Bideterministic Automata and Minimal Representations of Regular Languages

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Literature:

- Tamm, H., and Ukkonen, E. Bideterministic automata and minimal representations of regular languages. In *Proc. of CIAA 2003*, LNCS 2759, Springer, 2003, 61-71.
- Tamm, H., and Ukkonen, E. Bideterministic automata and minimal representations of regular languages. To appear in *Theoretical Computer Science*.

Finite automata

A finite automaton $A = (Q, \Sigma, \delta, I, F)$ where Q is a finite set of *states*, Σ is the *input alphabet*, $\delta : Q \times \Sigma \to 2^Q$ is the *transition function*, $I \subseteq Q$ is the set of *initial states* and $F \subseteq Q$ is the set of *final states*.

An automaton A is *deterministic* (DFA) if it has a unique initial state and if for every $q \in Q$ and every $a \in \Sigma$, $|\delta(q, a)| \leq 1$.

The general case of automata is *nondeterministic* (NFA).

The *reversal* of an automaton A is the automaton $A^R = (Q, \Sigma, \delta^R, F, I)$ where $\delta^R(p, a) = \{q \mid p \in \delta(q, a)\}$ for all $p \in Q$ and $a \in \Sigma$.

Minimal automata

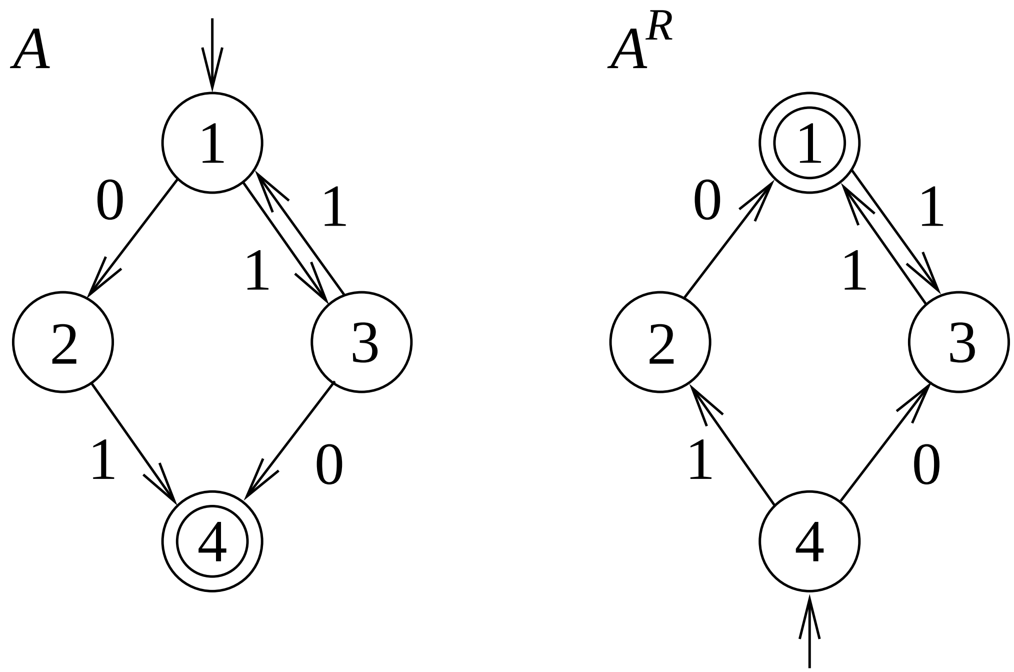
- Minimality with respect to the number of states
- Minimization of DFA is efficient, the result is unique
- Minimization of NFA is hard, the result is not necessarily unique
- Other methods to obtain small NFAs
- Sufficient conditions for minimality among NFAs?

Our results

- *bideterministic* automata are minimal among NFA
- sufficient conditions for a minimal DFA or the reversed automaton of the minimal DFA of the reversal language to be minimal NFA

Bideterministic automata

An automaton A is *bideterministic* if both A and its reversed automaton A^R are deterministic.



Bideterministic automata

- Known: If A is bideterministic then A is a minimal DFA. (Easy to show by Brzozowski's DFA minimization algorithm $min(A) = D((D(A^R))^R)$ where D is *determinizing by subset construction*)

- New result: If A is bideterministic then A is uniquely minimal among all automata accepting the same language.

NFA minimization of Kameda and Weiner

Kameda, T., and Weiner, P. On the state minimization of nondeterministic automata. *IEEE Trans. Comput.* **C-19**, 7 (1970), 617-627.

- Let $A = (Q, \Sigma, \delta, I, F)$ be an automaton,
  $B = D(A) = (Q', \Sigma, \delta', \{q'\}, F')$ and
  $C = D(A^R) = (Q'', \Sigma, \delta'', \{q''\}, F'')$.

  (The elements of Q' and Q'' are subsets of Q.)

- The *states map (SM)* of A is a matrix which contains a row for each nonempty state of B, and a column for each nonempty state of C. The (i, j) entry contains $q'_i \cap q''_j$ (or is blank if $q'_i \cap q''_j = \emptyset$), where $q'_i \in Q', q''_j \in Q''$.

NFA minimization of Kameda and Weiner (cont.)

- Two states of B (C) having the same pattern of blank entries in the corresponding rows (columns) of the SM of A can be merged. These rows (columns) are called *equivalent*.
- The *reduced states map (RSM)* is obtained from the SM by merging all equivalent rows and columns (by union of corresponding entries).
- The *reduced automaton matrix (RAM)* is formed from the RSM by replacing each nonblank entry with a 1.

Theorem. [Kameda and Weiner] Equivalent automata have a RAM that is unique up to permutation of the rows and columns.

NFA minimization of Kameda and Weiner (cont.)

- In a RAM, if all the entries at the intersections of a set of rows and columns are 1's then this set of 1's forms a *grid*.
- A set of grids forms a *cover* if every 1 in the RAM belongs to at least one grid in the set.

Theorem. [Kameda and Weiner] The states of A appear as a cover of the RAM of A.

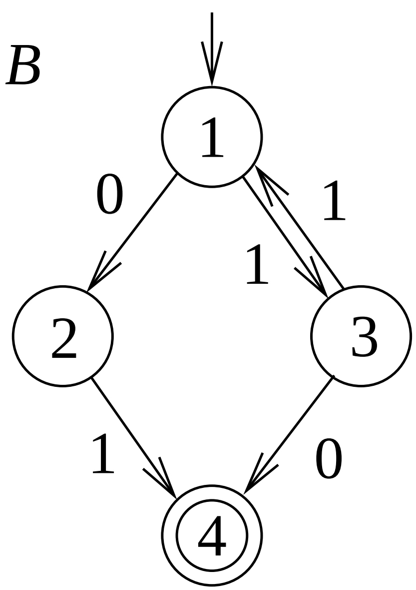
NFA minimization:

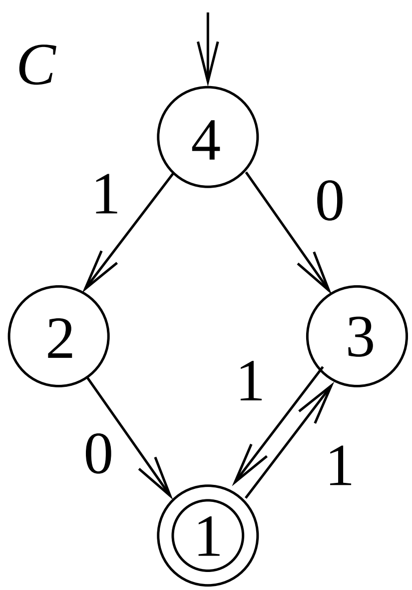
- By a special rule, an NFA is associated with any cover of the RAM
- This NFA may be not equivalent to the original automaton
- To find a minimal automaton, the covers of RAM are tested in increasing order of their sizes

SM of a bideterministic automaton

Let $A = (Q, \Sigma, \delta, \{q_0\}, \{q_f\})$ be a bideterministic automaton.
Then $B = D(A) = A$ and $C = D(A^R) = A^R$.
Let $Q = \{q_0, ..., q_{n-1}\}$.

- The SM of A consists of n rows and n columns, with n non-blank entries $\{q_0\}, ..., \{q_{n-1}\}$, exactly one such entry in every row and every column.





B \ C	{4}	{2}	{3}	{1}
{1}				{1}
{2}		{2}		
{3}			{3}	
{4}	{4}			

RSM and RAM of A

- The RSM of A is equal to SM of A.
- The RAM of A is formed from the RSM by replacing each nonblank entry with a 1.

$B \setminus C$	$\{4\}$	$\{2\}$	$\{3\}$	$\{1\}$
$\{1\}$				$\{1\}$
$\{2\}$		$\{2\}$		
$\{3\}$			$\{3\}$	
$\{4\}$	$\{4\}$			

$B \setminus C$	$\{4\}$	$\{2\}$	$\{3\}$	$\{1\}$
$\{1\}$				1
$\{2\}$		1		
$\{3\}$			1	
$\{4\}$	1			

Bideterministic automata are minimal

There are n grids in the RAM, each grid consisting of single 1; this set of grids is the only cover of the RAM.

$B \backslash C$	{4}	{2}	{3}	{1}
{1}				1
{2}		1		
{3}			1	
{4}	1			

Any automaton accepting $L(A)$ has at least as many states as is the number of grids in the minimum cover of RAM, therefore A is minimal. □

Bideterministic automata are uniquely minimal

- A bideterministic automaton is a minimal DFA which is unique.
- Are there any non-deterministic automata of the same size accepting the same language? No!
- Bideterministic automata are uniquely minimal.

Other sufficient conditions for minimality?

- For a language accepted by a bideterministic automaton the size of the minimal DFA is the smallest size of any automaton accepting that language.
- Other conditions that imply similar minimalities?

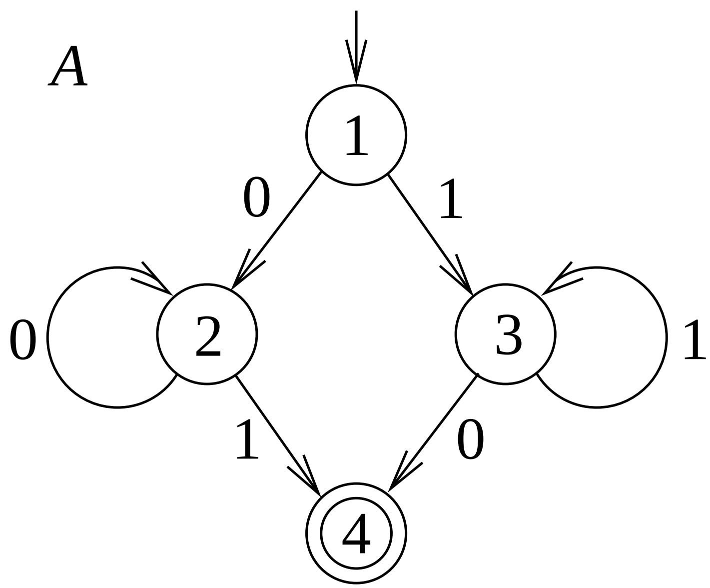
Partitions of automata state sets

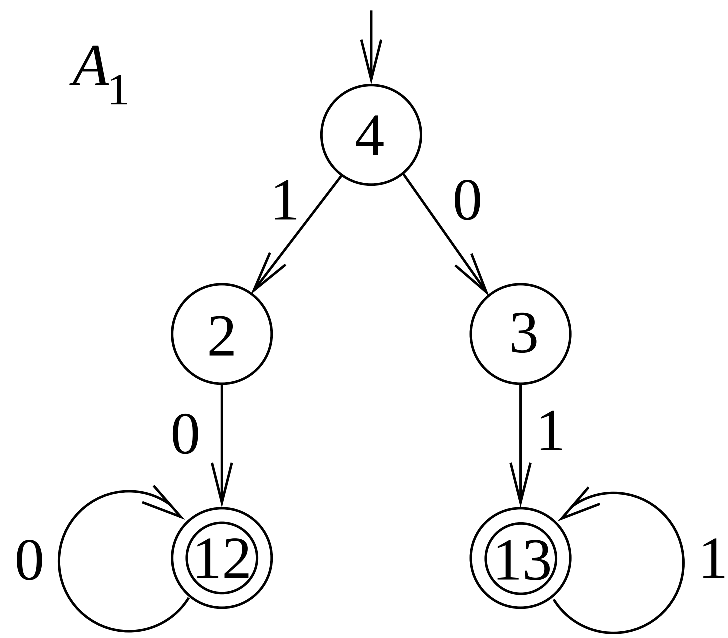
Let A be a minimal DFA and let $A_1 = D(A^R)$.
Then A_1 is the minimal DFA accepting $L(A^R)$ (by Brzozowski's minimization, as $A_1 = D(A^R) = D(D(A)^R) = D(D((A^R)^R)^R))$.

Let Q and Q'' be the state sets of A and A_1, resp.

Consider a partition $\{Q''_1, ..., Q''_k\}$ of Q'' such that any pair of states q''_1 and q''_2 of A_1 belongs to the same Q''_i, $i \in \{1, ..., k\}$, if and only if there exist states $q''_{i_1}, ..., q''_{i_l}$ of A_1 such that $q''_{i_1} = q''_1$, $q''_{i_l} = q''_2$ and $q''_{i_j} \cap q''_{i_{j+1}} \neq \emptyset$ for all $j = 1, ..., l-1$.

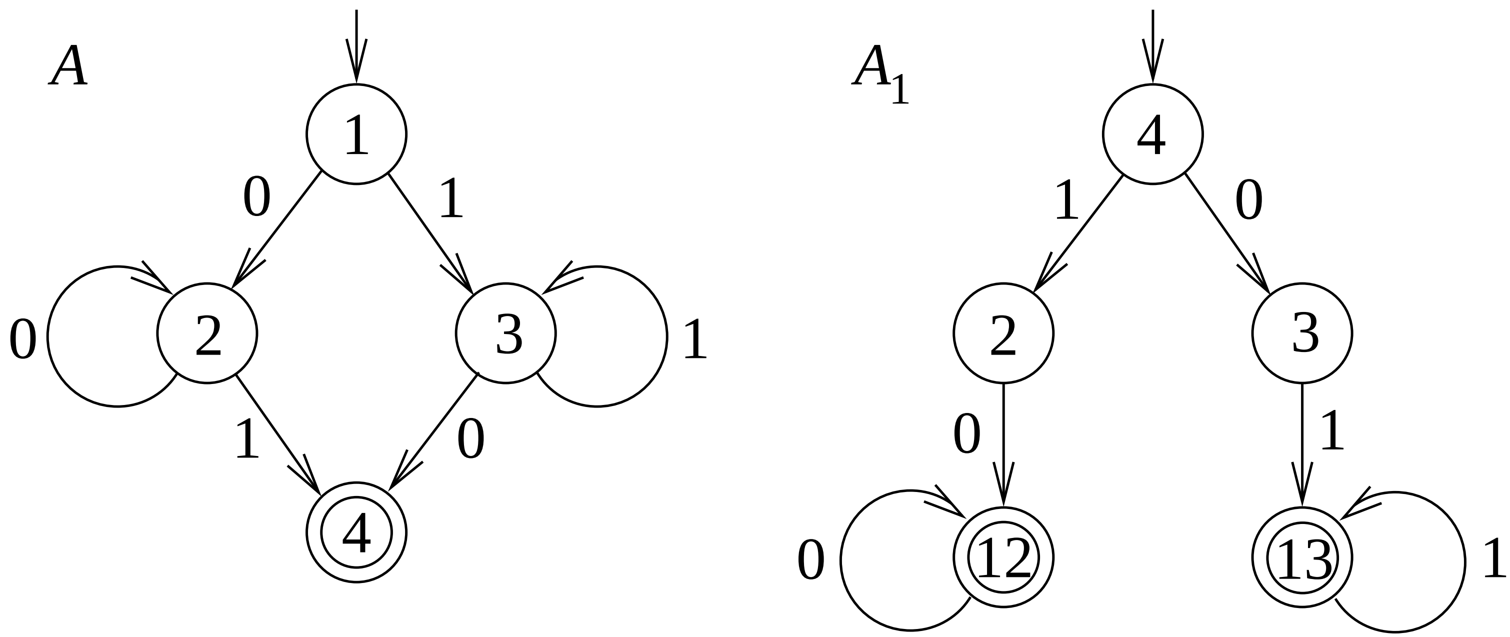
Example: a minimal DFA A and $A_1 = D(A^R)$





$Q''_1 = \{\{1,2\},\{1,3\},\{2\},\{3\}\}$
$Q''_2 = \{\{4\}\}$

Let $Q_i = \bigcup_{q''_j \in Q''_i} q''_j$ for $i = 1, ..., k$, then $\{Q_1, ..., Q_k\}$ forms a partition of Q.



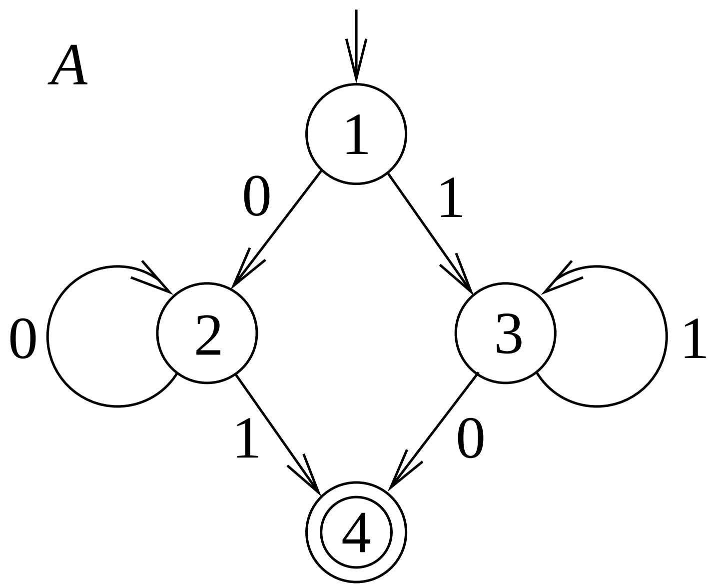
$Q''_1 = \{\{1, 2\}, \{1, 3\}, \{2\}, \{3\}\}$
$Q''_2 = \{\{4\}\}$

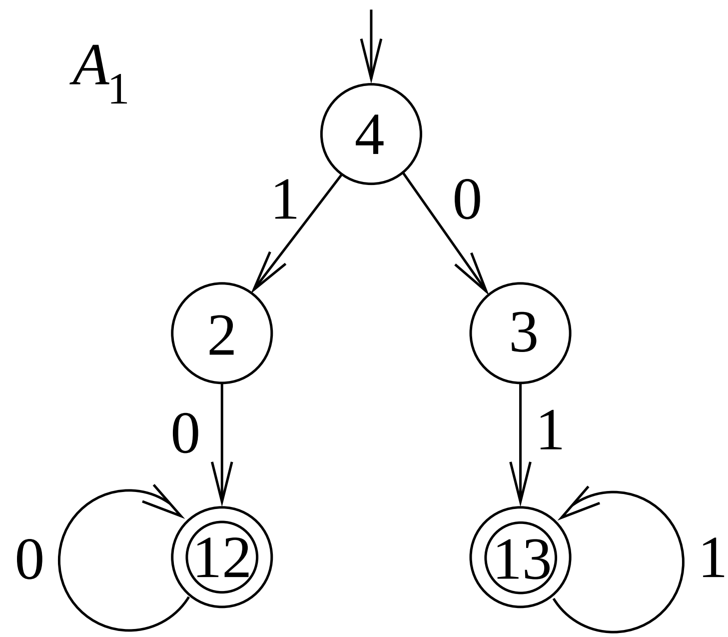
$Q_1 = \{1, 2, 3\}$
$Q_2 = \{4\}$

Another minimality result

Theorem. Let A be a minimal DFA and let $A_1 = D(A^R)$ such that either (i) every state of A_1 consists of at most two states of A, or (ii) each state of A occurs in at most two states of A_1.
Let $\{Q_1, ..., Q_k\}$ and $\{Q''_1, ..., Q''_k\}$ be the partitions of the states of A and A_1, respectively, as described above.
If $|Q_i| \leq |Q''_i|$ for all $i = 1, ..., k$, then A is a minimal automaton accepting $L(A)$.
If $|Q''_i| \leq |Q_i|$ for all $i = 1, ..., k$, then A_1^R is a minimal automaton accepting $L(A)$.

Example: a minimal DFA A and $A_1 = D(A^R)$





$Q_1'' = \{\{1,2\},\{1,3\},\{2\},\{3\}\}, \;\; Q_2'' = \{\{4\}\}$

$Q_1 = \{1,2,3\}, \;\; Q_2 = \{4\}$

Both conditions (i) and (ii) of the last Theorem hold.

As $|Q_1| = 3 < 4 = |Q_1''|$ and $|Q_2| = 1 = |Q_2''|$ then A is minimal.