Semantic spaces in Priestley form

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Semantics of programming languages:

is about developing techniques for designing and describing programming languages.

Semantics approaches include:

- axiomatic (the program logic) – an example is Hoare logic.

- operational – an example is Java Abstract Machine.

- denotational – gives mathematical meaning of language constructs.

Denotational semantics:

uses a category to interpret programming language constructs;

- data types $\Longleftrightarrow$ objects,
- programs $\Longleftrightarrow$ morphisms.

Restrictions on the category:

1. A map which assigns to every endomorphism f on an object M a point $m \in M$ such that $f(m) = m$ (a fix-point for f).
2. With every functor $G : A^{op} \times A \longrightarrow A$, there should exist an object M such that

$$G(M, M) \cong M.$$

Domains – Dana Scott (1969):

Sets, topological spaces, vectors spaces, and groups are not a good choice for denotational semantics.

Domains = ordered sets + certain conditions.

From now on:

- data types $\Longleftrightarrow$ domains,
- programs $\Longleftrightarrow$ functions between domains.

Scott topologies on domains to measure computability.

Scott topologies provides program logic (M. B. Smyth – 1983):

Based on geometric logic (logic of observable properties): Scott-open sets of a domain are interpreted as properties.

Suppose C is a continuous map (computable program) from a domain D_1 to a domain D_2. If P_2 is a property (a Scott-open subset) of D_2 then $P_1 := C^{-1}(P_2)$ is a property of D_1, by continuity of C.

Moreover, it is certain that if an input x to the program C satisfies P_1 then the output $C(x)$ will satisfy property P_2.

Stone duality
Marshall Harvey Stone (1936)

Totally disconnected compact spaces

$$\Updownarrow$$

Boolean algebras.

This was the starting point of a whole area of research known as Stone duality.

Dualities are generally good for translating problems form one space to another where it could be easier to solve.

Stone duality
Marshall Harvey Stone (1937)
Hillary Priestley (1970)

spectral spaces (T_0)

$\Updownarrow$ 1937

bounded distributive lattices.

$\Updownarrow$ 1970

Priestley spaces (Hausdorff)

Definition. *A* **Priestley space** *is a compact ordered space $\langle X; \mathcal{T}, \leq \rangle$ such that for every $x, y \in X$, if $x \ngeq y$ then there exists a clopen upper set U such that $y \in U$ and $x \notin U$.*

A **spectral space** is a stably compact space with a basis of compact open sets.

Stone duality and computer science
Samson Abramsky(1991)

Logical representation for bifinite domains (a particular Cartesian-closed category of domains).

In this framework,

- bifinite domains $\Longleftrightarrow$ propositional theories,
- functions $\Longleftrightarrow$ program logic axiomatising the properties of domains.

The domain interpretation via bifinite domains and the logical interpretation are Stone duals to each other and specify each other up to isomorphism.

Stably compact spaces

Abramsky's work was extended by Achim Jung et al to a class of topological spaces, stably compact spaces defined as follows.

Definition. *A stably compact space is a topological space which is sober, compact, locally compact, and for which the collection of compact saturated subsets is closed under finite intersections, where a saturated set is an intersection of open sets.*

These spaces contains coherent domains in their Scott topologies.

Coherent domains include bifinite domains and other interesting Cartesian-closed categories of domains such as FS.

Achim Jung's work in more detail

If $\langle X, \mathcal{T} \rangle$ is a stably compact space then its lattice $\mathcal{B}_X$ of observable properties is defined as follows:

$$\mathcal{B}_X = \{\langle O, K \rangle \mid O \in \mathcal{T}, K \in \mathcal{K}_X \text{ and } O \subseteq K\},$$

where $\mathcal{K}_X$ is the set of compact saturated subsets of X.

The computational interpretation is as follows. For a point $x \in X$ and a property $\langle O, K \rangle \in \mathcal{B}_X$:

- $x \in O \Longleftrightarrow x$ *satisfies* the property $\langle O, K \rangle$,
- $x \in X \setminus K \Longleftrightarrow x$ does not satisfy the property $\langle O, K \rangle$, and
- $x \in K \setminus O \Longleftrightarrow$ the property $\langle O, K \rangle$ is *unobservable for* x.

Proximity relation

On the lattice $\mathcal{B}_X$ of observable properties a binary relation (*strong proximity relation*)was defined as:

$$\langle O, K\rangle \prec \langle O', K'\rangle \overset{\text{def}}{\Longleftrightarrow} K \subseteq O'.$$

The computational interpretation of the strong proximity relation $\prec$ can be stated as follows:

$$\langle O, K\rangle \prec \langle O', K'\rangle$$

$$\Updownarrow$$

$(\forall x \in X)$ either $\langle O', K'\rangle$ is observably satisfied for x

or $\langle O, K\rangle$ is (observably) not satisfied for x.

Thus we can say that $\prec$ behaves like a classical implication.

$\mathcal{B}_X$ and $\prec$ abstractly:

Definition. *A binary relation* $\prec$ *on a bounded distributive lattice* $\langle L; \vee, \wedge, 0, 1\rangle$ *is called a* proximity *if, for every* $a, x, y \in L$ *and* $M \subseteq_{fin} L$*,*

$$
\begin{array}{ll}
(\prec\prec) & \prec \circ \prec = \prec, \\
(\vee\!-\prec) & M \prec a \iff \bigvee M \prec a, \\
(\prec -\wedge) & a \prec M \iff a \prec \bigwedge M, \\
(\prec -\vee) & a \prec x \vee y \Longrightarrow (\exists\ x', y' \in L)\ x' \prec x,\ y' \prec y \\
 & \quad \text{and } a \prec x' \vee y', \\
(\wedge\!-\prec) & x \wedge y \prec a \Longrightarrow (\exists\ x', y' \in L)\ x \prec x',\ y \prec y' \\
 & \quad \text{and } x' \wedge y' \prec a.
\end{array}
$$

A **strong proximity lattice** *is a bounded distributive lattice* $\langle L; \vee, \wedge, 0, 1\rangle$ *together with a proximity relation* $\prec$ *on* L*.*

The lattice order is always a proximity relation.

Approximable relations: Capturing continuous maps between stably compact spaces

Definition. *Let* $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ *and* $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ *be strong proximity lattices and let* $\vdash$ *be a binary relation from* L_1 *to* L_2*. The relation* $\vdash$ *is called* approximable *if for every* $a \in L_1, b \in L_2$*,* $M_1 \subseteq_{fin} L_1$ *and* $M_2 \subseteq_{fin} L_2$*,*

$$
\begin{array}{rl}
(\vdash - \prec_2) & \vdash \circ \prec_2 = \vdash, \\
(\prec_1 - \vdash) & \prec_1 \circ \vdash = \vdash, \\
(\vee - \vdash) & M_1 \vdash b \Longleftrightarrow \bigvee M_1 \vdash b, \\
(\vdash - \wedge) & a \vdash M_2 \Longleftrightarrow a \vdash \bigwedge M_2, \\
(\vdash - \vee) & a \vdash \bigvee M_2 \Longrightarrow (\exists\ N \subseteq_{fin} L_1)\ a \prec_1 \bigvee N \\
 & \quad \text{and } (\forall n \in N)(\exists\ m \in M_2)\ n \vdash m.
\end{array}
$$

MLS:
Logical representation of stably compact spaces

Definition. *Let* $\langle A; \vee, \wedge, \top, \bot\rangle$ *and* $\langle B; \vee, \wedge, \top, \bot\rangle$ *be two algebras of type* $\langle 2, 2, 0, 0\rangle$*. A binary relation* $\vdash$ *from finite subsets of* A *to those of* B *is a consequence relation if for every* $\phi, \psi \in A, \Gamma, \Gamma' \subseteq_{fin} A, \phi', \psi' \in B$ *and* $\Delta, \Delta' \subseteq_{fin} B$*,*

$(L\bot)$ $(\forall\ \Theta \subseteq_{fin} B)\ \bot \vdash \Theta$.

$(L\top)$ $\Gamma \vdash \Delta \iff \top, \Gamma \vdash \Delta$.

$(L\wedge)$ $\phi, \psi, \Gamma \vdash \Delta \iff \phi \wedge \psi, \Gamma \vdash \Delta$.

$(L\vee)$ $\phi, \Gamma \vdash \Delta$ *and* $\psi, \Gamma \vdash \Delta \iff \phi \vee \psi, \Gamma \vdash \Delta$.

$(R\bot)$ $\Gamma \vdash \Delta \iff \Gamma \vdash \Delta, \bot$.

$(R\top)$ $(\forall\ \Theta \subseteq_{fin} A)\ \Theta \vdash \top$.

$(R\wedge)$ $\Gamma \vdash \Delta, \phi'$ *and* $\Gamma \vdash \Delta, \psi' \Longleftrightarrow \Gamma \vdash \Delta, \phi' \wedge \psi'$.

$(R\vee)$ $\Gamma \vdash \Delta, \phi', \psi' \Longleftrightarrow \Gamma \vdash \Delta, \phi' \vee \psi'$.

(W) $\Gamma \vdash \Delta \Longrightarrow \Gamma', \Gamma \vdash \Delta, \Delta'$.

A **sequent calculus** is an algebra $\langle A; \vee, \wedge, \top, \bot \rangle$ together with a consequence relation $\Vdash$ on A.

Sequent calculi and **consequence relations** are, respectively, the objects and morphisms of **MLS**.

Aim of my PhD work

The primary aim is to introduce Priestley spaces to the world of semantics of programming languages.

This can be done by answering the following question:

How can Priestley duality for bounded distributive lattices be extended to strong proximity lattices?

Logically the answer is interesting because *theories* (or *models*) of $\mathcal{B}_X$ are represented by prime filters, which are the points of the Priestley dual space of $\mathcal{B}_X$ as a bounded distributive lattice.

Apartness relations:

To answer the question we equip Priestley spaces with the following relation:

Definition. *A binary relation* $\propto$ *on a Priestley space* $\langle X; \leq, \mathfrak{T}\rangle$ *is called an* apartness *if, for every* $a, c, d, e \in X$,

($\propto\mathfrak{T}$) $\propto$ *is open in* $\langle X; \mathfrak{T}\rangle \times \langle X; \mathfrak{T}\rangle$

($\downarrow\propto\uparrow$) $a \leq c \propto d \leq e \Longrightarrow a \propto e$,

($\propto\forall$) $a \propto c \Longleftrightarrow (\forall b \in X)\ a \propto b$ *or* $b \propto c$,

($\propto\uparrow\uparrow$) $a \propto ({\uparrow}c \cap {\uparrow}d) \Longrightarrow (\forall b \in X)\ a \propto b,\ b \propto c$ *or* $b \propto d$,

($\downarrow\downarrow\propto$) $({\downarrow}c \cap {\downarrow}d) \propto a \Longrightarrow (\forall b \in X)\ d \propto b,\ c \propto b$ *or* $b \propto a$.

The relation $\not\geq$ is always an apartness.

The answer is:

The dual of a strong proximity lattice L is the corresponding Priestley space of prime filters, equipped with the apartness,

$$F \propto_{\prec} G \overset{\text{def}}{\iff} (\exists\ x \in F)(\exists\ y \notin G)\ x \prec y.$$

Vice versa, the dual of a Priestley space X with apartness $\propto$ is the lattice of clopen upper sets equipped with the strong proximity,

$$A \prec_{\propto} B \overset{\text{def}}{\iff} A \propto (X \setminus B).$$

Up to isomorphism, the correspondence is one-to-one.

Concerning the morphisms...

We proof that:

Continuous order-preserving maps that reflect the apartness relation are in one-to-one correspondence with lattice homomorphisms that preserve the strong proximity relation.

Let X_1 and X_2 be Priestley spaces with apartness relation. Then (weakly) separating relations from X_1 to X_2 are in one-to-one correspondence with (weakly) approximable relations from the dual of X_1 to the dual of X_2.

Separating relations:

Definition. *Let* $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ *and* $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ *be Priestley spaces with apartness relations* $\propto_1$ *and* $\propto_2$*, respectively, and let* $\ltimes$ *be a binary relation from* X_1 *to* X_2*. The relation* $\ltimes$ *is called* separating *(or a* separator*) if it is open in* $\mathcal{T}_1 \times \mathcal{T}_2$ *and if, for every* $a, b \in X_1, d, e \in X_2$ *and* $\{d_i \mid 1 \leq i \leq n\} \subseteq X_2$*,*

$$
\begin{aligned}
(\downarrow_1\ltimes\uparrow_2) &\quad a \geq_1 b \ltimes d \geq_2 e \Longrightarrow a \ltimes e,\\
(\forall\ltimes) &\quad b \ltimes d \Longleftrightarrow (\forall c \in X_1)\ b \propto_1 c \text{ or } c \ltimes d,\\
(\ltimes\forall) &\quad b \ltimes d \Longleftrightarrow (\forall c \in X_2)\ b \ltimes c \text{ or } c \propto_2 d,\\
(\ltimes n\uparrow) &\quad b \ltimes \bigcap {\downarrow} d_i \Longrightarrow (\forall c \in X_1)\ b \propto_1 c\\
&\qquad \text{or } (\exists\ i)\ c \ltimes d_i.
\end{aligned}
$$

The relation $\ltimes$ *is called* weakly separating (or weak separator) *if it satisfies all of the above conditions, but not necessarily* $(\ltimes n\uparrow)$*.*

Priestley and stably compact spaces

What is the direct relationship between the Priestley spaces equipped with apartness relations stably compact spaces?

The answer is the following:

Theorem. *Let* $\langle X; \leq, \mathfrak{T}\rangle$ *be a Priestley space with apartness* $\propto$*. Then* $\langle core(X), \mathfrak{T}'\rangle$*, where*

$$core(X) = \{x \in X \mid \{y \in X \mid x \propto y\} = X \setminus {\downarrow}x\}$$

and

$$\mathfrak{T}' = \{O \cap core(X) \mid O \text{ is an open lower subset of } X\},$$

is a stably compact space.

Moreover, every stably compact space can be obtained in this way and is a retract of a Priestley space with apartness.

Concerning morphisms again ...

We show that continuous maps between stably compact spaces are equivalent to separators between Priestley spaces equipped with apartness.

More results...

We present direct functors between the category **MLS** *and the category* **PSws** *of Priestley spaces with apartness and weak separators. These functors are then used to prove the equivalence of these categories.*

We introduce Priestley semantics (in **PSws***) for* **MLS***'s concepts and facts such as compatibility, Gentzen's cut rule, round ideals and filters, and consistency.*

More results (2)

We show how some domain constructions such as lifting, sum, product, and Smyth power domain can be done in the Priestley form:

Product: Suppose $\langle Y_1, \mathcal{T}_{Y_1}\rangle$ and $\langle Y_2, \mathcal{T}_{Y_2}\rangle$ are stably compact spaces and $\langle X_1; \mathcal{T}_{X_1}, \leq_1\rangle$ and $\langle X_2; \mathcal{T}_{X_2}, \leq_2\rangle$ are Priestley spaces equipped with apartness relations $\propto_1$ and $\propto_2$ respectively. Let $S_1[\![\cdot]\!] : \mathit{core}(X_1) \longrightarrow Y_1$ and $S_2[\![\cdot]\!] : \mathit{core}(X_2) \longrightarrow Y_2$ be homeomorphic maps.

Then for the Priestley spaces $\langle X_1 \times X_2; \mathcal{T}_1 \times \mathcal{T}_2\rangle$ equipped with the apartness:

$$\langle x, y\rangle \propto \langle x', y'\rangle \overset{\text{def}}{\Longleftrightarrow} x \propto_1 x' \text{ or } y \propto_2 y'.$$

the map

$$S[\![\cdot]\!] : \mathit{core}(X) \longrightarrow Y_1 \times Y_2; \langle x, y\rangle \longmapsto \langle S_1[\![x]\!], S_2[\![y]\!]\rangle.$$

is a homeomorphism.

Thanks for your attention!