# An Elementary Proof of a $3n - o(n)$ Lower Bound on Circuit Complexity of Affine Dispersers

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Estonian Theory Days
08 October 2011

# Boolean Circuits

Inputs:

$x_1, x_2, \ldots, x_n, 0, 1$

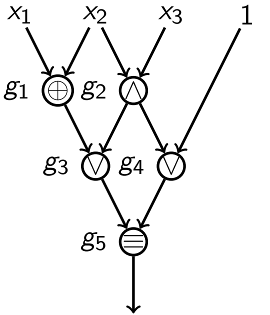
Gates:

binary functions

Fan-out:

unbounded

$$
\begin{aligned}
g_1 &= x_1 \oplus x_2 \\
g_2 &= x_2 \wedge x_3 \\
g_3 &= g_1 \vee g_2 \\
g_4 &= g_2 \vee 1 \\
g_5 &= g_3 \equiv g4
\end{aligned}
$$



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- For $t = 2^n/(10n)$, $F(n,t)$ is approximately $2^{2^n/5}$, which is $\ll 2^{2^n}$.
- Thus, the circuit complexity of almost all Boolean functions on $n$ variables is exponential in $n$. Still, we do not know any explicit function with super-linear circuit complexity.

# Known Lower Bounds

| | circuit size | formula size |
|---|---|---|
| full binary basis $B_2$ | $3n - o(n)$ [Blum] | $n^{2-o(1)}$ [Nechiporuk] |
| basis $U_2 = B_2 \setminus \{\oplus, \equiv\}$ | $5n - o(n)$ [Iwama et al.] | $n^{3-o(1)}$ [Hastad] |
| monotone basis $M_2 = \{\vee, \wedge\}$ | exponential [Razborov; Alon, Boppana; Andreev; Karchmer, Wigderson] | |

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| $2n - c$ | [Kloss and Malyshev, 65] |
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## This Talk

In this talk, we will present a new proof of a $3n - o(n)$ lower. The proof is much simpler than Blum's proof, however the function used is much more complicated.

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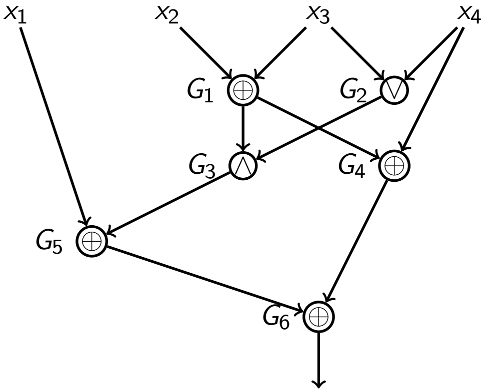
## Remark

This method is very unlikely to produce non-linear lower bounds.

# Example



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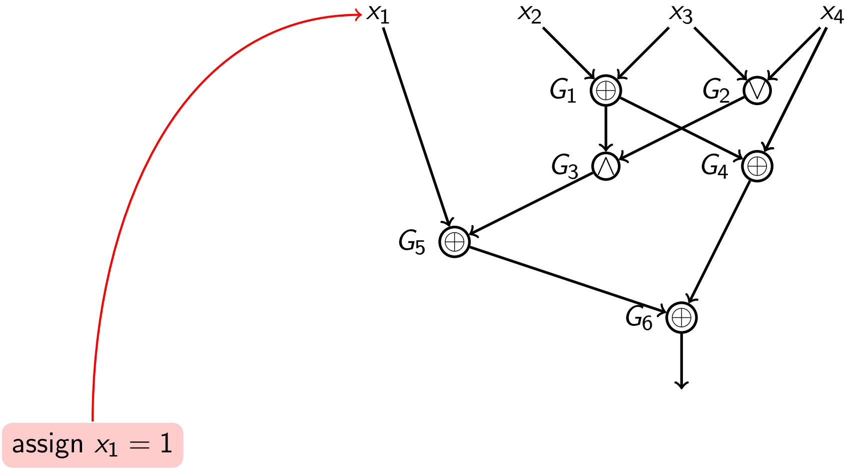

$x_1$
$x_2$
$x_3$
$x_4$
$G_1$ $\oplus$
$G_2$ $\vee$
$G_3$ $\wedge$
$G_4$ $\oplus$
$G_5$ $\oplus$
$G_6$ $\oplus$
assign $x_1 = 1$

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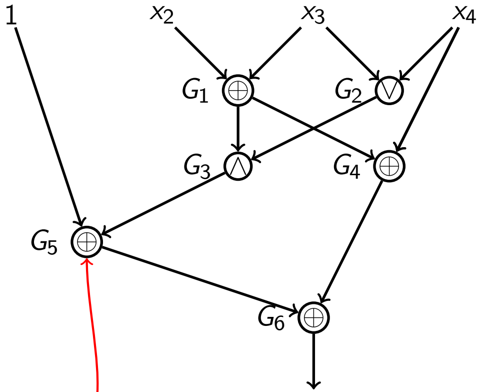


$G_5$ now computes $G_3 \oplus 1 = \neg G_3$

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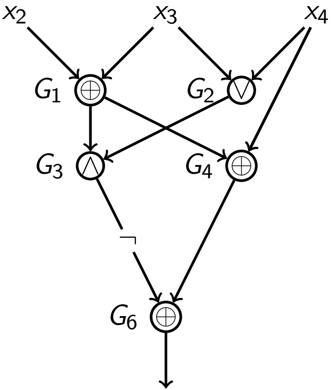

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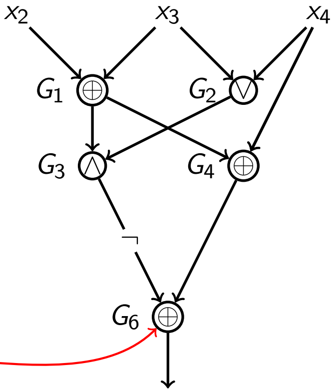


now we can change the binary function assigned to $G_6$

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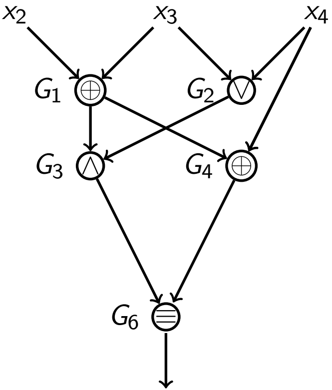

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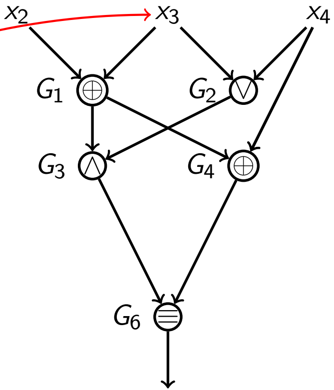


now assign $x_3 = 0$

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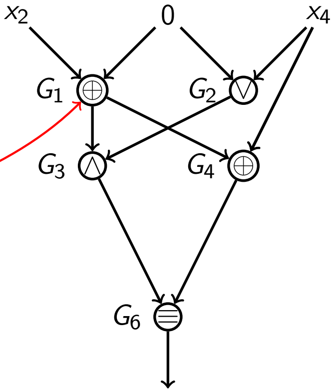


$G_1$ then is equal to $x_2$

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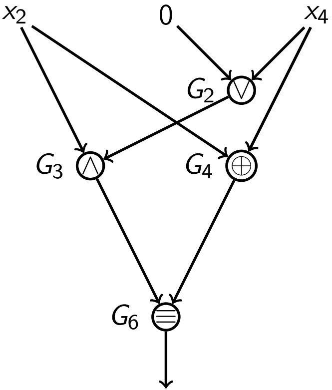

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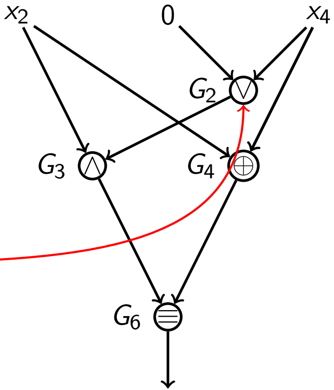


$G_2 = x_4$

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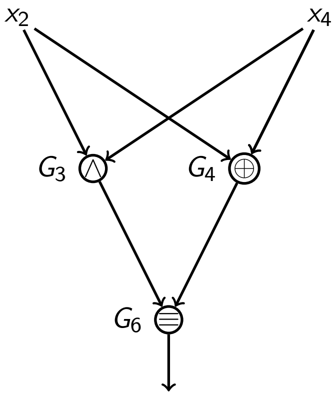
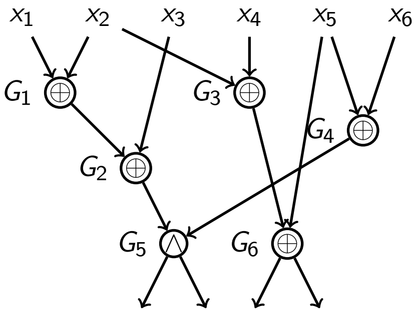


# A Typical Bottleneck



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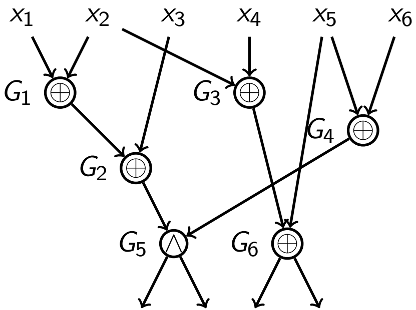

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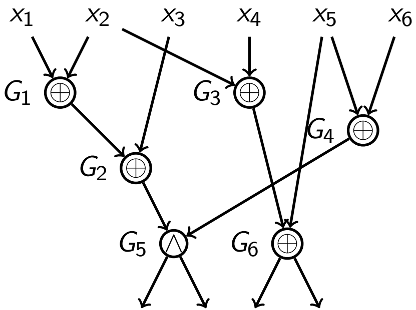

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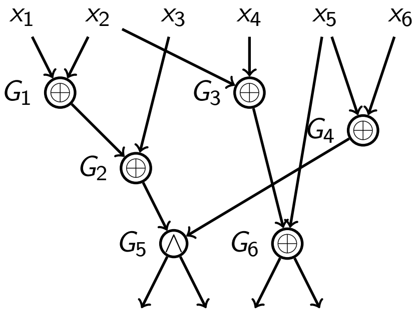

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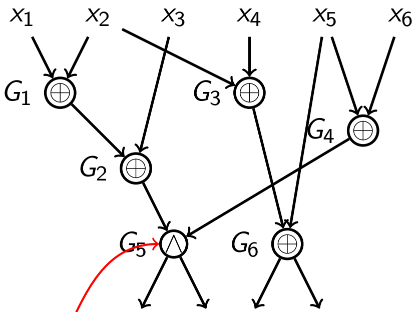
consider a substitution $x_1 \oplus x_2 \oplus x_3 = 0$: under it $G_5$ trivializes



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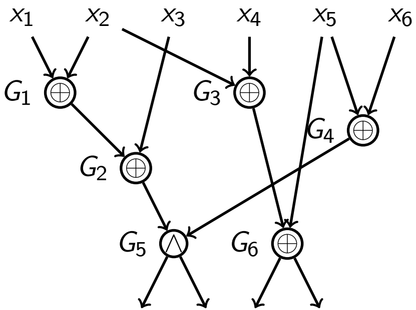
- Only recently, an explicit affine disperser for $d = o(n)$ was constructed [Ben-Sasson and Kopparty, 09].

# Proof Idea



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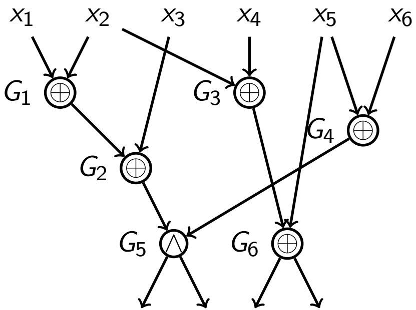
take the first gate which is not a XOR of out-degree 1



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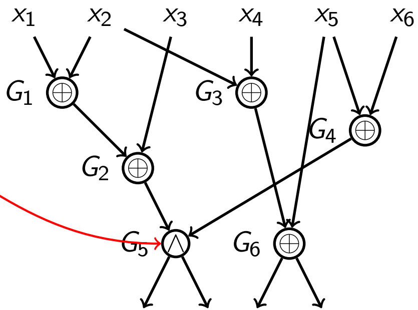

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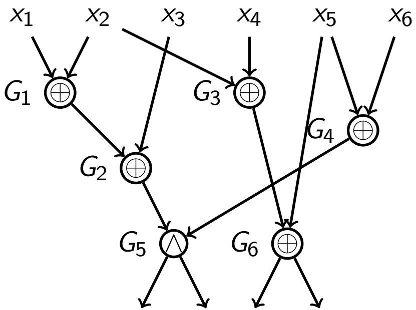

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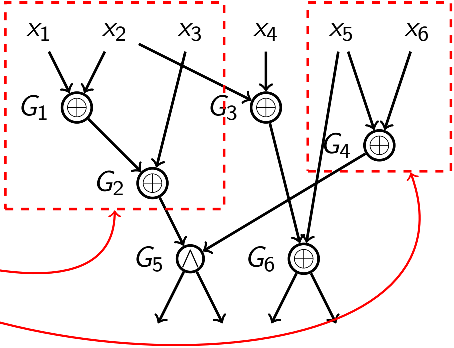

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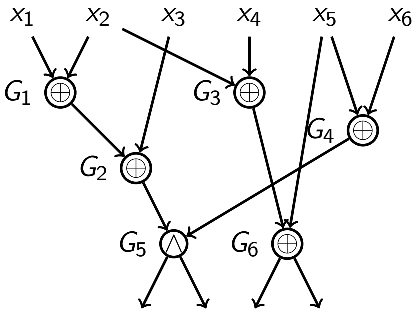

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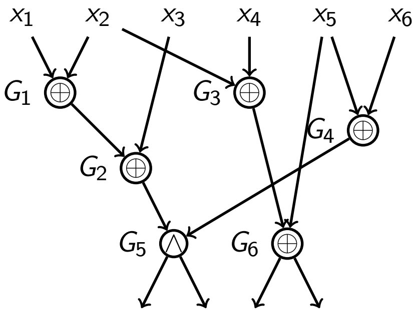

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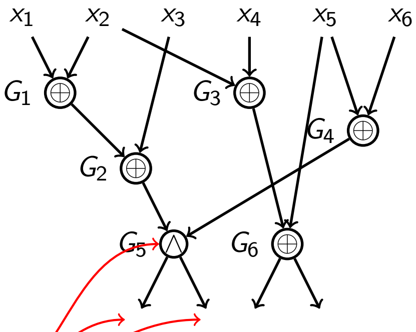

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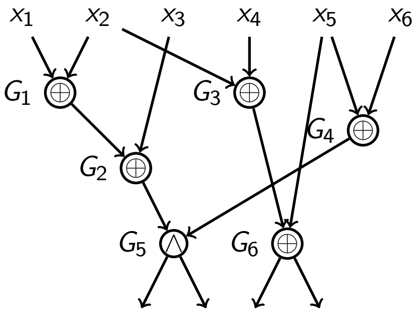

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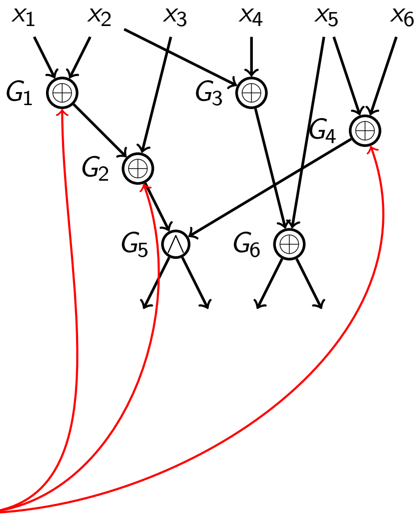

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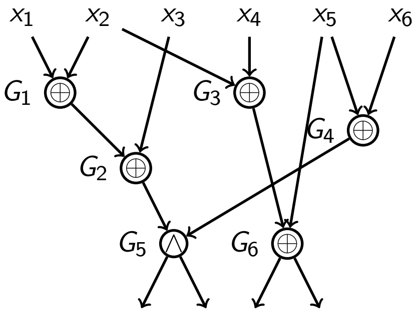
by a short case analysis it is possible to show that this way one can always eliminate 3 gates; since we can make $n - o(n)$ such substitutions a lower bound $3n - o(n)$ follows



# Thank you for your attention!