Parametricity and Strong Dinaturality

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Motivation

- Parametric polymorphism is one of the key features of modern (functional) languages:
  - most commonly in Hindley-Milner style, where type variables are quntified in top level,
  - but more recently also in less restricted form (eg. rank-2 or rank-n polymorphism).
- ”Common knowledge” about polymorphism:
  - you get theorems for free! [Wadler, 1989] (but there are some pessimists);
  - polymorphic functions are (di)natural transformations (or maybe vice versa?!);
  - if you have PhD, get a real job! [Eppendhal, 2004]

Logical Relations and Parametricity

- The usual reading of a type is that it's a set of values (maybe with some some extra structure):
  - eg. the type Int is the set of integers;
  - the type $A \times B$ is the set of pairs, where components are from the sets corresponding to types A and B respectively;
  - etc.
- An alternative is to take that a type is a relation:
  - base types are interpreted as identity relations;
  - eg. $(x, y) \in \mathsf{Int} \iff x = y$;
  - every type constructor is interpreted as a corresponding action on relations.
- Relational reading is the key for parametricity results and free theorems.

Logical Relations and Parametricity

Definition

- For any relations $\mathcal{A} : A \leftrightarrow A'$, $\mathcal{B} : B \leftrightarrow B'$, the relation $\mathcal{A} \times \mathcal{B} : (A \times B) \leftrightarrow (A' \times B')$ is defined by:

$$((x, y), (x', y')) \in \mathcal{A} \times \mathcal{B} \quad \text{iff} \quad (x, x') \in \mathcal{A} \ \& \ (y, y') \in \mathcal{B}$$

- For any relations $\mathcal{A} : A \leftrightarrow A'$, $\mathcal{B} : B \leftrightarrow B'$, the relation $\mathcal{A} \to \mathcal{B} : (A \to B) \leftrightarrow (A' \to B')$ is defined by:

$$(f, f') \in \mathcal{A} \to \mathcal{B} \quad \text{iff} \quad (x, x') \in \mathcal{A} \ \Rightarrow \ (fx, f'x') \in \mathcal{B}$$

- For any relation transformer $\mathcal{F} : F \leftrightarrow F'$, the relation $\forall \mathcal{X}. \mathcal{F}(\mathcal{X}) : (\forall X. F(X)) \leftrightarrow (\forall X. F'(X))$ is defined by:

$$(g, g') \in \forall \mathcal{X}. \mathcal{F}(\mathcal{X}) \quad \text{iff} \quad \mathcal{A} : A \leftrightarrow A' \ \Rightarrow \ (g_A, g'_{A'}) \in \mathcal{F}(\mathcal{A})$$

Logical Relations and Parametricity

Parametricity

If T is a closed type and $t : T$ is a closed term, then $(t, t) \in \mathcal{T}$, where $\mathcal{T} : T \leftrightarrow T$ is the relation corresponding to the type T.

Theorems for free

Given a closed type T

- construct the corresponding relation $\mathcal{T} : T \leftrightarrow T$;
- instanciate relation transformers with graph relations;
- and simplify.

Definition

Given a function $g : A \rightarrow B$, the graph relation $\langle g \rangle : A \leftrightarrow B$ is defined by $\langle g \rangle = \{(u, g\ u) \mid \forall u : A\}$

Logical Relations and Parametricity

Example

$$
\begin{aligned}
(t,\, t) \in \forall \mathcal{X}.\, \mathcal{X} \to \mathcal{X} \quad & \Leftrightarrow \quad \forall \mathcal{R} : A \leftrightarrow B.\ (t_A,\, t_B) \in \mathcal{R} \to \mathcal{R} \\
& \Leftrightarrow \quad \forall \mathcal{R} : A \leftrightarrow B.\ \forall x : A,\ y : B. \\
& \qquad\qquad (x,\, y) \in \mathcal{R} \ \Rightarrow\ (t_A\, x,\, t_B\, y) \in \mathcal{R} \\
& \Rightarrow \quad \forall g : A \to B.\ \forall x : A,\ y : B. \\
& \qquad\qquad (x,\, y) \in \langle g \rangle \ \Rightarrow\ (t_A\, x,\, t_B\, y) \in \langle g \rangle \\
& \Leftrightarrow \quad \forall g : A \to B.\ \forall x : A,\ y : B. \\
& \qquad\qquad y = g\, x \ \Rightarrow\ t_B\, y = g(t_A\, x) \\
& \Leftrightarrow \quad \forall g : A \to B.\ \forall x : A.\ t_B(g\, x) = g(t_A\, x) \\
& \Leftrightarrow \quad \forall g : A \to B.\ t_B \circ g = g \circ t_A
\end{aligned}
$$

Note

The equation says that t is a natural transformation $\mathsf{Id} \to \mathsf{Id}$.

Natural transformations

Definition

Let $G, H : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\tau : G \to H$ is a family of maps $\tau_X : G(X) \to H(X)$ in $\mathcal{D}$ such that, for every map $f : X \to Y$ in $\mathcal{C}$, the following square commutes:

$$\begin{array}{ccc} G(X) & \xrightarrow{\tau_X} & H(X) \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle H(f)} \\ G(Y) & \xrightarrow[\tau_X]{} & H(Y) \end{array}$$

Note

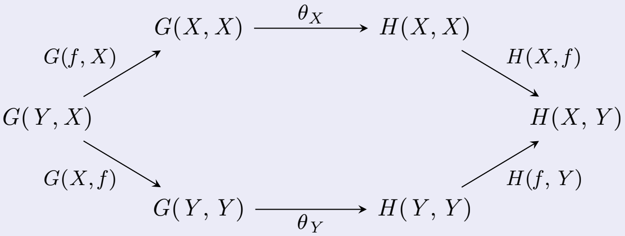
- Type may have mixed variant type variables.
- Separate the co- and contravariant instances and use diagonalization to recover the original type.
- Eg. $\forall X.\ X \to X = \forall X.\ H(X, X)$, where $H(A, B) = A \to B$.

Dinaturality

Definition

Let $G, H : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be functors.

A dinatural transformation $\theta : G \to H$ is a family of maps $\theta_X : G(X, X) \to H(X, X)$ in $\mathcal{D}$ such that, for every map $f : X \to Y$ in $\mathcal{C}$, the following hexagon commutes:



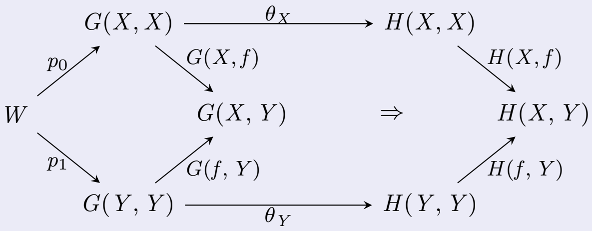
Strong dinaturality

Definition

A strong dinatural transformation $\theta : G \to H$ is a family of maps $\theta_X : G(X,X) \to H(X,X)$ in $\mathcal{D}$ such that, for every

- map $f : X \to Y$ in $\mathcal{C}$,
- object W in $\mathcal{D}$ and
- maps $p_0 : W \to G(X,X)$, $p_1 : W \to G(Y,Y)$ in $\mathcal{D}$,

if the square commutes, then so does the hexagon:



From parametricity to strong dinaturality

Covariant types

$$
\begin{array}{rcll}
F(A) & ::= & A & \mid \; C \\
& \mid & F(A) \times F(A) & \mid \; F(A) + F(A) \\
& \mid & G'(A) \to F(A) & \mid \; \forall X.\, F([X, A]) \\
G'(A) & ::= & C & \\
& \mid & G'(A) \times G'(A) & \mid \; G'(A) + G'(A) \\
& \mid & F(A) \to G'(A) &
\end{array}
$$

From parametricity to strong dinaturality

Contravariant types

$$
\begin{array}{rcll}
G(A) & ::= & C & \\
& \mid & G(A) \times G(A) & \mid \quad G(A) + G(A) \\
& \mid & F'(A) \to G(A) & \mid \quad \forall X.\, G([X, A]) \\
F'(A) & ::= & A & \mid \quad C \\
& \mid & F'(A) \times F'(A) & \mid \quad F'(A) + F'(A) \\
& \mid & G(A) \to F'(A) &
\end{array}
$$

From parametricity to strong dinaturality

Definition

A functor $H : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is weakly cartesian if forall $f : A \to B$ the bifunctoriality diagram is a weak pullback:

$$\begin{array}{ccc} H(B,A) & \xrightarrow{H(f,A)} & H(A,A) \\ {\scriptstyle H(B,f)}\downarrow & & \downarrow{\scriptstyle H(A,f)} \\ H(B,B) & \xrightarrow{H(f,B)} & H(A,B) \end{array}$$

Weakly cartesian types

$$\begin{array}{rcl|l} H(A,B) & ::= & G'(A) & F'(B) \\ & | & H(A,B) \times H(A,B) & H(A,B) + H(A,B) \\ & | & C \to H(A,B) & \end{array}$$

From parametricity to strong dinaturality

Definition

A type K is `Eq2R` if forall closed terms $a : K(A, A)$, $b : K(B, B)$ and functions $g : A \to B$

$$K(A, g)\, a = K(g, B)\, b \quad \Longrightarrow \quad (a,\ b) \in \mathcal{K}\langle g \rangle$$

`Eq2R` types

$$\begin{array}{lclcl} K(A, B) & ::= & B & \mid & C \\ & \mid & K(A, B) \times K(A, B) & \mid & K(A, B) + K(A, B) \\ & \mid & H(B, A) \to K(A, B) & \mid & \forall X.\, K([X, A], [X, B]) \end{array}$$

From parametricity to strong dinaturality

Definition

A type K is R2Eq if forall closed terms $a : K(A, A)$, $b : K(B, B)$ and functions $g : A \to B$

$$(a,\ b) \in \mathcal{K}\langle g \rangle \quad \Longrightarrow \quad K(A, g)\, a = K(g, B)\, b$$

R2Eq types

$$\begin{array}{rcl|l} L(A, B) & ::= & B & C \\ & | & L(A, B) \times L(A, B) & L(A, B) + L(A, B) \\ & | & K(B, A) \to L(A, B) & \forall X.\, L([X, A], [X, B]) \end{array}$$

From parametricity to strong dinaturality

Theorem

Let K and L be System F types containing one free type variable and let $t : \forall X.K(X,X) \to L(X,X)$ be a closed term of closed type.

If K is `Eq2R` *and L is* `R2Eq`*, then t is a strongly dinatural transformation.*

From parametricity to strong dinaturality

Theorem

Let $F : \mathcal{C} \to \mathcal{C}$ be a functor and $C \in \mathcal{C}$ an object. If F has an initial algebra μF then:

$$\mathsf{SDinat}(\mathsf{Hom}(F-,-), \mathsf{Hom}(C,-)) \cong \mathsf{Hom}(C, \mu F)$$

Corollary

Let F be a type expression with one covariant type variable, derivable from nonterminal F'. Then

$$\forall X.\ (F(X) \to X) \to X \cong \mu F$$

From parametricity to strong dinaturality

Example

- Fixpoints are not definable in System F :

$$\begin{aligned}\forall X.\,(X \to X) \to X &\cong \forall X.\,(\mathsf{Id}(X) \to X) \to X \\ &\cong \mu\,\mathsf{Id} \quad \cong \quad 0\end{aligned}$$

- Polymorphic identity:

$$\begin{aligned}\forall X.\; X \to X &\cong \forall X.\,(\mathbf{1}(X) \to X) \to X \\ &\cong \mu\,\mathbf{1} \quad \cong \quad 1\end{aligned}$$

- Empty type:

$$\begin{aligned}\forall X.\; X &\cong \forall X.\, 1 \to X \\ &\cong \forall X.\,(\mathbf{0}(X) \to X) \to X \\ &\cong \mu\,\mathbf{0} \quad \cong \quad 0\end{aligned}$$

Conclusions and Further Work

- We have identified a class of types whose terms are strongly dinatural in every parametric model.
- The class is large enough to cover several important applicatons;
  - eg. Church encoding of initial algebras.
- Possible directions for the future work include:
  - to investigate the relationship with other formalisms (eg. structural polymorphism [Freyd, 1993], polynomial polymorphism [Jay, 1995], cospan diparametricity [Eppendahl, 2005]);
  - to try to find less syntactic characterization of the suitable classes of types.