Ernst and the King

Myths and Facts about Chess and Game Theory

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Abstract

- The very first theorem in game theory is about Chess.
- This theorem is often mis-interpreted, mis-quoted, mis-understood.
- This talk aims to make some order in the chaos.

Bibliography

- Schwalbe, U. and Walker, P. (2001) Zermelo and the Early History of Game Theory. *Games and Economic Behavior* **34**, 123–137.
- Zermelo, E. (1913) Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels. Proc. 5th Congress of Mathematicians, Cambridge, 501–504.
- Kőnig, D. (1927) Über eine Schlussweise aus dem Endlichen ins Unendliche. *Acta Sci. Math. Szeged* **3**, 121–130.
- Kalmár, L. (1929) Zur Theorie der abstrakten Spielen. *Acta Sci. Math. Szeged* **4**, 65–85.
- Chess riddles from Mariano Tomatis' website.

Premise

- I usually speak sincerely.
- I am a very poor chess player.
- I can play two games with any two players and be sure to finish with a non-negative balance—*i.e.*, two draws or at least one win.

How do I do?

Urban Legend

"A friend of a friend told me that, if you play White in a chess game and you *never ever make any errors*, then you are sure to win! It's *mathematical!* It has been proven by a German professor, *Zermelon* or something like that, by analyzing the game *from the end to the beginning!*"

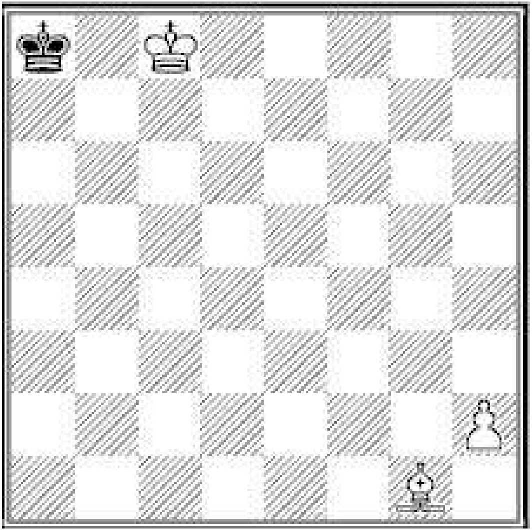
(Your average chess amateur)

A Note on Backward Analysis in Chess

A class of Chess problems hes the following form:

- given a position,
- deduce the previous position.

Example Backward Analysis Problem (Smullyan, 1994)



. . . OK, So Where Is the Proof?

So, How Was the Urban Legend Born?

Misinformation.

- People often say something false for interest.
- People as often say something wrong and just believe it.

Misunderstanding.

- People often make errors.
- This may happen even in university level publications!

Wishful thinking.

- People often understand what they want instead of what it is.
- Several mediocre players would like to win a chess match.

Is First-Player Advantage Real?

Evidence

- Statistics tell that White's winning percentage—i.e, percent of wins plus half percent of draws—is between 52 and 56 percent.

Counter-evidence

- The similar game of Checkers has been solved two years ago—with a result expected by the experts but probably not the amateurs...

In fact, first-player advantage might well be psychological rather than tactical.

Checkers is Solved

A perfect game of Checkers ends in a draw.

- Schaeffer *et al.* (2007) Checkers is solved. *Science* **317**, pp. 1518–1522.

Incidentally:
Consensus among Chess masters is that perfect play from both sides should end in a draw.

Properties of Games

A game is

- zero-sum, if the total of wins always equal the total of losses;
- perfect information, if there is no hidden information and no role of chance.

As such:

- Chess and Checkers are zero-sum, perfect information games.
- Backgammon and Blackjack are zero-sum, imperfect information games.
- Lottery is a non-zero-sum game.
  (If we don't count the organizer as a player.)

So, What Does Mathematics Really Say?

"The following considerations are independent on the special rules of the game of Chess and are valid in principle just as well for all similar games of reason, in which two opponents play against each other with the exclusion of chance events; for the sake of determinateness they shall be exemplified by Chess as the best known of all games of this kind."

(E. Zermelo, 1913; translation by Schwalbe and Walker)

Positions and Moves

A position p takes into account all relevant variables, such as

- the displacement of pieces on the checkboard
- the player that has to make the next move
- castling
- pawn promotion
- etc.

A move is a transition $p_I \longrightarrow p_F$ allowed by the rules of the game.
Note: in Chess, there are finitely many positions.
Note: with these conventions, Fool's mate

$$\text{f3 e5; g4 Qh4\#}$$

consists of four moves.

Fool's Mate

Problem 1:

What properties must position p possess
to ensure a mate in r moves?

Problem 2:

If p is a winning position,
then how long does it take for White
to win from p?

Note: Zermelo was mostly interested in Problem 2.

Zermelo's Approach: Endgames

An endgame from position p is a sequence—possibly infinite—of positions

$$\eta = (p_0 = p, p_1, p_2, \ldots) = \eta(p)$$

such that

1. $p_i \to p_{i+1}$ is a move for all i, and
2. if η is finite, $\eta = (p, \ldots, p_n)$, then p_n is either a checkmate or a stalemate.

A position p is

- winning if, starting from p, White can win whatever game Black plays.
- non-losing if, starting from p, White can always avoid defeat whatever game Black plays.

Zermelo's Argument

p is winning for White in at most r moves

iff

$\exists U_r(p) \neq \emptyset$ of endgames from p with the following properties:

1. Every $\eta \in U_r(p)$ has at most $r+1$ elements, the last being a win for White.
2. If $\eta = (p, p_1, \dots) \in U_r(p)$, Black must move at p_i, and $p_i \to p'_{i+1}$ is a move, then $\exists \eta' = (p, \dots, p_i, p'_{i+1}, \dots) \in U_r(p)$.

That is,

1. White has a strategy to win from p in r moves or less, modeled by $U_r(p)$, and
2. White's strategy cannot be ruined by Black's game.

Zermelo's Argument (cont.)

Properties 1 and 2 are stable for union.

1. Consider the union $\overline{U}_r(p)$ of all the $U_r(p)$'s.
2. Then p is winning in $\leq r$ moves iff $\overline{U}_r(p) \neq \emptyset$.

If $r_1 \leq r_2$ then $\overline{U}_{r_1}(p) \subseteq \overline{U}_{r_2}(p)$.

1. Suppose p is winning.
2. Let $\rho = \min\{r \mid \overline{U}_r(p) \neq \emptyset\} = \rho(p)$.
3. Let $\tau = \max_{q \text{ winning}} \rho(q)$.
4. Then $\tau \leq t$ where t is the number of positions.

Finally, as an upper bound,

$$p \text{ is winning iff } U(p) = \overline{U}_\tau(p) \neq \emptyset.$$

... and What If $\overline{U}_\tau(p) = \emptyset$?

Problem 1bis:

What properties must position p possess
to delay defeat for at least s moves?

Zermelo's answer: existence of a set $V_s(p) \neq \emptyset$ of endgames from p with

1. Every $\eta \in V_s(p)$ has at least $s + 1$ elements.
2. If $\eta = (p, p_1, \ldots) \in V_s(p)$, Black must move at p_i, and $p_i \to p'_{i+1}$ is a move, then $\exists \eta' = (p, \ldots, p_i, p'_{i+1}, \ldots) \in V_s(p)$.

Zermelo's Argument (concl.)

Properties 1 and 2 are stable for union.

1. Consider the union $\overline{V}_s(p)$ of all the $V_s(p)$'s.
2. Then p is non-losing for $\geq s$ moves iff $\overline{V}_s(p) \neq \emptyset$.

If $s_1 \leq s_2$ then $\overline{V}_{s_1}(p) \supseteq \overline{V}_{s_2}(p)$.

1. This implies $\overline{V}_s(p) \neq \emptyset$ either for all s (if p is non-losing) or for finitely many (if p is losing).
2. Suppose p is losing.
3. Then Black can win from p in at most τ moves.
4. This is the same as saying that $\overline{V}_{\tau+1}(p) = \emptyset$.

Consequently,

$$p \text{ is non-losing iff } V(p) = \overline{V}_{\tau+1}(p) \neq \emptyset.$$

Let's Look Again at That Upper Bound...

Zermelo:
"If more than t moves are needed, then one of the positions is repeated, thus White could just play the first time the game he/she plays the second time."

Kőnig:
"But why should Black play the same game as second time?"

Kőnig's Infinity Lemma

Let $\{E_n\}$ be an infinite sequence of finite nonempty sets.
Let $\mathcal{R}$ be a binary relation on $E = \bigcup_n E_n$ such that

$$\forall n \in \mathbb{N} \, \forall y \in E_{n+1} \, \exists x \in E_n \mid x \mathcal{R} y \, .$$

Then

$$\forall n \in \mathbb{N} \, \exists a_n \in E_n \mid a_n \mathcal{R} a_{n+1} \forall n \in \mathbb{N} \, .$$

Equivalently:

- Every finitely-branching infinite tree has an infinite path.

An application to game theory:

- If p is winning, then there exists $N = N(p)$ such that White can win from p in at most N moves.

(Kőnig, 1927; suggested by von Neumann)

Kőnig's Approach: Beginnings

A beginning of game from p is a licit finite sequence of moves $\beta = (w_1, b_1, \ldots, w_n)$, beginning and ending with a move from White.

Let $B(p)$ be the set of beginnings from p

Then p is winning if $\exists S \subseteq B(p)$ s.t.

1. $\exists \beta = (w_1) \in S$.
2. If $\beta = (w_1, b_1, \ldots, w_n) \in S$ and b_n is licit after w_n, then $\exists w_{n+1} \mid \beta|(b_n, w_{n+1}) \in S$.
3. If γ is a game not ending in a stalemate and if $\gamma[1 : 2n - 1] \in S \, \forall n \in \mathbb{N}$ then $\gamma \in S$.

Kőnig's Proof

1. Suppose p is winning.
2. Suppose $N(p)$ does not exist.
3. Let E_n be the set of beginnings from p of length $2n-1$.
4. Then $0 < |E_n| < \infty \, \forall n$.
5. Let $\beta_n \mathcal{R} \beta_{n+1}$ iff $\beta_i \in E_i \cap S$ and $\beta_{n+1} = \beta_n; (b_n, w_{n+1})$.
6. Then $\forall \beta_{n+1} \exists \beta_n \mid \beta_n \mathcal{R} \beta_{n+1}$.
7. Choose $a_n \in E_n$ so that $a_n \mathcal{R} a_{n+1} \, \forall n$.
8. Then $a = \lim_{n \to \infty} a_n$ is a game
   - not a victory for White
   - not ending in a stalemate
   - such that every prefix of length $2n-1$ is in S

   against condition 3 on S.

A Note on Kőnig's Proof

- There is no need that the number of positions is finite.
- It is only necessary that finitely many positions are reachable at any moment.

In fact, Kőnig notices that in a game

- on an infinite chessboard,
- with the rules of Chess, and
- with the same moves as on a normal chessboard (i.e., the Queen, Rook, and Bishop move at most seven squares at a time)

there would still exist an $N(p)$.

… But the Upper Bound Holds Anyway!

Kőnig showed Zermelo his argument.
Zermelo replied with a new proof that at most t moves are sufficient.

1. Let m_r be the number of positions that
   - are winning for White, and
   - allow a shortest mate in exactly r moves.
2. Then $\sum_{r \in \mathbb{N}} m_r \leq t$.
3. Let $\lambda = \min\{l \mid m_r = 0 \, \forall r > l\}$, so that $\sum_{r \in \mathbb{N}} m_r = \sum_{r=0}^{\lambda} m_r$.
4. If $m_r > 0$, then $m_{r-1} > 0$ as well.
   (Take one of the m_r positions and make one move according to a shortest mate in r moves.)
5. As such, $\lambda \leq \sum_{r \in \mathbb{N}} m_r \leq t$.

... But What About (Non-)Repetition?

Kőnig:
"Zermelo's argument on non-repetition is not convincing."

Kalmár:
"But non-repetition is possible anyway!"

Kalmár's Approach: Script Games

Let $\mathcal{G}$ be a two-player, zero-sum, perfect information game.
The script game of $\mathcal{G}$ is the game $S_{\mathcal{G}}$ on the histories of $\mathcal{G}$, with the same rules for moves as $\mathcal{G}$
A tactic *in the strict sense* for a player in $\mathcal{G}$ is a tactic that does not restrict the other player.
A tactic *in the weak sense* in $\mathcal{G}$ is a tactic *in the strict sense* in $S_{\mathcal{G}}$

Proposition

1. A winning position *in the strict sense* is also winning *in the weak sense*.
2. A position is losing *in the strict sense* iff it is losing *in the weak sense*.

Kalmár's Characterization

Theorem

The set of winning positions *in the weak sense* is the smallest set M such that every position p behaves as follows.

1. If White has to move and $\exists p \to p' \mid p' \in M$, then $p \in M$.
2. If Black has to move and $\forall p \to p' \mid p' \in M$, then $p \in M$.

Theorem

- Every set U with properties 1 and 2 above contains the set of winning positions *in the strict sense* for White.
- The set of winning positions without repetitions has properties 1 and 2 above.

Consequently, each winning position allows victory without repetition.

The Zermelo-Kőnig-Kalmár Theorem

Let $\mathcal{G}$ be a two-player, zero-sum, perfect information game.

1. Every position p belongs to exactly one of the following.
   1.1 The set $\mathcal{G}_A$ of winning positions for the first player.
   1.2 The set $\mathcal{G}_B$ of winning positions for the second player.
   1.3 The set $\mathcal{G}_D$ of drawing positions.
2. For every $p \in \mathcal{G}_A$ there exists a winning strategy for the first player, depending only on $\mathcal{G}$.
3. For every $p \in \mathcal{G}_B$ there exists a winning strategy for the second player, depending only on $\mathcal{G}$.
4. For every $p \in \mathcal{G}_D$ there exists a non-losing strategy for each player, depending only on $\mathcal{G}$.

Note:

- No restriction on number of positions.
- No restriction on number of reachable positions either!

An Application to Concurrency Theory

Consider a family Proc of processes built on a set Act of actions.
Process P may perform action a and evolve into process P'.
$P, Q \in \mathrm{Proc}$ are bisimilar if there is a symmetric relation $\mathcal{R}$ such that

1. $P \mathcal{R} Q$, and
2. if $P \xrightarrow{a} P'$ then $\exists Q' \in \mathrm{Proc}$ s.t. $P' \mathcal{R} Q'$ and $Q \xrightarrow{a} Q'$.

Bisimilarity means "being able to simulate each other".

An Application to Concurrency Theory (cont.)

Consider the following game on Proc.

- There are two player, the attacker and the defender.
- Positions are pairs (P, Q) of processes.
- At each move:
  - The attacker performs a transition on a term, by some action.
  - The defender performs a transition on the other term, by the same action.
- The attacker wins if the defender cannot move.
- The defender wins if the attacker cannot move or the game is infinite.

An Application to Concurrency Theory (end)

Theorem

1. P and Q are bisimilar iff the defender has a winning strategy from (P, Q).
2. P and Q are not bisimilar iff the attacker has a winning strategy from (P, Q).

Proof.

- If P and Q are bisimilar, the defender can win by always choosing a term bisimilar to the one chosen by the attacker.
- If the defender has a winning strategy, define $\mathcal{R}$ according to that strategy.
- The vice versa follows immediately from ZKK theorem.

Urban Legend, Debunked

"White can always win, it's *mathematical!*"

- Until now, there is no such *mathematical* proof.
- Indeed, there is both favorable and contrary evidence.

"Zermelo proved that White always wins."

- No.
- That wasn't his main concern either!
  (In fact, it was von Neumann's.)

"Zermelo used backward analysis."

- No.
- Nor did Kőnig.
- Nor did Kalmár.

THANK YOU FOR ATTENTION!

Any questions?

Solutions to the riddle

Play two games with non-negative outcome.

- Set up two keyboards A and B with two different players.
- Play Black on A and White on B.
- When White on A moves, play same move on B.
- When Black on B moves, play same move on A.

Backward analysis problem.

- Black King originally in a7.
- White Knight in b6.
- Other pieces as in figure.
- Moves: Na8 (puts King under check by the Bishop in g1) Ka8.