Optimal Index Codes with Near-Extreme Rates

Vitaly Skachek

(joint work with **Son Hoang Dau** and **Yeow Meng Chee**)

Estonian Theory Days
October 25th, 2013

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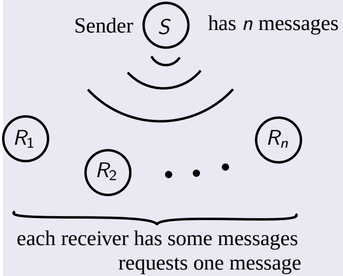
Index Coding with Side Information (ICSI)



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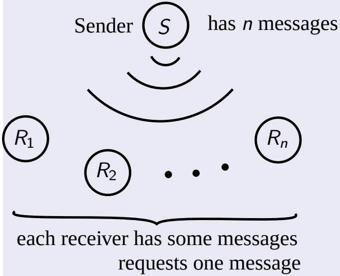

Receiver	Demand	Side Info.
R_1	x_1	$\{x_2\}$
R_2	x_2	$\{x_3\}$
R_3	x_3	$\{x_1, x_4\}$
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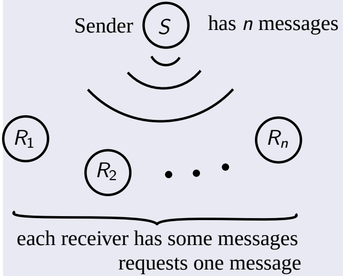

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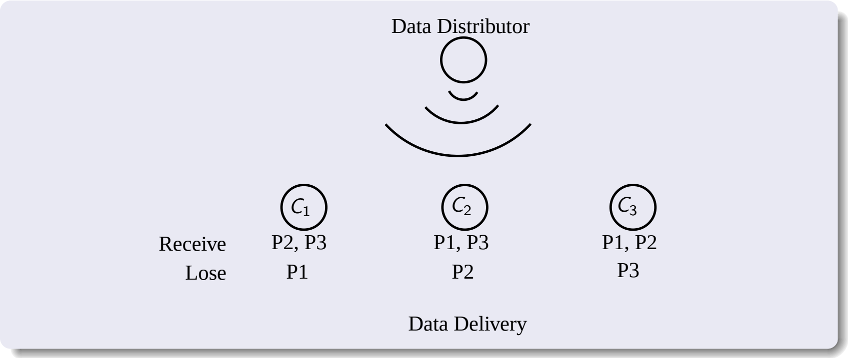
Questions: How can S satisfy all the demands in a minimum number of transmissions?

Motivation



Data Distributor
C_1
C_2
C_3
Receive
Lose
P2, P3
P1
P1, P3
P2
P1, P2
P3
Data Delivery

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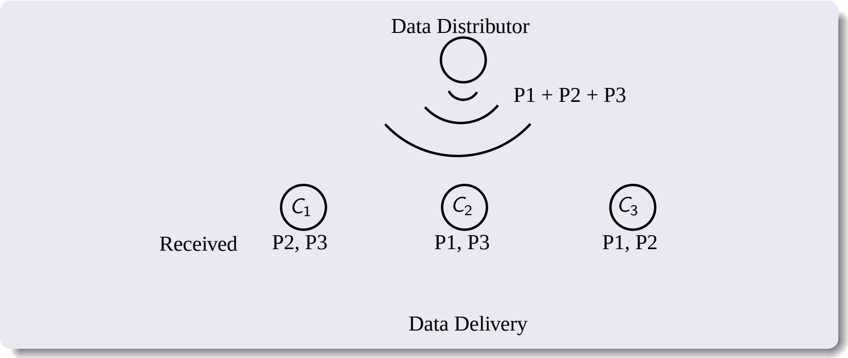
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P1 + P2 + P3
C_1
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Received
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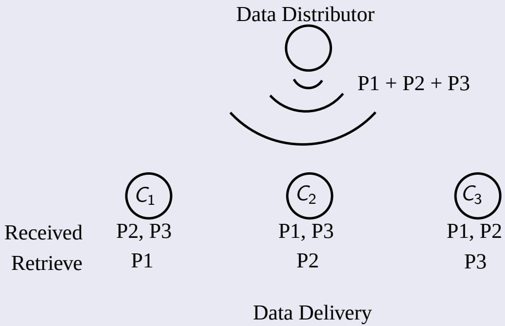

Data Distributor
P1 + P2 + P3
C_1
C_2
C_3
Received
P2, P3
P1, P3
P1, P2
Retrieve
P1
P2
P3
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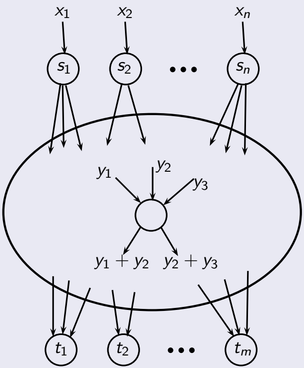
Network Coding

Network Coding (NC): Ahlswede *et al.*, 2000



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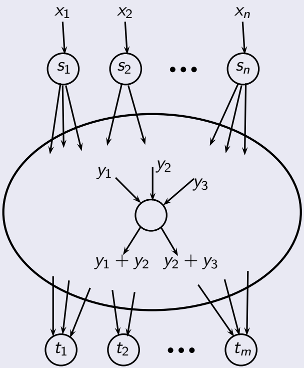

Index Coding and Network Coding

1. Index coding proposed by Birk and Kol (1998)
2. ICSI is a special case of *non-multicast* network coding
3. ICSI and NC are equivalent (El Rouayheb, Sprintson, Georghiades, 2008; Effros, El Rouayheb, Langberg, 2012)

Literature Overview

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- Bar-Yossef *et al.* (2006); Berliner and Langberg (2011): polynomial time computation of minranks for some families of graphs

Other works

- Alon *et al.*(2008)
- Lubetzky, Stav (2009)
- Dau, Skachek, Chee (2011)
- Haviv, Langberg (2012)
- Ong, Lim, Ho (2012)
- Brahma, Fragouli (2012)
- Tehrani, Dimakis (2012)
- Neely, Tehrani, Zhang (2012)
- Shum, Dai, Sung (2012)
- Maleki, Cadambe, Jafar (2012)
- Arbabjolfaei, Bandemer, Kim, Sasoglu (2013)
- Shanmugam, Dimakis, Caire (2013)

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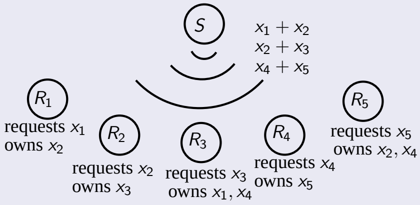
An Example

Consider the following ICSI instance:



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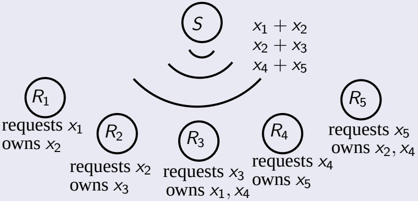

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Definitions and Notation

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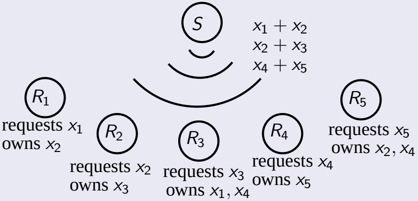

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1. R_1 decodes: $x_1 = x_2 + (x_1 + x_2)$
2. R_2 decodes: $x_2 = x_3 + (x_2 + x_3)$
3. R_3 decodes: $x_3 = x_1 + (x_1 + x_2) + (x_2 + x_3)$
4. R_4 decodes: $x_4 = x_5 + (x_4 + x_5)$
5. R_5 decodes: $x_5 = x_4 + (x_4 + x_5)$

Describe an ICSI Instance via Side Information Digraphs

Definitions and Notation

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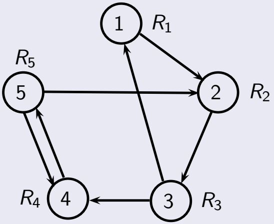

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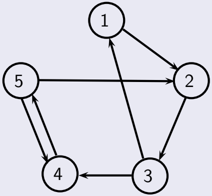

A digraph of minrank 3

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2. The *minrank* of $\mathcal{D}$ over $\mathbb{F}_q$ is defined to be

$$\mathrm{minrk_q}(\mathcal{D}) \triangleq \min \left\{ \mathrm{rank}(\mathbf{M}) :\ \mathbf{M} \in \mathbb{F}_q^{n\times n} \text{ and } \mathbf{M} \text{ fits } \mathcal{D} \right\}.$$

$$\mathbf{A}+\mathbf{I} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \implies \mathbf{M} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & \underline{\mathit{0}} & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & \underline{\mathit{0}} & 0 & 1 & 1 \end{pmatrix}$$

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Graphs and Digraphs of Extreme MinRanks

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Theorem

For any digraph $\mathcal{D}$ we have

$$\alpha(\mathcal{D}) \leq \mathsf{minrk_q}(\mathcal{D}) \leq \mathsf{cc}(D).$$

Graphs and Digraphs of Near-Extreme MinRanks

Summary

Min-Rank	Graph $\mathcal{G}$	Digraph $\mathcal{D}$
1	$\mathcal{G}$ is complete (trivial)	$\mathcal{D}$ is complete (trivial)
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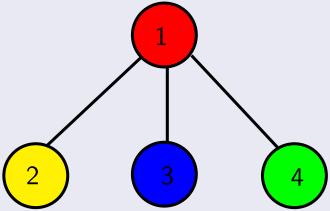
Digraphs of minranks two

$\mathrm{minrk}_2(\mathcal{D}) = 2$ iff $\mathcal{D}$ is not complete and $\overline{\mathcal{D}}$ is fairly 3-colorable



A 4-coloring of a graph

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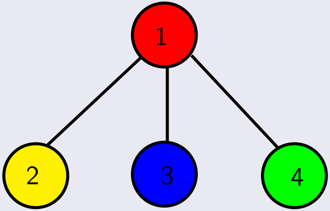

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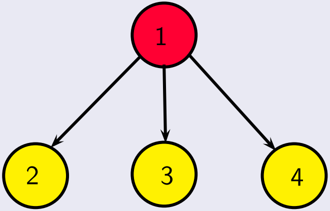
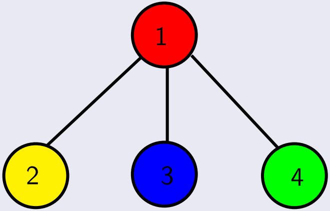


A fair 2-coloring of a digraph

A fair coloring: out-neighbors of each vertex have the same color



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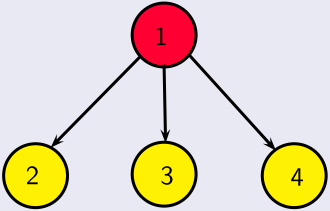

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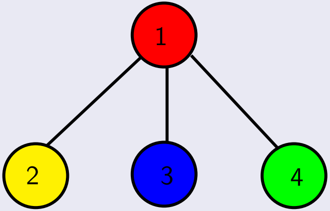

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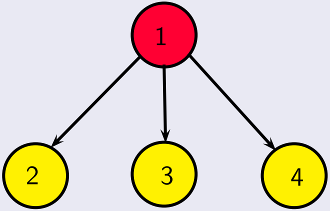

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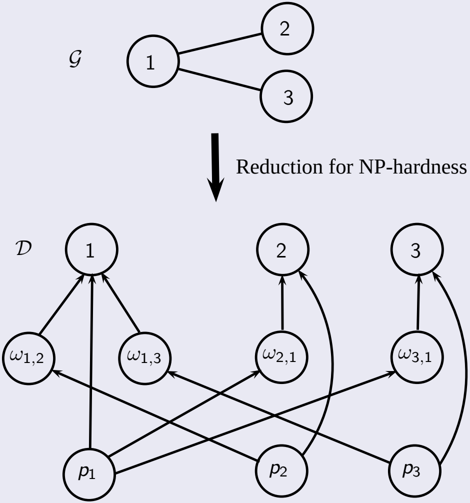
Corollary

Deciding whether $\mathrm{minrk}_2(\mathcal{D}) = 2$ *is an NP-complete problem*

Reduction from k-Coloring to Fair k-Coloring



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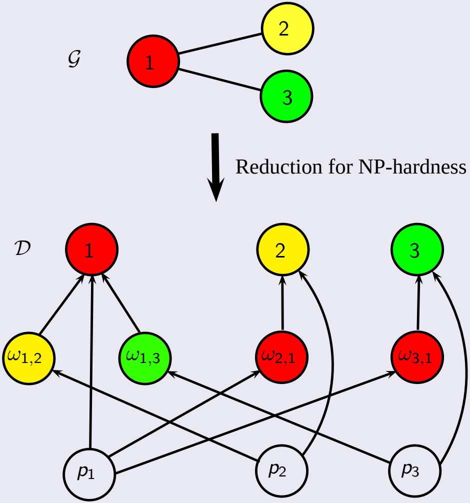

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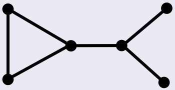
Theorem

Suppose $\mathcal{G}$ is a connected graph of order $n \geq 6$. Then $\mathrm{minrk_q}(\mathcal{G}) = n - 2$ *iff $\mathcal{G}$ has a maximum matching of size two and does not contain a subgraph isomorphic to the graph depicted below*



The forbidden subgraph F

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Digraphs of Extreme MinRanks

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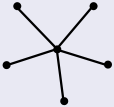

Open Problems

Determine the hardness of the problem of deciding whether $\mathrm{minrk_q}(\mathcal{D}) = 2$ for $q > 2$?

Characterize families of graphs of order n with minrank $n - k$, for a constant $k > 2$