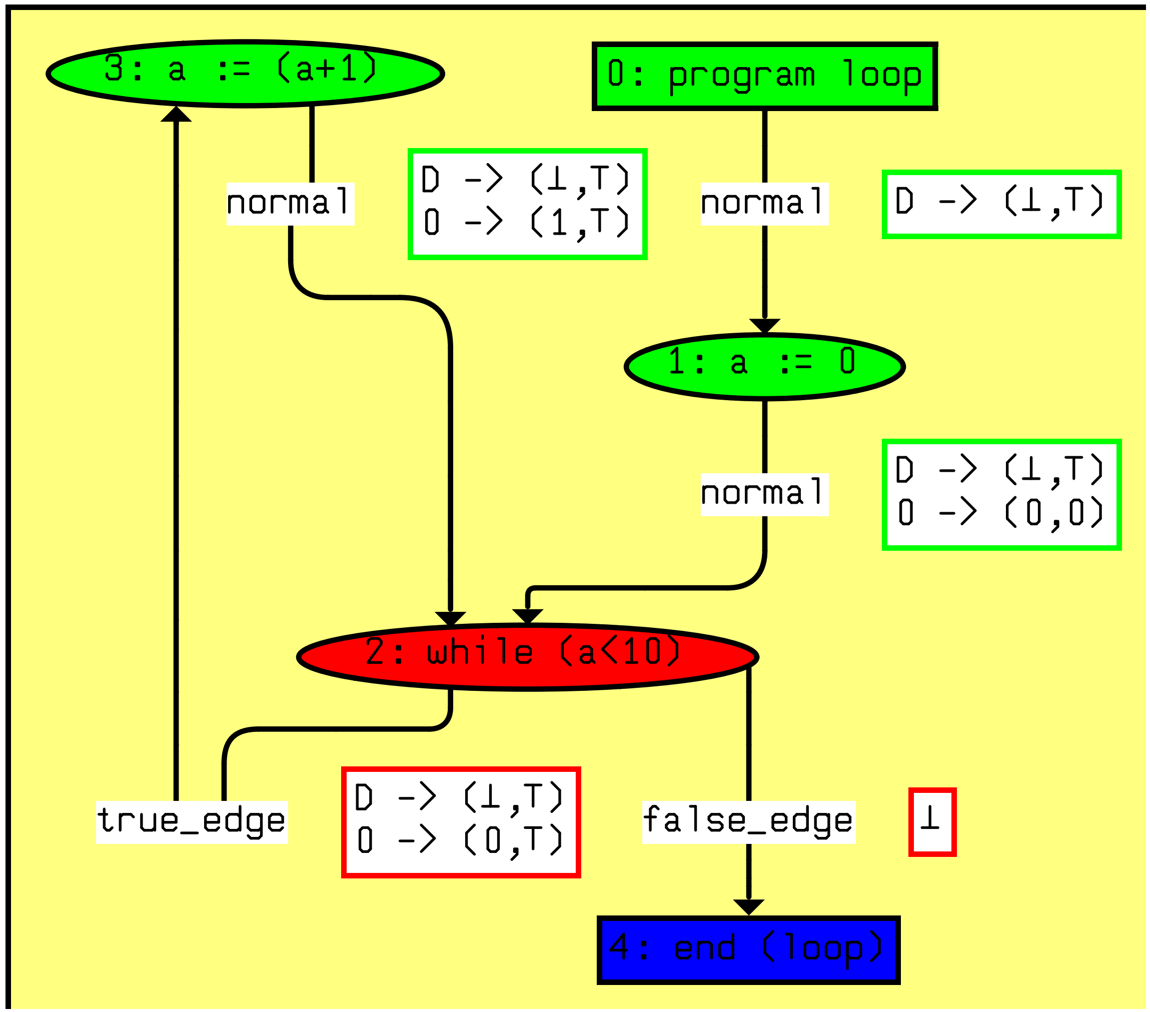

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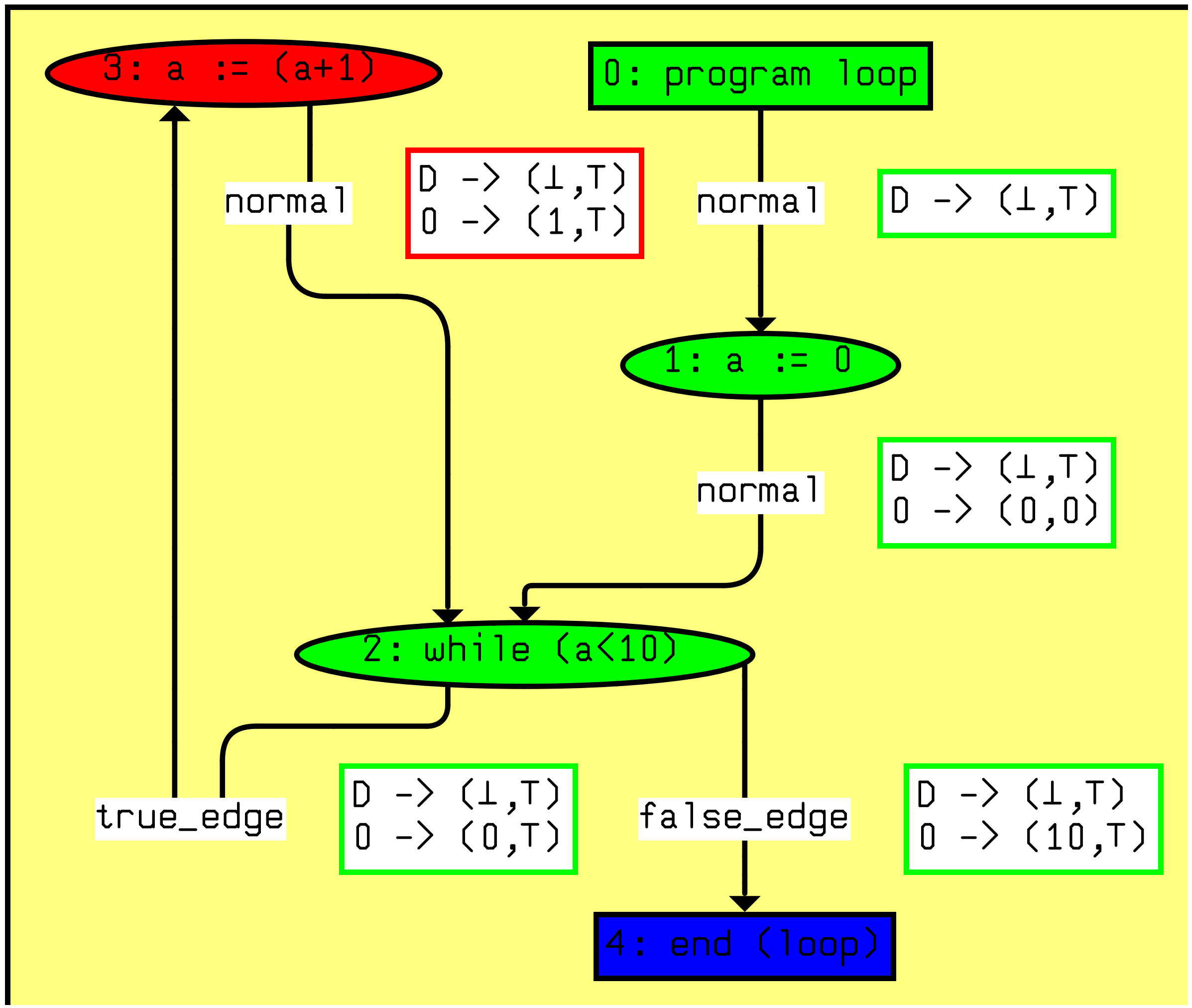

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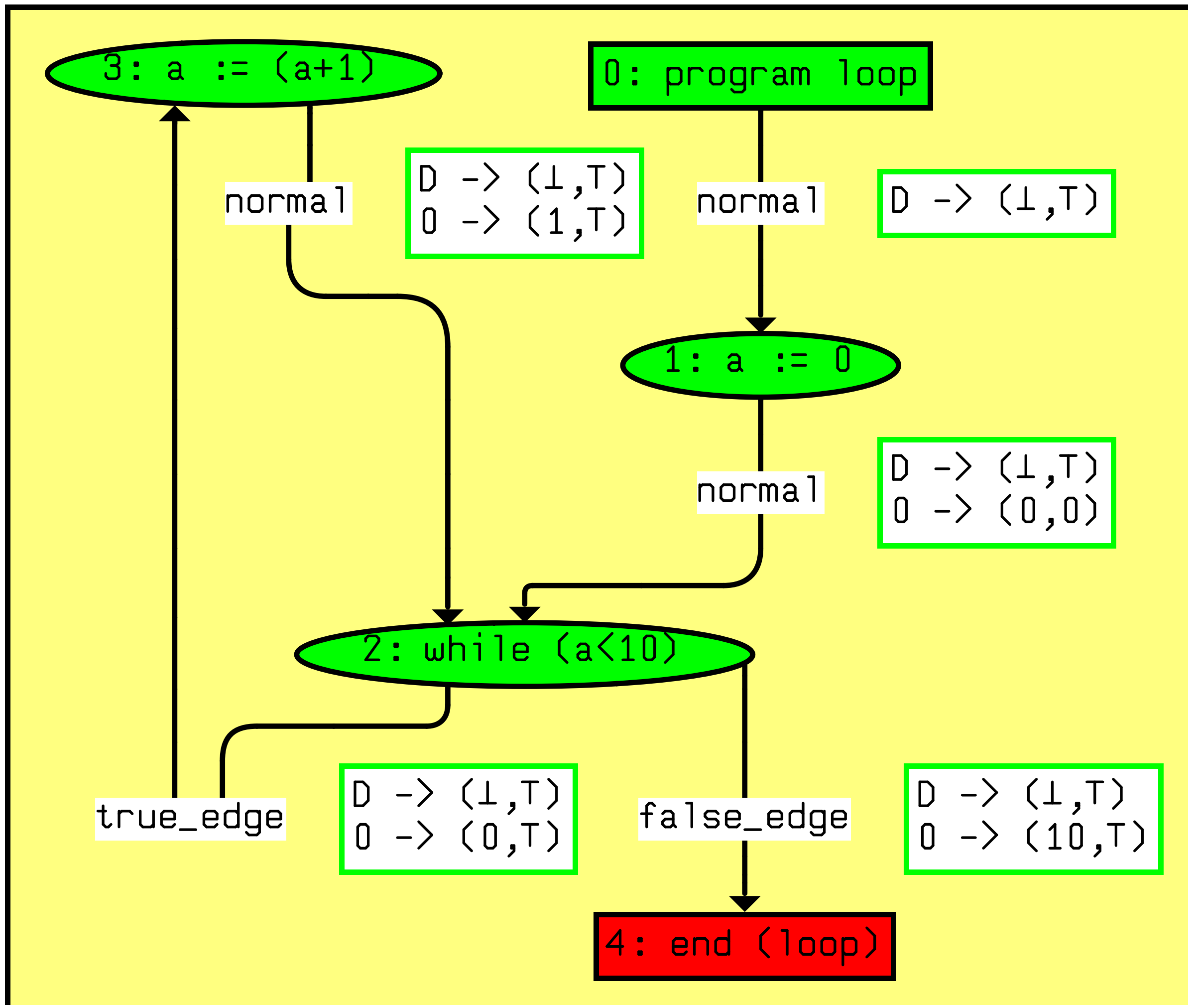


Narrowings

- Since F is reductive at the point reached through the upward iteratoin, we can fine-tune the result.
- Downward sequence may also not reach the fix-point, so we might need a similar procedure, called *narrowings*, to ensure termination.
- The only difference:

$$\forall x, y \in L : (y \sqsubseteq x) \implies (y \sqsubseteq (x \vartriangle y) \sqsubseteq x)$$

Not the dual of widening

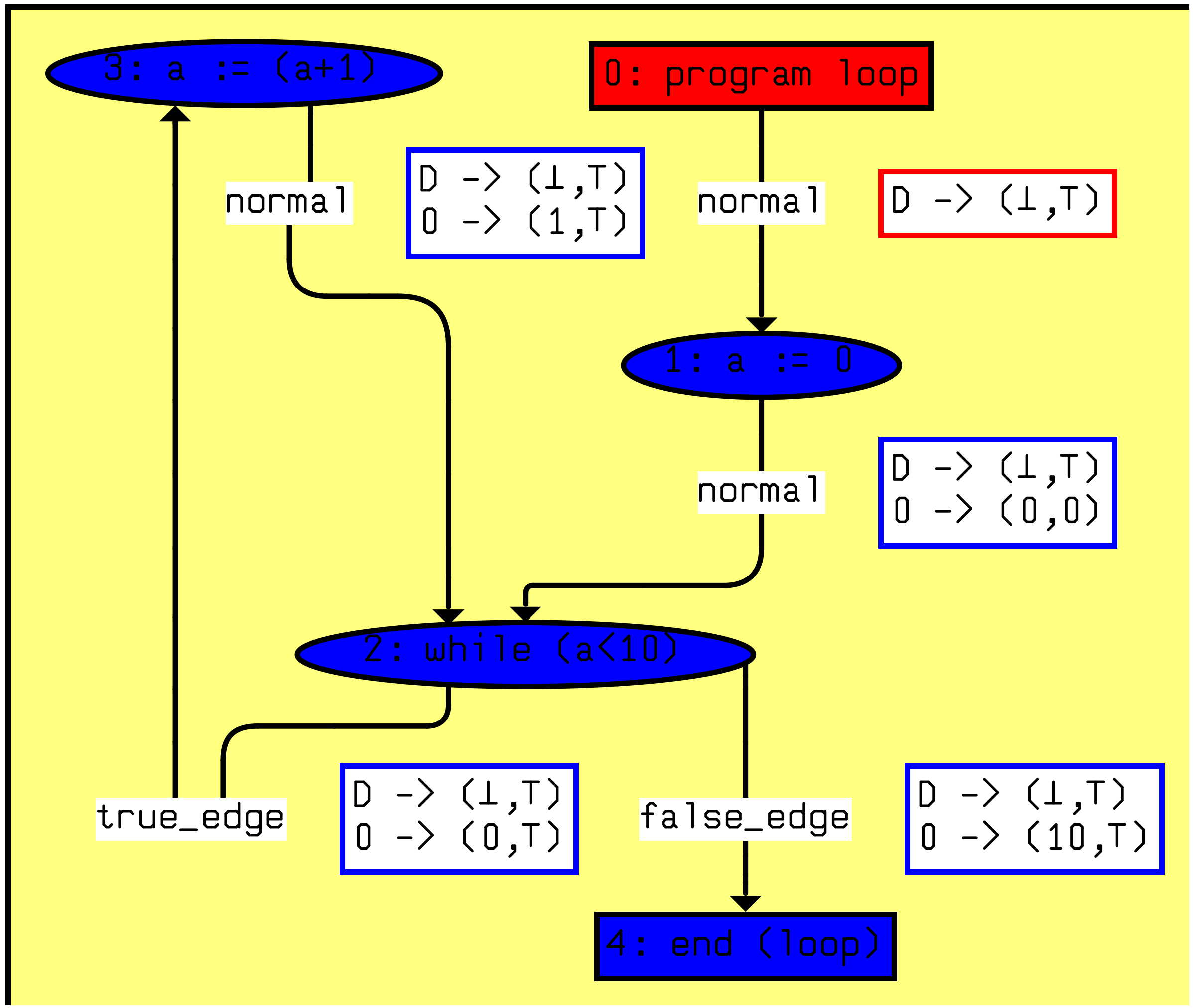
- The difference:

$$\forall x, y \in L : (y \sqsubseteq x) \implies (y \sqsubseteq (x \mathbin{\triangle} y) \sqsubseteq x)$$

means it is not a lower bound operator.

- While narrowings keep the sequence in Red, the dual of widenings would step out of Red and eventually into Ext.

Fine-tuning the result



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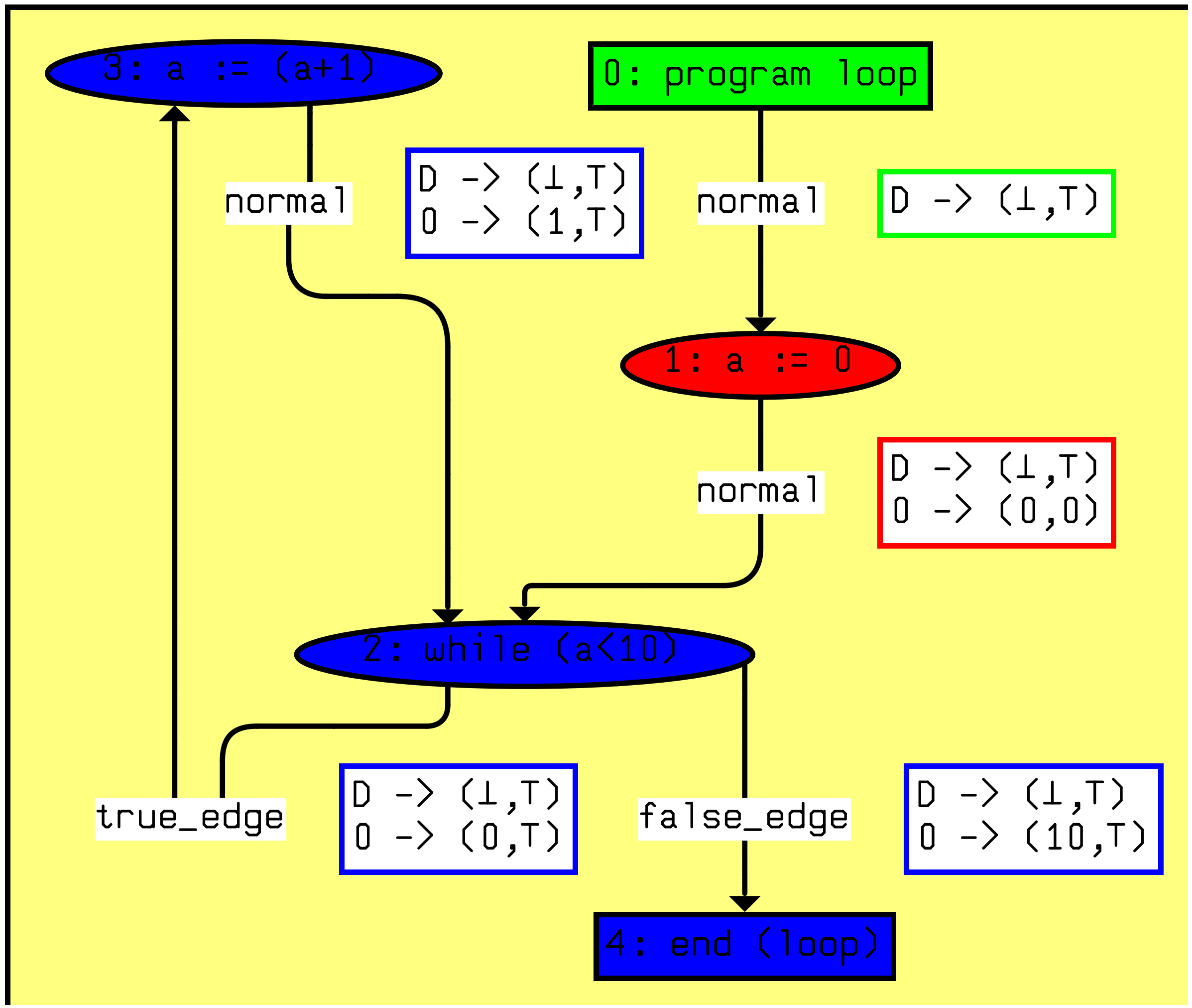

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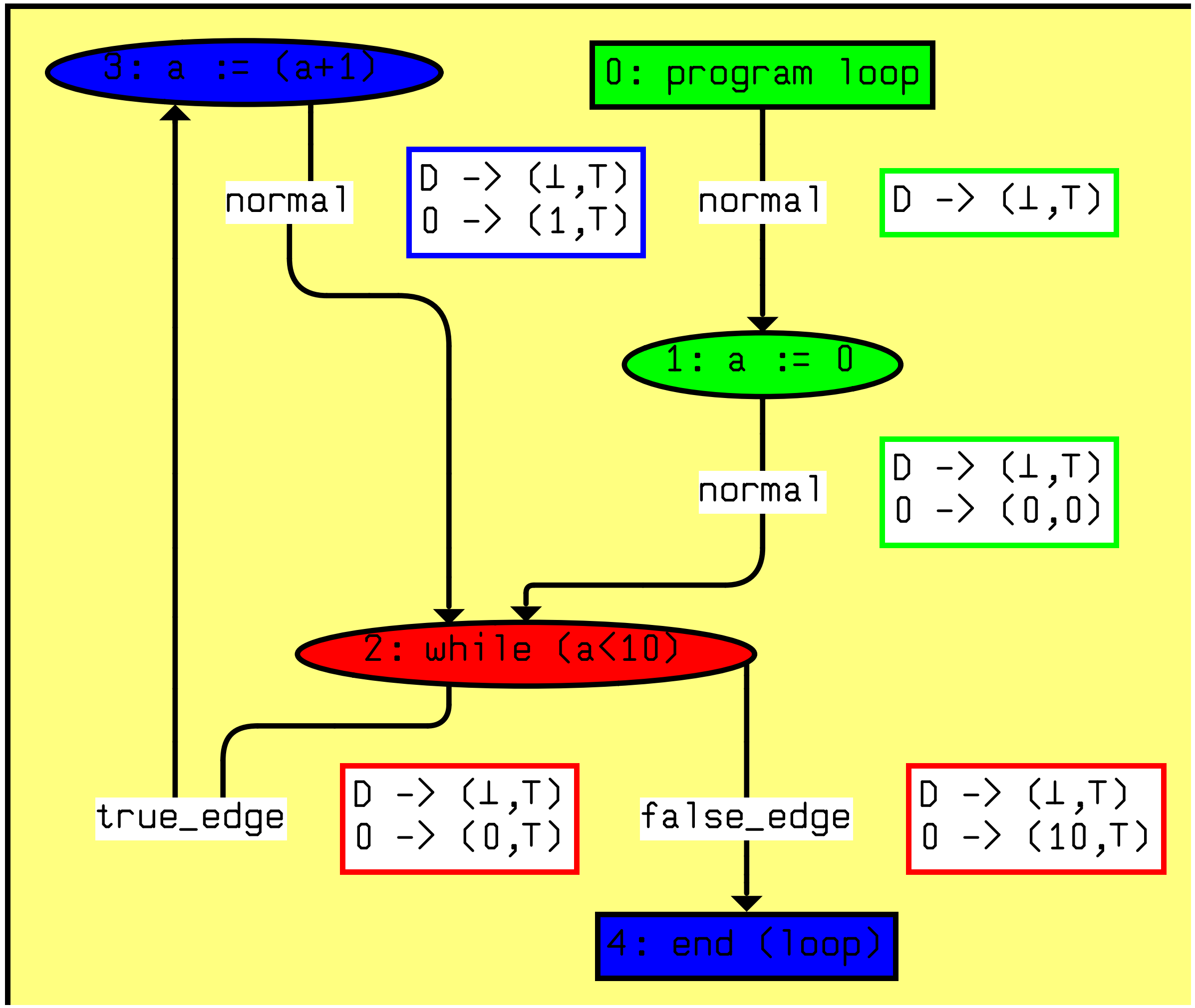

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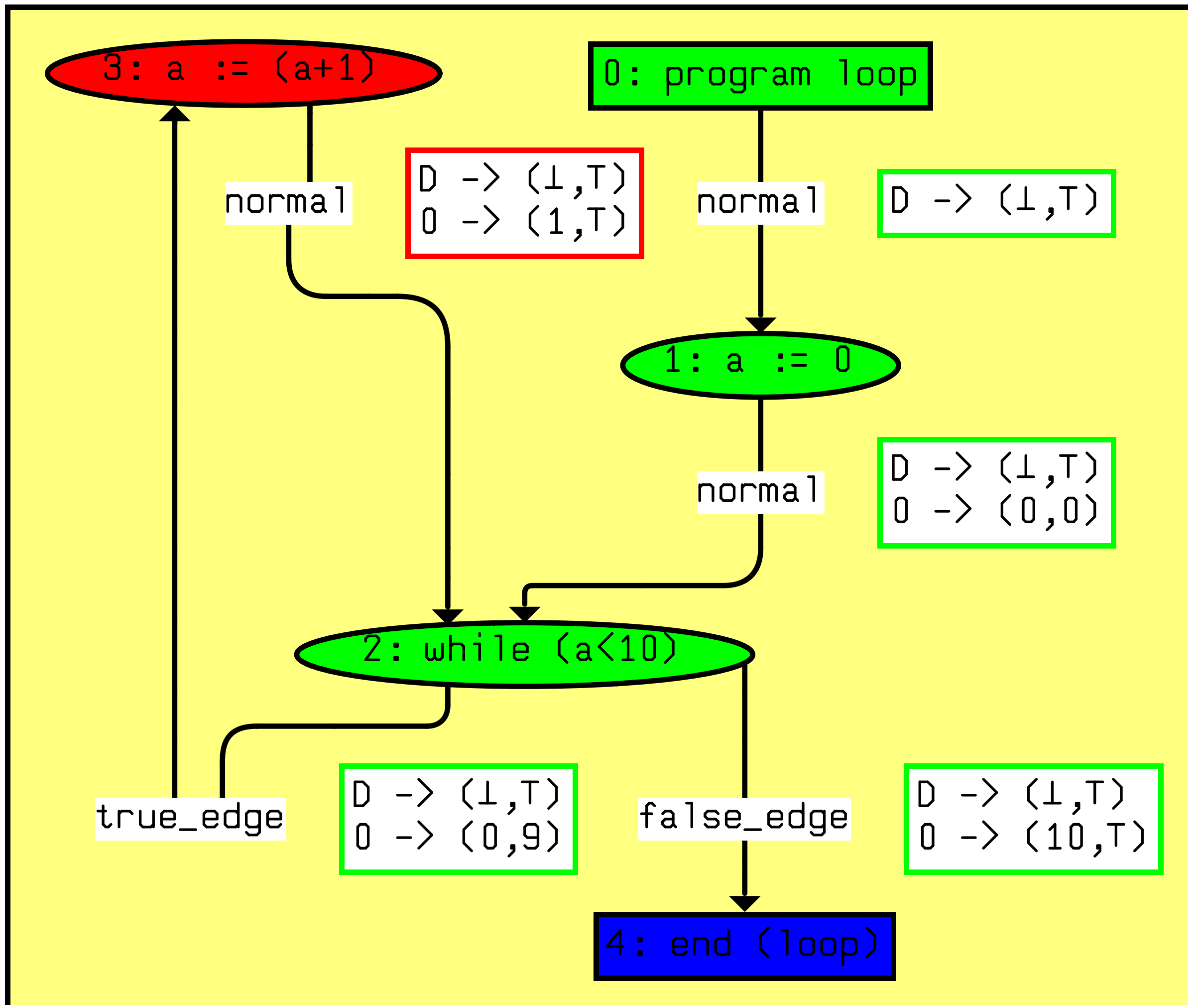

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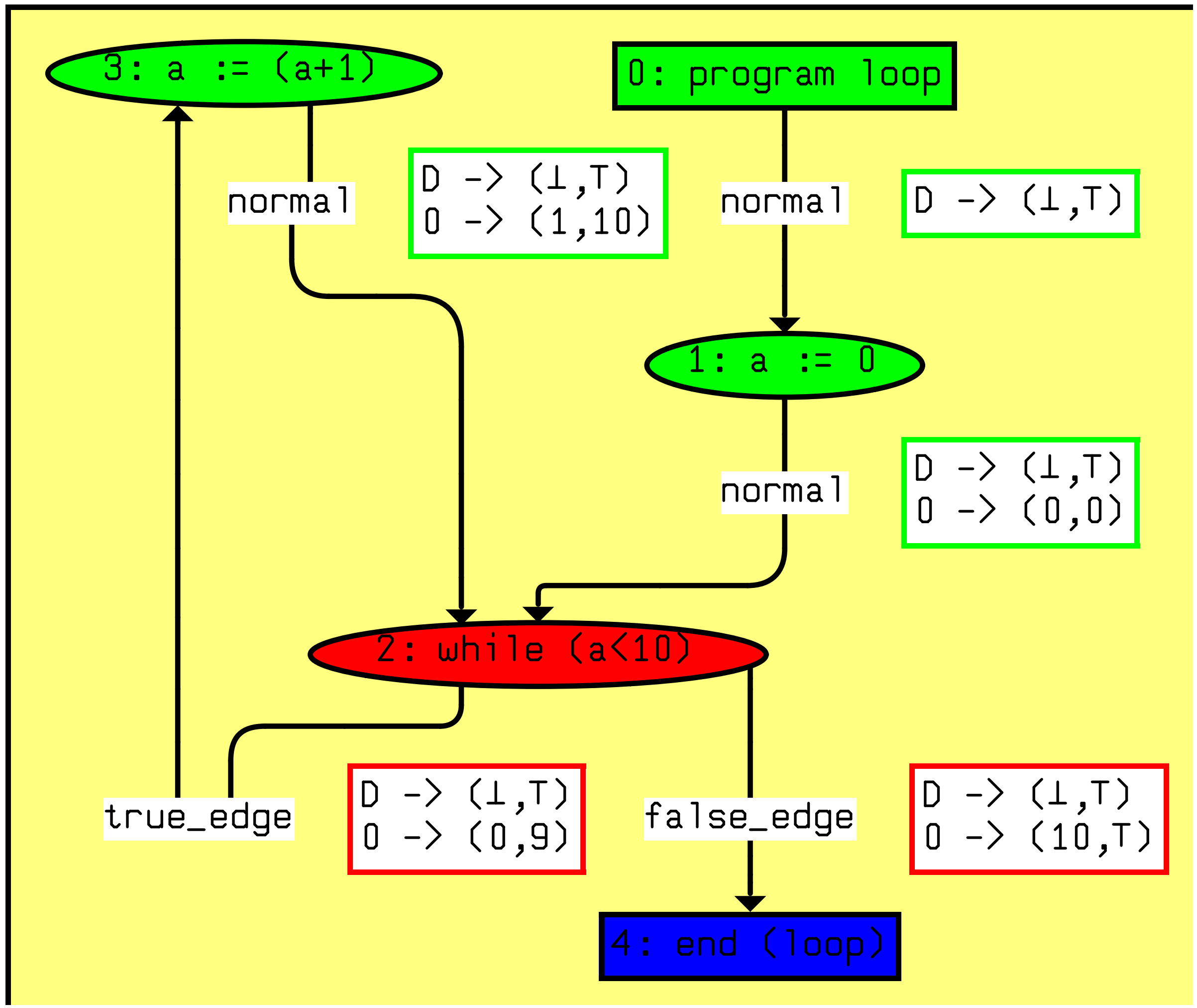

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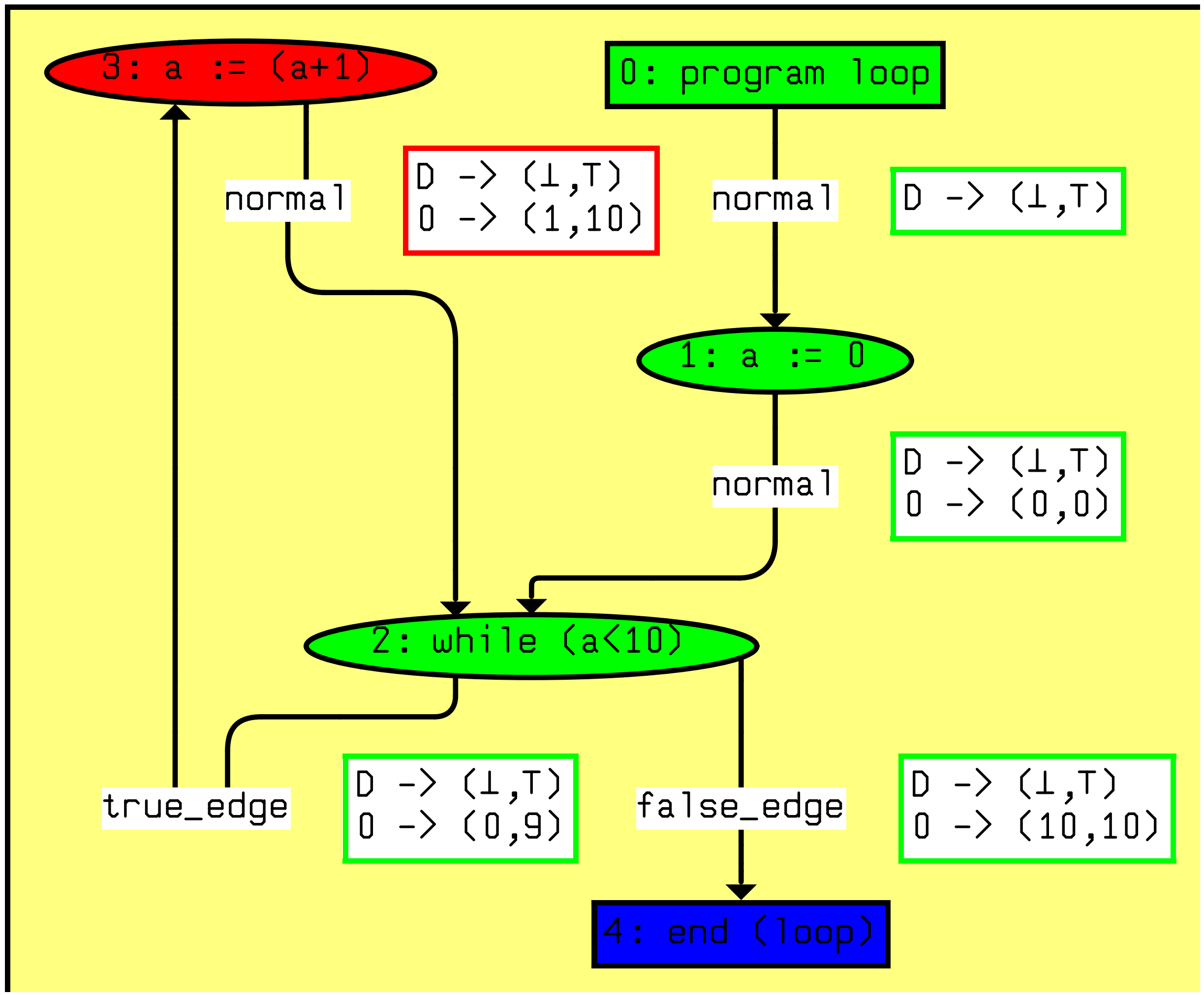

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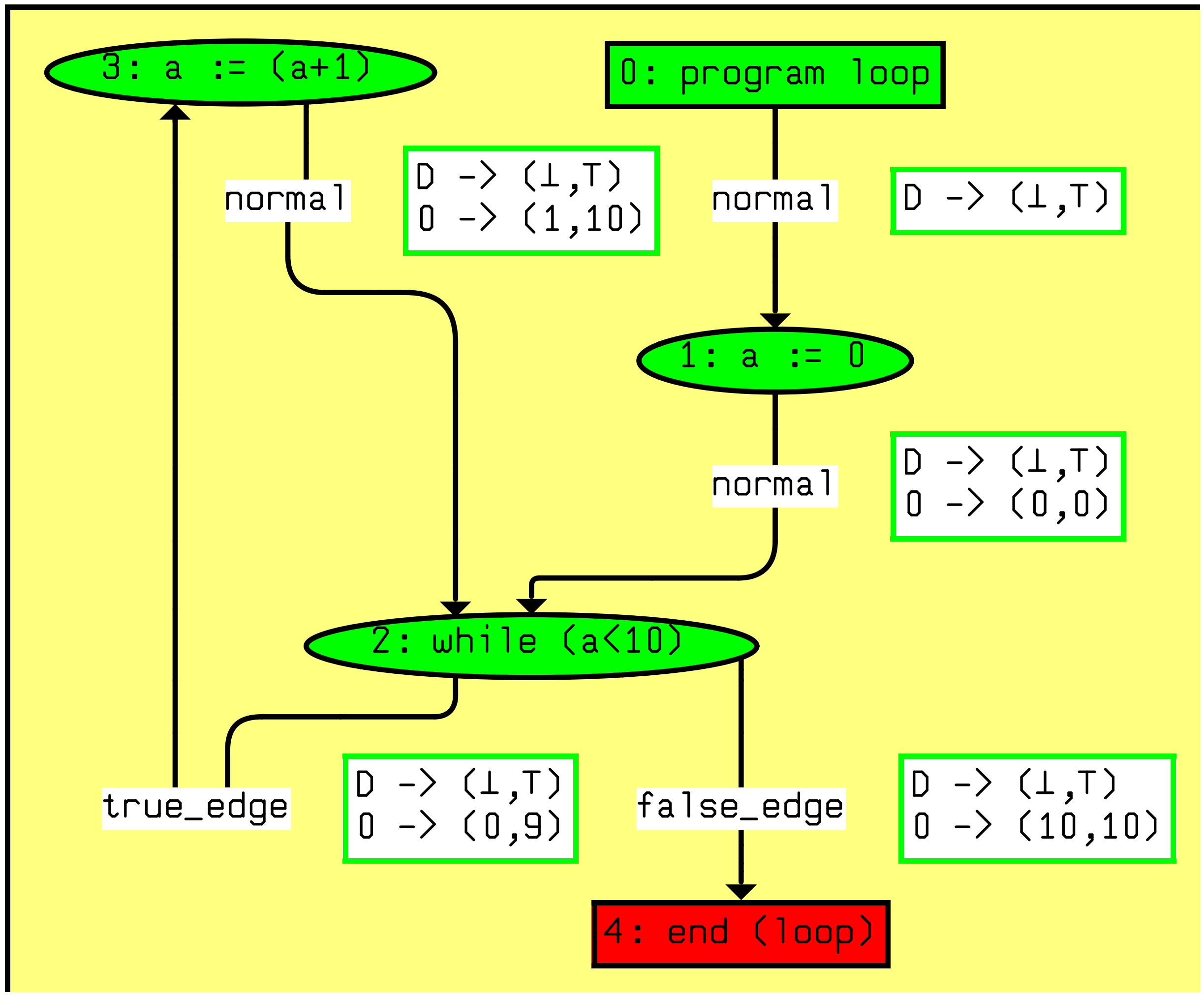

Examples

Questions