Computers Without Batteries?

Rewriting Cellular Automata into Block Automata

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June 5, 2009

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Introduction

- Cellular automata (CA) are schematics for digital circuits
- Intrinsic of CA is the use of 1-to-N signal replication
- Physically, this entails employing free energy
- Question:

  It is possible to achieve the same
  global functional behavior
  without draining the power supply of free energy?
- Possible drawback: a more complex structure

Outline of the talk

- Physical issues in locally-defined models of computation
- Kari's theorem on reversible 1D CA
- Our result for 1D non-surjective CA
- Conclusions and suggestions for higher-dimensional CA

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A recipe for CA

Ingredients:

- an integer dimension $d \geq 1$
- a finite set Q of states
- a neighborhood index $\mathcal{N} : \{1, \ldots, N\} \to \mathbb{Z}^d$
- a local map $f : Q^N \to Q$

"Bake" the global map

$$c_x^{t+1} = f\left(c^t_{x+\mathcal{N}(1)}, \ldots, c^t_{x+\mathcal{N}(n)}\right)$$

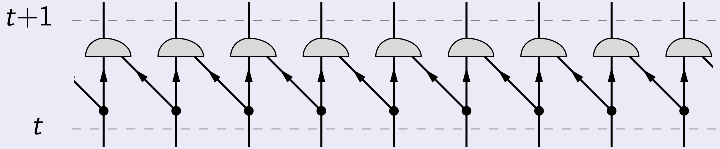
Enjoy!

The shift map

$\sigma(c)_x = c_{x+1} \; \forall c \in Q^{\mathbb{Z}} \, , x \in \mathbb{Z}$

CA (thermo)dynamics

Space-time diagram of a CA



Heat balance of CA

- Many-to-one discipline increases entropy—produces heat
- Fanout nodes perform signal replication—require power
- Heat must be dissipated via a heat sink
- Power must be supplied by an external source

Three classes of CA

Reversible (r.e.)	Properly Surjective	Non−Surjective (r.e.)

- $\mathcal{C}^+$—there exists a CA modeling the inverse dynamics
- $\mathcal{C}^0$—each configuration has a preimage, some have more
- $\mathcal{C}^-$—some finite blocks have no preimage

Hedlund's theorem

$$\mathcal{C}^+ = \{F \in C(Q^{\mathbb{Z}}, Q^{\mathbb{Z}}) \mid \exists F^{-1} \text{ and } F \circ \sigma = \sigma \circ F\}$$

Block automata

A "watertight compartments" computation

- Space is partitioned into equally-shaped blocks
- Each block updates at the same time
- Each block updates independently of the others

Block automata (BA) may be thought of as

zero-range, coarse-grained CA

Lattice gases: A two-steps discipline

Collision

- Strictly pointwise process
- Same number for inputs and outputs
- Same types for inputs and outputs

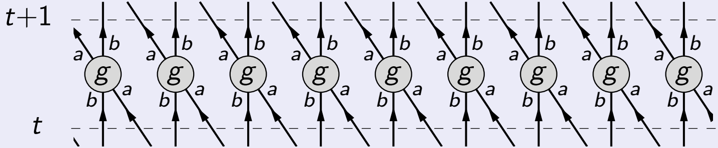
Propagation

- Each signal to one neighbour
- No replication
- No reuse

Lattice gases are combinations of shifts and BA with one-point blocks

Lattice gas (thermo)dynamics

Space-time diagram of a lattice gas



Heat balance of a lattice gas

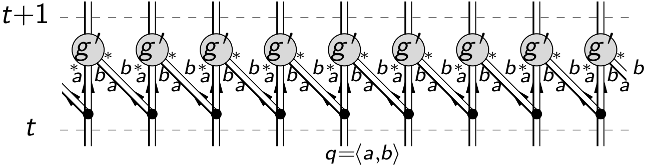
- No fanout nodes, no signal destruction.
- The energy-requiring task is skipped.
- No need to dissipate heat if collision invertible.
- No need to provide power.

There. . .

Every BA can be re-written as a CA

- Same dimension
- Old blocks become new points
- Neighborhood reduced to the point itself

Also every LG can be re-written as a CA



...and back again?

From $\mathcal{C}^+$ to $\mathcal{L}$

Kari, 1996

- OK in dimensions 1 and 2
- Equivalent to a conjecture on high-dimensional shift spaces for $d \geq 3$.

From $\mathcal{C}^0$ to $\mathcal{L}$? Impossible!

In a (possibly non-homogeneous) BA or LG, the global map is [in|sur]jective iff each of the local maps is.
(Toffoli, Capobianco and Mentrasti, Lemma 3)

From $\mathcal{C}^-$ to $\mathcal{L}$?

Never really checked before as far as we know

Kari's construction for 1D reversible CA

Let $\mathcal{A}$ be a reversible 1D CA and let F be its global rule.
Let $[-r, \ldots, r]$ contain the neighborhoods of both $\mathcal{A}$ and $\mathcal{A}^{-1}$.
Put

$$\begin{aligned} R_{\mathcal{A}} &= \{(c_0, \ldots, c_{2r-1}, F(c)_{-r}, \ldots, F(c)_{r-1}) \mid c \in Q^{\mathbb{Z}}\} \\ L_{\mathcal{A}} &= \{(F(c)_0, \ldots, F(c)_{2r-1}, c_{-r}, \ldots, c_{r-1}) \mid c \in Q^{\mathbb{Z}}\} \end{aligned}$$

Observe that $L_{\mathcal{A}} = R_{\mathcal{A}^{-1}}$ and vice versa.

Lemma A

$|R_{\mathcal{A}}| \cdot |L_{\mathcal{A}}| = |Q|^{6r}$

Reason why: the two $4r$-tuples have a total of $2r$ constraints, for a total of $6r$ degrees of freedom.

A group-theoretic note

Put $h_+(\mathcal{A}) = \dfrac{|R_\mathcal{A}|}{|Q|^{3r}}$ and $h_-(\mathcal{A}) = \dfrac{|L_\mathcal{A}|}{|Q|^{3r}}$.

This is well-posed: increasing r by 1 adds to each tuple 4 elements with 1 constraint.

By Lemma A, $h_-(\mathcal{A}) \cdot h_+(\mathcal{A}) = 1$.

Lemma B

$h_\pm \in \mathrm{Hom}(\mathcal{C}^+, \mathbb{Q}_+)$. Also, $h_+(\mathcal{A}) = (h_-(\mathcal{A}))^{-1}$.

Reason why: $h_\pm(\mathcal{A}_1; \mathcal{A}_2) \leq h_\pm(\mathcal{A}_1) \cdot h_\pm(\mathcal{A}_2)$ and observation above.

Lift $h_\pm$ to $\mathrm{Hom}(\Gamma, \mathbb{Q}_+)$ where

$\Gamma = \{F \in C(Q^\mathbb{Z}, Q^\mathbb{Z}) \mid \exists F^{-1} \text{ and } \exists n \geq 1 \mid F \circ \sigma^n = \sigma^n \circ F\}$

- Every BA is in $\ker(h_-)$.
- $\mathrm{im}(h_-)$ is generated by the prime factors of $|Q|$.

Two BA layers always suffice

Kari's main lemma

Every CA in $\ker(h_-)$ is composition of two BA.

Reason why:

- $\mathcal{A} \in \ker(h_-)$ means $|R_{\mathcal{A}}| = |L_{\mathcal{A}}| = |Q|^{3r}$.
- Let $b_X : X_{\mathcal{A}} \to |Q|^{3r}$ be a bijection.
- With F global rule of $\mathcal{A}$, put

$$\begin{aligned} f_{R,\mathcal{A}}(c_0, \ldots, c_{6r-1}) &= (c_{4r}, \ldots, c_{6r-1}, F(c)_{3r}, \ldots, F(c)_{5r-1}) \\ f_{L,\mathcal{A}}(c_0, \ldots, c_{6r-1}) &= (F(c)_r, \ldots, F(c)_{3r-1}, c_0, \ldots, c_{2r-1}) \end{aligned}$$

- Then the following are permutations of $|Q|^{6r}$ objects:

$$\begin{aligned} \pi_1 &= (b_L \circ f_{L,\mathcal{A}}) \otimes (b_R \circ f_{R,\mathcal{A}}) \\ \pi_2 &= (b_R \circ f_{L,\mathcal{A}^{-1}}) \otimes (b_L \circ f_{R,\mathcal{A}^{-1}}) \end{aligned}$$

- But $F = p_2^{-1} \circ p_1$, with p_i block permutation induced by π_i.

Partial shifts

If $Q = Q_1 \times Q_2 \times \ldots \times Q_k$, consider $\sigma_i : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ given by

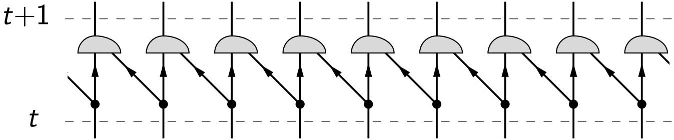
$$(\sigma_i(c)_x)_j = \begin{cases} (c_{x+1})_i & \text{iff } i = j \\ (c_x)_j & \text{iff } i \neq j \end{cases}$$

- Every partial shift is a CA.
- Q can always be re-written as above, with each $|Q_i|$ a prime that divedes $|Q|$.
- $h_-(\sigma_i) = |Q_i|$.

Theorem

Every reversible 1D CA is composition of two BA and partial shifts.

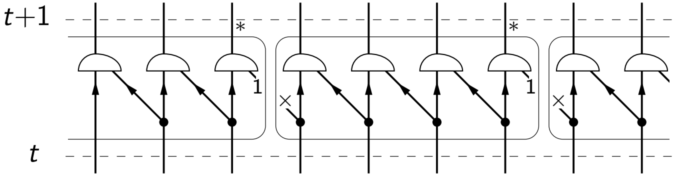
The AND CA on two neighbours



101 is not reachable:

⋯	1	0	1	·	⋯
⋯	1	1	1	1	⋯
⋯	·	1	·	·	⋯

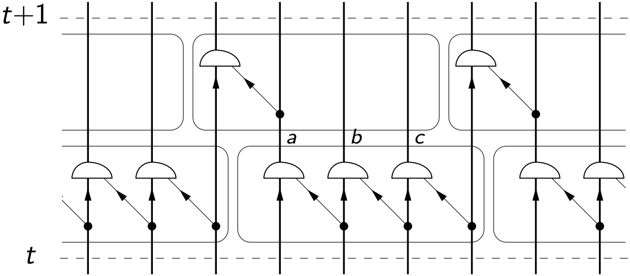
First attempt: Just partition into blocks



Problems

- Either we force a value on a line...
- ...or we allow superluminal speed

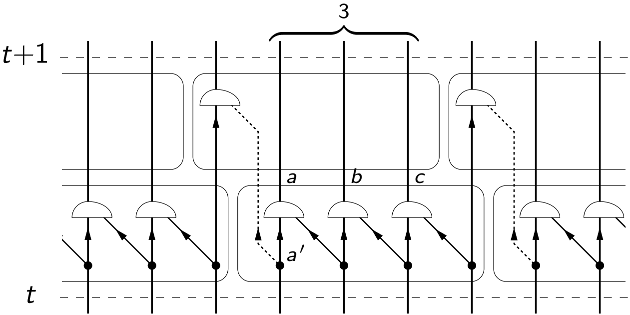
Second attempt: Add another layer



Problems

- This is a composition of BA. . .
- . . . but its dynamics is wrong

Third attempt: Add a duplication channel



Problems

- This has the correct dynamics...
- ...but violates the input-output constraint!

What is left?

Exploiting information loss

Variety

$\nu_{\mathcal{A}}(n) =$ nr. of patterns on $\{0, \ldots, n-1\}$ obtainable by applying the CA rule.

$$V(n) = \log_{|Q|} \nu(n)$$

The CA is in $\mathcal{C}^-$ iff $V(n) < n$ for some n.

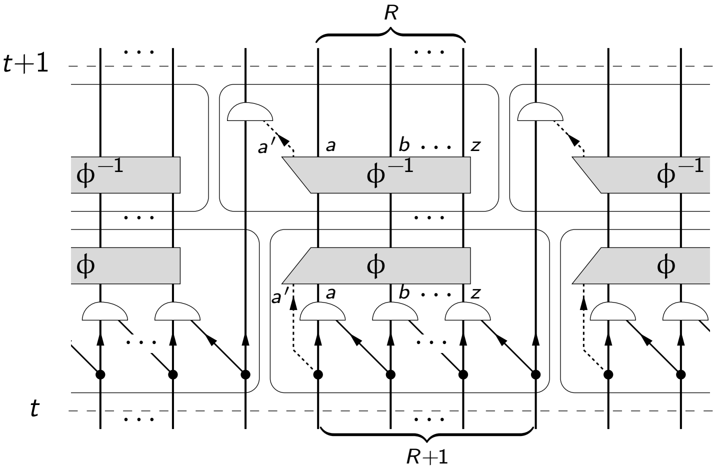
In our case

n	$\nu(n)$	$V(n)$	$n - V(n)$
1	2	1	0
2	4	2	0
3	7	2.81	0.19
4	12	3.58	0.42

n	$\nu(n)$	$V(n)$	$n - V(n)$
5	21	4.39	0.61
6	37	5.21	0.79
7	65	6.02	0.98
8	**114**	**6.83**	**1.17**

Idea: use that free bit to encode the boundary

Fourth attempt: add a codec—we made it at last!



Rewriting AND as a LG—“text only” version

Input: a non-surjective CA

1. Find R so that $R + N$ signals can be compressed into R
2. Partition the space into blocks of $R + N$ cells each
3. Compute as many points as possible
4. Add another layer
5. Replicate the signal entering the neighboring block
6. Encode the output and replicated signal into R bits
7. Send information from first layer to second
8. Decode the compressed output-and-replication
9. Compute the remaining points

But: Can we always perform step 1?

Yes, we can!

Fekete's lemma

If $f : \{1, 2, \ldots\} \to [0, +\infty)$ satisfies $f(n+m) \leq f(n) + f(m) \; \forall \, n, m$, then

$$\lim_{n \to \infty} \frac{f(n)}{n} = \inf_{n \geq 1} \frac{f(n)}{n}$$

Every CA in $\mathcal{C}^-$ can be re-written with 2 BA layers (TCM'08)

- Let $1 > \delta > \inf_{n \geq 1} V(n)/n$
- A possible output sized $n + m$ is a junction of two sized n, m
  $\Rightarrow$ V is subadditive
  $\Rightarrow$ $R + N - V(R+N) \geq (R+N)(1-\delta) > N$ for R large enough
- Use blocks of size $R + N$

Hints for reversible CA

Kari's theorem in dimension 2

Every reversible 2D CA is the composition of at most four BA layers.

The proof is based on the following

Lemma C

Let X and Y be shifts of finite type.
Suppose a power of $X \times Y$ is conjugate to a full shift.
Then a power of X and a power of Y are conjugate to full shifts.

Kari proves that an analogous conjecture in dimension d is equivalent to his theorem in dimension $d + 1$.

Hints for non-surjective CA

Multivariate Fekete's lemma (Capobianco, 2008)

Let $\mathbb{Z}_+^d = \{1, 2, \ldots\}^d$ be pre-ordered by $x \leq y$ iff $x_i \leq y_i \, \forall i$.
If $f : \{1, 2, \ldots\}^d \to [0, +\infty)$ is subadditive in each variable, then

$$\lim_{z \in \mathbb{Z}_+^d} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} = \inf_{z \in \mathbb{Z}_+^d} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d}$$

Neighborhood size is never an issue

Put $\Lambda(r_1, \ldots, r_d) = r_1 \cdots r_d - V(r_1, \ldots, r_d)$. TFAE.

1. The CA is non-surjective.
2. For every $K, n_1, \ldots, n_d \geq 0$ there exist $t_1, \ldots, t_d \in \mathbb{Z}_+$ such that, if $r_j \geq t_j$ for every j, then

$$\Lambda(r_1, \ldots, r_d) \geq (r_1 + n_1) \cdots (r_d + n_d) - r_1 \cdots r_d + K \; .$$

Computation without batteries!

State of the art

- OK for $d = 1$
- One basic case for $d = 2$
- Open—and promising—for $d > 1$

Future work

- Find construction schemes for any d
- Use at most $d + 1$ layers in dimension d (cf. Kari 1999)
- Prove or disprove Kari's conjecture about $\mathcal{C}^+$

Thank you for attention!

Any questions?