Revisiting wreath products of automata

Jaan Penjam

Institute of Cybernetics

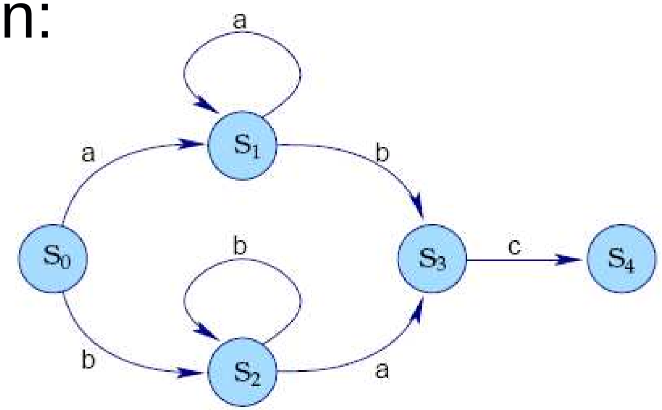
(joint work with Uno Kaljulaid, Varmo Vene)

Outline

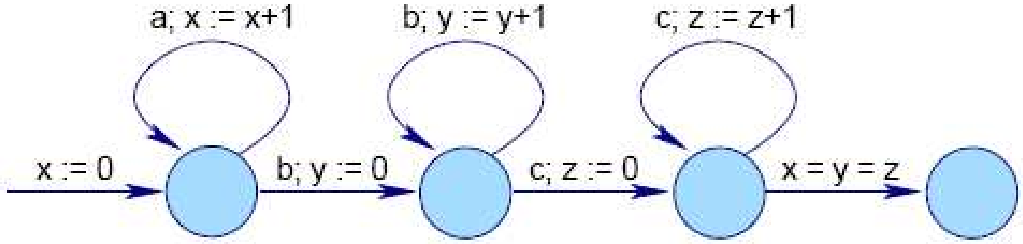
- FA as a Category
- Generalized Automaton
- Wreath products
- Wreath products of categories
- Wreath products of functors
- Conclusions

Automata

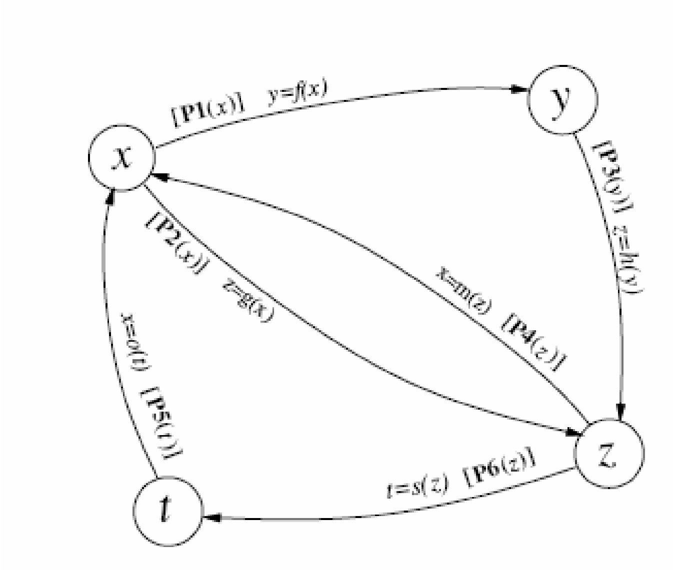
Finite automaton:



Attribute automaton:



Automaton with memory

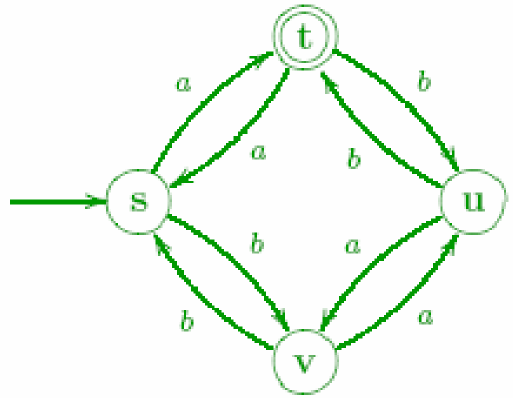


$M = (S, T)$, where :

- S is a set of states with two distinguished subsets: $S_i \subseteq S$ (initial states), and $S_f \subseteq S$ (final states); for every state $s \in S$ is associated with a memory (an attribute) a_s with its domain A_s;
- $T \subseteq S \times S$ is the set of transitions. Every transition $t = (s, s') \in T$ associated with
  enabling predicate $P_t : A_s \longrightarrow \mathbf{bool}$, and
  transformational function $f_t : A_s \longrightarrow A_{s'}$.

FA as a Category

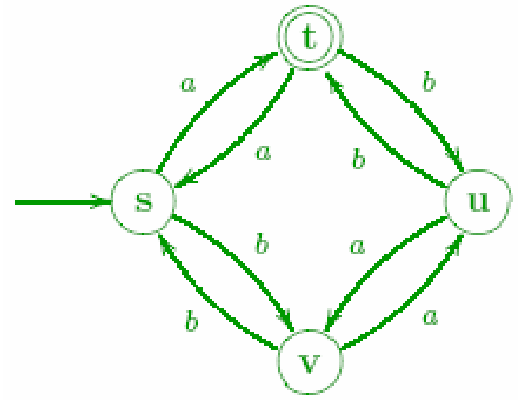
$$\begin{aligned} S &= \{\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}\} \\ \Sigma &= \{a, b\} \end{aligned}$$



Semigroup action: $A = (S, \Sigma; \circ)$ where $s \circ (uv) = (s \circ u) \circ v$ holds for all $s \in S$ and $u, v \in \Sigma^*$

FA as a Category

$$
\begin{aligned}
S &= \{\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}\} \\
\Sigma &= \{a, b\}
\end{aligned}
$$



Category Automaton: $\mathbf{A} = (\mathrm{Obj}(\mathbf{A}), \mathrm{Mor}(\mathbf{A}))$ where

- $\mathrm{Obj}(\mathbf{A}) = S$
- $\mathrm{Mor}_{\mathbf{A}}(s, s) = \{s \xrightarrow{\varepsilon} s,\ s \xrightarrow{aa} s,\ s \xrightarrow{abab} s, \ldots\}$
- $\mathrm{Mor}_{\mathbf{A}}(s, t) = \{s \xrightarrow{a} t,\ s \xrightarrow{abb} t,\ s \xrightarrow{aaa} t, \ldots\}$
- $\ldots$

Generalized Automaton

A general automaton is a system $\mathfrak{A} = (\mathbf{A}, A, \Box)$ where

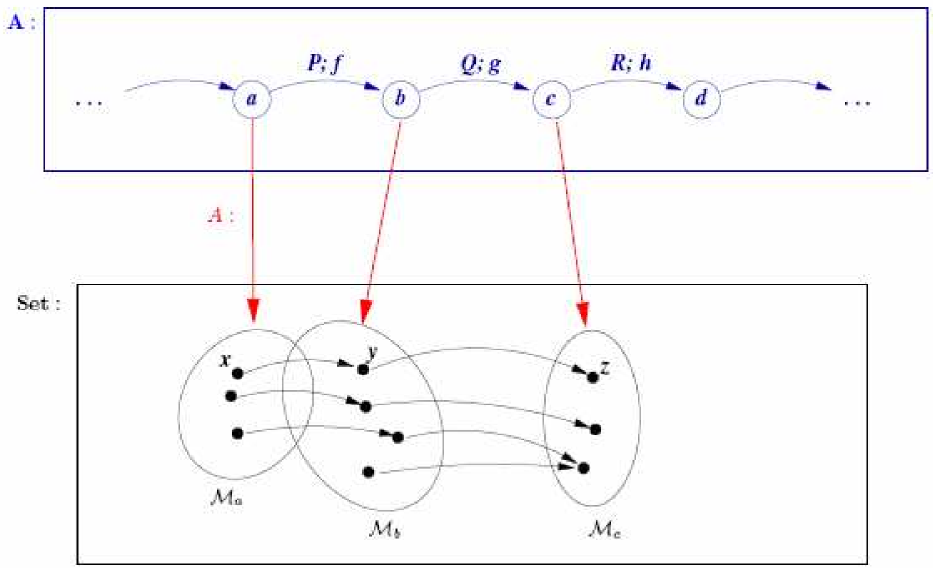
- $\mathbf{A}$ is a (small) category
- A is a functor $\mathbf{A} \longrightarrow \mathbf{Set}$
- $\Box$ is a (partial) feedback operation

$$\Box : \Big(\coprod_{a \in Obj(\mathbf{A})} a^A \times \mathrm{Mor}(\mathbf{A}) \Big) \longrightarrow \mathrm{Mor}(\mathbf{A}),$$

satisfying the condition

$$x \,\Box (f \cdot g) = f^A(x) \Box g$$

Generalized Automaton



$$\left.\begin{array}{rcl} g & = & x\Box f \\ h & = & y\Box g \\ y & = & f^A(x) \end{array}\right\} \Rightarrow x\,\Box(f\cdot g) = h = y\Box g = f^A(x)\Box g$$

Wreath products

$$\left.\begin{array}{rcl}\mathfrak{A} & = & (\mathbf{A}, A, \square) \\ \mathfrak{B} & = & (\mathbf{B}, B, \boxdot)\end{array}\right\} \Rightarrow \mathfrak{C} = (\mathbf{C}, C, \boxtimes) = (\mathbf{A} \operatorname{wr} \mathbf{B},\ A \operatorname{wr} B,\ \boxtimes)$$

Wreath products

$$\left.\begin{aligned} \mathfrak{A} &= (\mathbf{A}, A, \square) \\ \mathfrak{B} &= (\mathbf{B}, B, \boxdot) \end{aligned}\right\} \Rightarrow \mathfrak{C} = (\mathbf{C}, C, \boxtimes) = (\mathbf{A} \operatorname{wr} \mathbf{B},\ A \operatorname{wr} B,\ \boxtimes)$$

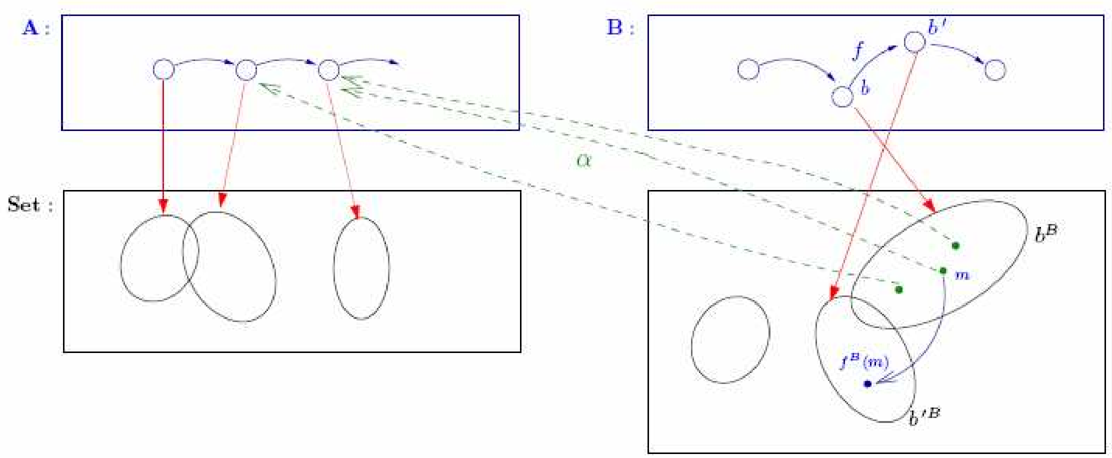
1 2 3

Wreath products of categories (1)

$$\boxed{1. \quad \mathbf{C} = \mathbf{A}\mathrm{wr}\mathbf{B}}$$

$$\mathrm{Obj}(\mathbf{C}) \stackrel{\mathrm{def}}{=} \{(\alpha, b) \mid b \in \mathrm{Obj}(\mathbf{B}), \quad \alpha : b^B \longrightarrow \mathrm{Obj}(\mathbf{A})\}$$

$$\mathrm{Mor}_{\mathbf{C}}((\alpha, b), (\alpha', b')) \stackrel{\mathrm{def}}{=} \{(\Phi, f) \quad | \quad f \in \mathrm{Mor}_{\mathbf{B}}(b, b'), \quad \Phi = \bigcup_{m \in b^B} \mathrm{Mor}_{\mathbf{A}}(\alpha(m), \alpha'(f^B(m)))\}$$



Wreath products of categories (2)

$$\mathbf{C}: \begin{tikzcd} (\alpha,b) \arrow[r, "{(\Phi, f)}"] \arrow[rr, dashed, bend right, "{(\Phi, f) = (\Psi, g)}"'] & (\alpha',b') \arrow[r, "{(\Psi, g)}"] & (\alpha'',b'') \end{tikzcd}$$

$(\Phi, f)\cdot(\Psi, g) = (\Phi * {}^{f}\Psi, f\cdot g),$ where the collection $\Phi * {}^{f}\Psi$ of morphisms in $\mathbf{A}$ is defined by the rule

$$\forall m \in b^B, \qquad (\Phi * {}^{f}\Psi)(m) = \Phi(m)\cdot \Psi(f^B(m)).$$

Wreath products of categories (2)

$$\mathbf{C}: \begin{tikzcd} (\alpha,b) \arrow[r, "(\Phi, f)"] \arrow[rr, dashed, bend right, "(\Phi, f) = (\Psi, g)"'] & (\alpha',b') \arrow[r, "(\Psi, g)"] & (\alpha'',b'') \end{tikzcd}$$

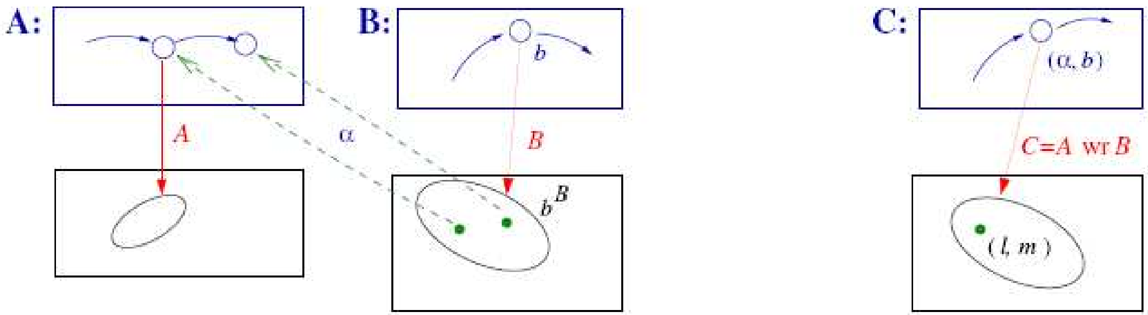
$(\Phi, f) \cdot (\Psi, g) = (\Phi * {}^f\Psi, f \cdot g)$, where the collection $\Phi * {}^f\Psi$ of morphisms in $\mathbf{A}$ is defined by the rule

$$\forall m \in b^B, \qquad (\Phi * {}^f\Psi)(m) = \Phi(m) \cdot \Psi(f^B(m)).$$

Lemma $\mathbf{C}$ is a category.

Wreath products of functors

$$\boxed{2. \quad C = A \,\mathrm{wr}\, B : \mathbf{A} \,\mathrm{wr}\, \mathbf{B} \longrightarrow \mathbf{Set}}$$

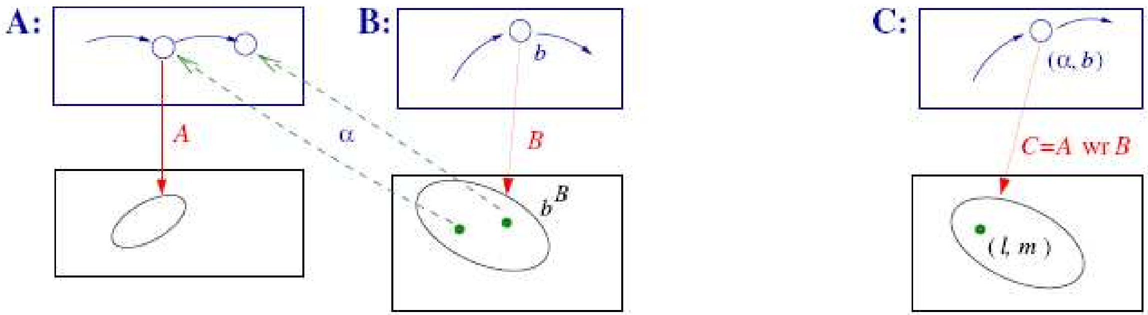


$$(\alpha, b)^C \stackrel{\text{def}}{=} \{(l, m) \mid m \in b^B, \quad l \in [\alpha(m)]^A\}$$

$$(\Phi, f)^C \stackrel{\text{def}}{=} (\Phi^A, f^B) \in \mathrm{Mor}_{\mathbf{Set}}((\alpha, b), (\alpha', b'))$$

Wreath products of functors

$$\boxed{2. \quad C = A \,\mathrm{wr}\, B : \mathbf{A} \,\mathrm{wr}\, \mathbf{B} \longrightarrow \mathbf{Set}}$$



$$(\alpha, b)^C \stackrel{\text{def}}{=} \{(l, m) \mid m \in b^B, \quad l \in [\alpha(m)]^A\}$$

$$(\Phi, f)^C \stackrel{\text{def}}{=} (\Phi^A, f^B) \in \mathrm{Mor}_{\mathbf{Set}}((\alpha, b), (\alpha', b'))$$

Lemma $A \,\mathrm{wr}\, B$ is a functor from $\mathbf{C}$ to $\mathbf{Set}$.

Composed control operator

$$\boxed{3. \quad \boxtimes : (\coprod c^C \times \mathrm{Mor}(\mathbf{C})) \longrightarrow \mathrm{Mor}(\mathbf{C})}$$

$$(l, m) \boxtimes (\Phi, f) \stackrel{\text{def}}{=} (l \Box \Phi(m), m \boxdot f)$$

Composed control operator

$$\boxed{3. \quad \boxtimes : (\coprod c^{C} \times \mathrm{Mor}(\mathbf{C})) \longrightarrow \mathrm{Mor}(\mathbf{C})}$$

$$(l, m) \boxtimes (\Phi, f) \stackrel{\mathrm{def}}{=} (l \Box \Phi(m), m \boxdot f)$$

Lemma: $(l, m) \boxtimes ((\Phi, f) \cdot (\Psi, g)) = (((\Phi, f)^{AwrB})(l, m)) \boxtimes (\Psi, g)$

Well-foundedness of the construction

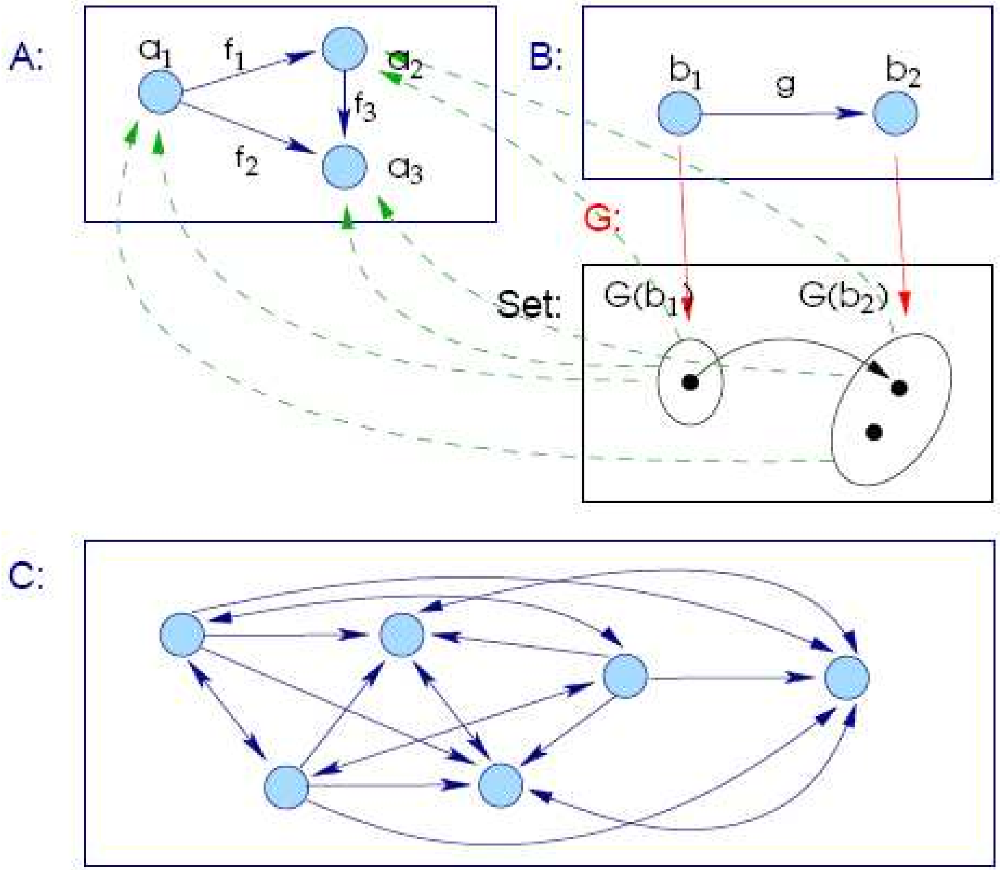
Theorem. $(\mathbf{C}, C, \boxtimes)$ is a generalized automaton

Well-foundedness of the construction

Theorem. $\left(\mathbf{C}, C, \boxtimes\right)$ is a generalized automaton

Conclusions

- Categorical defnition of generalized automata (GA) is given
- Wreath product of GAs is introduced and its correctness is shown
- Specialization of this wreath product give some intuitively simple (parallel, serial etc.) and more complex compositions of automata


A:
B:
C:
G:
Set:
a_1
a_2
a_3
f_1
f_2
f_3
b_1
b_2
g
$G(b_1)$
$G(b_2)$