Some recent developments in the theory of átomata

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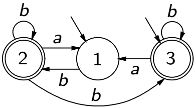
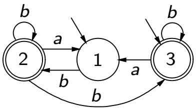
Languages and automata

- We consider regular languages $L \subseteq \Sigma^*$ and nondeterministic finite automata (NFAs).
- An NFA is a quintuple $\mathcal{N} = (Q, \Sigma, \delta, I, F)$, where Q is a finite, non-empty set of states, Σ is a finite non-empty alphabet, $\delta : Q \times \Sigma \rightarrow 2^Q$ is the transition function, $I \subseteq Q$ is the set of initial states, and $F \subseteq Q$ is the set of final states.



Languages and automata

- The language accepted by an NFA $\mathcal{N}$ is $L(\mathcal{N}) = \{w \in \Sigma^* \mid \delta(I, w) \cap F \neq \emptyset\}$.
- Two NFAs are equivalent if they accept the same language.
- The (right) language of a state q of $\mathcal{N}$ is $L_{q,F}(\mathcal{N}) = \{w \in \Sigma^* \mid \delta(q, w) \cap F \neq \emptyset\}$.



For example, $L_{1,\{2,3\}}(\mathcal{N}) = \{b, bb, bab, bbb, \ldots\}$

Quotients and DFA

- The (left) quotient of a language L by a word w is the language $w^{-1}L = \{x \in \Sigma^* \mid wx \in L\}$.
- A deterministic finite automaton (DFA) is $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$, with the transition function $\delta : Q \times \Sigma \to Q$, and the initial state q_0.
- There is a unique minimal DFA for every regular language.
- In the minimal DFA, the right language of any state is a quotient of L.

Atoms

Let $L_0 = L, L_1, \ldots, L_{n-1}$ be the quotients of a regular language L.

An atom of L is any non-empty language of the form

$$A = \widetilde{L_0} \cap \widetilde{L_1} \cap \cdots \cap \widetilde{L_{n-1}},$$

where $\widetilde{L_i}$ is either L_i or $\overline{L_i}$.

- A language has at most 2^n atoms.
- An atom is initial if it has L_0 (rather than $\overline{L_0}$) as a term.
- An atom is final if and only if it contains ε.
- There is exactly one final atom, the atom $\widehat{L_0} \cap \widehat{L_1} \cap \cdots \cap \widehat{L_{n-1}}$, where $\widehat{L_i} = L_i$ if $\varepsilon \in L_i$, $\widehat{L_i} = \overline{L_i}$ otherwise.

Átomaton

We use a one-to-one correspondence $A_i \leftrightarrow \mathbf{A}_i$ between atoms A_i of a language L and the states $\mathbf{A}_i$ of the NFA $\mathcal{A}$ defined below.

Let $L = L_0 \subseteq \Sigma^*$ be any regular language with the set of atoms $Q = \{A_0, \ldots, A_{m-1}\}$, initial set of atoms $I \subseteq Q$, and final atom A_{m-1}.

The átomaton of L is the NFA $\mathcal{A} = (\mathbf{Q}, \Sigma, \delta, \mathbf{I}, \{\mathbf{A}_{m-1}\})$, where

- $\mathbf{Q} = \{\mathbf{A}_i \mid A_i \in Q\}$,
- $\mathbf{I} = \{\mathbf{A}_i \mid A_i \in I\}$,
- $\mathbf{A}_j \in \delta(\mathbf{A}_i, a)$ if and only if $aA_j \subseteq A_i$, for all $A_i, A_j \in Q$.

Some properties of átomaton

- Let $A_0, \ldots, A_{m-1}$ be the atoms and let $\mathcal{A}$ be the átomaton of L.
- The right language of state $\mathbf{A}_i$ of $\mathcal{A}$ is the atom A_i, that is, $L_{\mathbf{A}_i, \{\mathbf{A}_{m-1}\}}(\mathcal{A}) = A_i$, for all $i \in \{0, \ldots, m-1\}$.
- The language accepted by $\mathcal{A}$ is L.
- Let $\mathcal{D}$ be the minimal DFA of L. The átomaton $\mathcal{A}$ is isomorphic to $\mathcal{D}^{RDR}$ (where R - reversal, D - determinization).
- The reverse automaton $\mathcal{A}^R$ of $\mathcal{A}$ is a minimal DFA for the reverse language of L.
- The determinized automaton $\mathcal{A}^D$ of $\mathcal{A}$ is a minimal DFA of L.

Partial atoms

- Let $\mathcal{N} = (Q, \Sigma, \delta, I, F)$ be an NFA accepting L, with state set $Q = \{q_0, \dots, q_{k-1}\}$.
- Let $L_i = L_{q_i,F}(\mathcal{N})$, $i \in \{0, \dots, k-1\}$ be the partial quotients of $\mathcal{N}$.
- A partial quotient L_i is initial if $q_i \in I$, it is final if $q_i \in F$.
- A partial atom of L is any non-empty language of the form

$$\widetilde{L_0} \cap \widetilde{L_1} \cap \dots \cap \widetilde{L_{k-1}},$$

  where $\widetilde{L_i}$ is either L_i or $\overline{L_i}$.
- A partial atom is initial if it has some initial partial quotient L_i as a term.
- A partial atom is final if and only if it contains ε.
- There is exactly one final partial atom, $\widehat{L_0} \cap \widehat{L_1} \cap \dots \cap \widehat{L_{k-1}}$, where $\widehat{L_i} = L_i$ if $\varepsilon \in L_i$, and $\widehat{L_i} = \overline{L_i}$ otherwise.

Partial átomaton

Let the set of partial atoms of $\mathcal{N}$ be $X = \{X_0, \dots, X_{l-1}\}$, the set of initial partial atoms I_X, and the final partial atom X_{h-1}.

We use a one-to-one correspondence $X_i \leftrightarrow \mathbf{X}_i$ between partial atoms X_i and the states $\mathbf{X}_i$ of the NFA $\mathcal{X}$ defined below.

The partial átomaton of $\mathcal{N}$, is the NFA defined by $\mathcal{X} = (\mathbf{X}, \Sigma, \eta, \mathbf{I}_X, \{\mathbf{X}_{h-1}\})$, where $\mathbf{X} = \{\mathbf{X}_i \mid X_i \in X\}$, $\mathbf{I}_X = \{\mathbf{X}_i \mid X_i \in I_X\}$, and $\mathbf{X}_j \in \eta(\mathbf{X}_i, a)$ if and only if $aX_j \subseteq X_i$, for all $\mathbf{X}_i, \mathbf{X}_j \in \mathbf{X}$ and $a \in \Sigma$.

Properties of partial atoms

- Partial atoms are pairwise disjoint, that is, $X_i \cap X_j = \emptyset$ for all $i, j \in \{0, \ldots, l-1\}$, $i \neq j$.
- Any quotient $w^{-1}L$ of L by $w \in \Sigma^*$ is a (possibly empty) union of partial atoms.
- Any quotient $w^{-1}X_i$ of a partial atom X_i by $w \in \Sigma^*$ is a (possibly empty) union of partial atoms.
- Partial atoms define a partition of Σ^*.

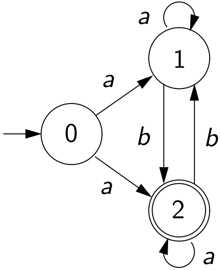
Some properties of partial átomaton

Let $X = \{X_0, \ldots, X_{l-1}\}$ be the set of partial atoms, and let $\mathcal{X}$ be the partial átomaton of $\mathcal{N}$.

- The right language of state $\mathbf{X}_i$ of $\mathcal{X}$ is partial atom X_i.
- The language accepted by $\mathcal{X}$ is L.
- Partial átomaton $\mathcal{X}$ of $\mathcal{N}$ is isomorphic to $\mathcal{N}^{RDR}$.

Proposition Let $\varphi : \mathbf{X} \to S$ be the mapping assigning to state $\mathbf{X}_i$, given by $X_i = L_{i_0} \cap \cdots \cap L_{i_{g-1}} \cap \overline{L_{i_g}} \cap \cdots \cap \overline{L_{i_{k-1}}}$ of $\mathcal{X}$, the set $\{q_{i_0}, \ldots, q_{i_{g-1}}\}$. Then φ is an NFA isomorphism between $\mathcal{X}$ and $\mathcal{N}^{RDR}$.

Example: constructing partial atoms



$$
\begin{aligned}
L_0 &= a(L_1 \cup L_2) \\
L_1 &= aL_1 \cup bL_2 \\
L_2 &= aL_2 \cup bL_1 \cup \varepsilon
\end{aligned}
$$

Since $L_2 = \overline{L_1}$, $L_1 \cap L_2 = \emptyset$ and $\overline{L_1} \cap \overline{L_2} = \emptyset$.

Also, $L_1 \cap \overline{L_2} = L_1$ and $\overline{L_1} \cap L_2 = L_2$.

Partial atoms are the non-empty intersections:

$$
\begin{aligned}
X_0 &= L_0 \cap L_1 \cap \overline{L_2} = aL_1 = a(L_0 \cap L_1 \cap \overline{L_2}) \cup a(\overline{L_0} \cap L_1 \cap \overline{L_2}), \\
X_1 &= L_0 \cap \overline{L_1} \cap L_2 = aL_2 = a(L_0 \cap \overline{L_1} \cap L_2) \cup a(\overline{L_0} \cap \overline{L_1} \cap L_2), \\
X_2 &= \overline{L_0} \cap L_1 \cap \overline{L_2} = bL_2 = b(L_0 \cap \overline{L_1} \cap L_2) \cup b(\overline{L_0} \cap \overline{L_1} \cap L_2), \\
X_3 &= \overline{L_0} \cap \overline{L_1} \cap L_2 = bL_1 \cup \varepsilon = b(L_0 \cap L_1 \cap \overline{L_2}) \cup b(\overline{L_0} \cap L_1 \cap \overline{L_2}) \cup \varepsilon.
\end{aligned}
$$

Example: constructing partial átomaton

$$
\begin{aligned}
X_0 &= L_0 \cap L_1 \cap \overline{L_2} = aL_1 = a(L_0 \cap L_1 \cap \overline{L_2}) \cup a(\overline{L_0} \cap L_1 \cap \overline{L_2}),\\
X_1 &= L_0 \cap \overline{L_1} \cap L_2 = aL_2 = a(L_0 \cap \overline{L_1} \cap L_2) \cup a(\overline{L_0} \cap \overline{L_1} \cap L_2),\\
X_2 &= \overline{L_0} \cap L_1 \cap \overline{L_2} = bL_2 = b(L_0 \cap \overline{L_1} \cap L_2) \cup b(\overline{L_0} \cap \overline{L_1} \cap L_2),\\
X_3 &= \overline{L_0} \cap \overline{L_1} \cap L_2 = bL_1 \cup \varepsilon = b(L_0 \cap L_1 \cap \overline{L_2}) \cup b(\overline{L_0} \cap L_1 \cap \overline{L_2}) \cup \varepsilon.
\end{aligned}
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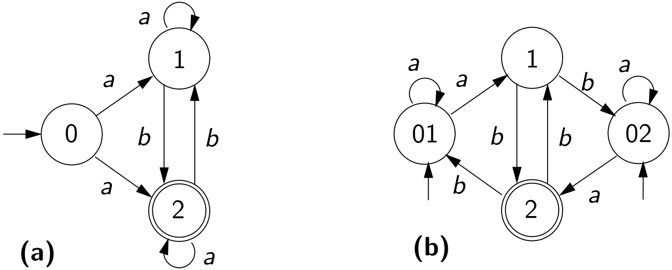

(a) An NFA $\mathcal{N}$; (b) partial átomaton $\mathcal{X}$ of $\mathcal{N}$.

Atoms and partial atoms

Let $\mathcal{N}$ be an NFA accepting L, with partial atoms $X = \{X_0, \ldots, X_{l-1}\}$, and partial átomaton $\mathcal{X}$.

Let the set of atoms of L be $A = \{A_0, \ldots, A_{m-1}\}$.

- For every X_i there exists some atom A_j, such that $X_i \subseteq A_j$.
- Every atom A_j is a disjoint union of some X_i's.
- The partial átomaton $\mathcal{X}$ of $\mathcal{N}$ is the átomaton of L if and only if X is the set of atoms A.

Atomicity of states

We call a state q_i of an NFA $\mathcal{N}$ atomic if its right language L_i is a union of atoms of L.

Consider the DFA $\mathcal{N}^{RD}$ (not necessarily minimal).
Let $S_0, \ldots, S_{r-1}$ be the sets of equivalent states of $\mathcal{N}^{RD}$.
We use the equivalence classes $S_0, \ldots, S_{r-1}$ to detect which states of $\mathcal{N}$ are atomic.

Theorem A state q_i of an NFA $\mathcal{N}$ is atomic if and only if the subset $S_i' = \{s_j \in S \mid q_i \in s_j\}$ of states of $\mathcal{N}^{RD}$ is a union of some equivalence classes of $\mathcal{N}^{RD}$.

Atomic NFAs

We define an NFA $\mathcal{N}$ to be atomic if all states of $\mathcal{N}$ are atomic.

Corollary An NFA $\mathcal{N}$ is atomic if and only if $\mathcal{N}^{RD}$ is minimal.

Corollary An NFA $\mathcal{N}$ is atomic if and only if the partial atoms of $\mathcal{N}$ are the atoms of L.

Generalization of Brzozowski's Theorem

Theorem (Brzozowski, 1962). If an NFA $\mathcal{N}$ has no empty states and $\mathcal{N}^R$ is deterministic, then $\mathcal{N}^D$ is minimal.

Our generalization:

Theorem (2011) For any NFA $\mathcal{N}$, $\mathcal{N}^D$ is minimal if and only if $\mathcal{N}^R$ is atomic.