Cut-elimination and Proof-search for Bi-Intuitionistic Logic Using Nested Sequents

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Introduction: Sequent Calculus and Proof Search

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  - Forcing of compound formulae defined w.r.t. $\leq$

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- $p \vdash q, r \to ((p \mathrel{-\!\!<} q) \wedge r)$ is not cut-free derivable in Rauszer's G1
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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Id} \qquad q \vdash ?}{p, r \vdash p \mathrel{-\!\!<} q}\,\mathrel{-\!\!<}_R \qquad \dfrac{}{p, r \vdash r}\,\text{Id}}{p, r \vdash (p \mathrel{-\!\!<} q) \wedge r}\,\wedge_R}{p \vdash r \to ((p \mathrel{-\!\!<} q) \wedge r)}\,\to_R}{p \vdash q, r \to ((p \mathrel{-\!\!<} q) \wedge r)}\,w_R$$

- Derivation using cut:

$$\dfrac{\dfrac{\dfrac{}{p \vdash q, p}\,\text{Id} \qquad \dfrac{}{q \vdash q}\,\text{Id}}{p \vdash q, p \mathrel{-\!\!<} q}\,\mathrel{-\!\!<}_R \qquad \dfrac{\dfrac{\dfrac{}{p \mathrel{-\!\!<} q, r \vdash p \mathrel{-\!\!<} q}\,\text{Id} \qquad \dfrac{}{p \mathrel{-\!\!<} q, r \vdash r}\,\text{Id}}{p \mathrel{-\!\!<} q, r \vdash (p \mathrel{-\!\!<} q) \wedge r}\,\wedge_R}{p \mathrel{-\!\!<} q \vdash r \to ((p \mathrel{-\!\!<} q) \wedge r)}\,\to_R}{p \vdash q, r \to ((p \mathrel{-\!\!<} q) \wedge r)}\,\text{cut}$$

Uustalu's Example: Semantic Motivation

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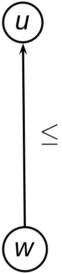

$u \Vdash p$

$u \Vdash r$

$u \nVdash (p \mathbin{-\!\!<} q) \wedge r$

$w \Vdash p$

$w \nVdash q$

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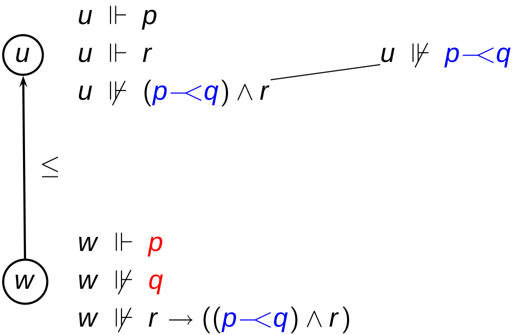

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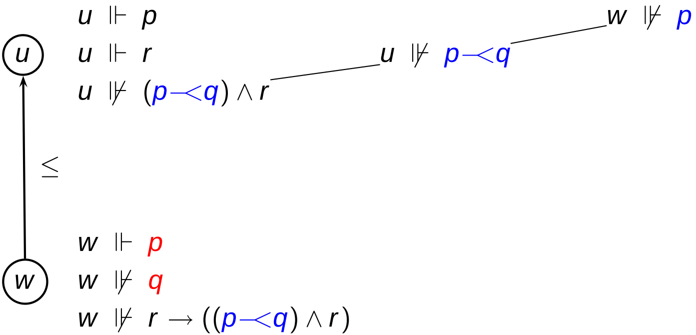

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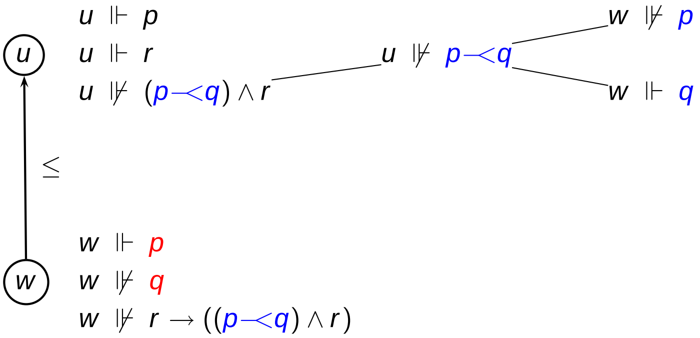

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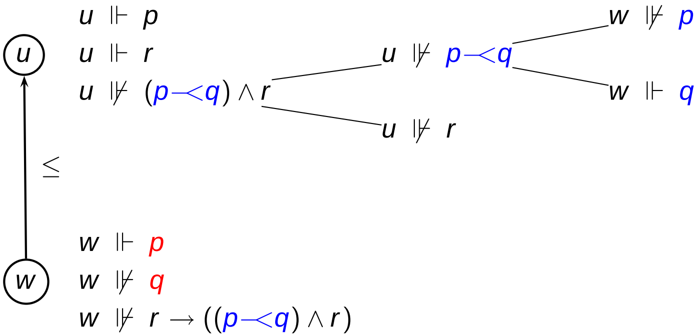

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- Examples:
  - $r \vdash p {-\!\!<} q$
  - $(p < q), r \vdash (q > r), ((p < q) > w)$

Identity, cut and structural rules

Identity and cut:

$$\frac{}{X, A \vdash A, Y}\ id \qquad \frac{X_1 \vdash Y_1, A \quad A, X_2 \vdash Y_2}{X_1, X_2 \vdash Y_1, Y_2}\ cut$$

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Structural rules:

$$\frac{X \vdash Y}{X, A \vdash Y}\ w_L \qquad \frac{X \vdash Y}{X \vdash A, Y}\ w_R \qquad \frac{X, A, A \vdash Y}{X, A \vdash Y}\ c_L \qquad \frac{X \vdash A, A, Y}{X \vdash A, Y}\ c_R$$

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$$\frac{(X_1 < Y_1), X_2 \vdash Y_2}{X_1, X_2 \vdash Y_1, Y_2}\ s_L \qquad \frac{X_1 \vdash Y_1, (X_2 > Y_2)}{X_1, X_2 \vdash Y_1, Y_2}\ s_R$$

$$\frac{X_2 \vdash Y_2, Y_1}{X_1, (X_2 < Y_2) \vdash Y_1}\ < \qquad \frac{X_1, X_2 \vdash Y_2}{X_1 \vdash Y_1, (X_2 > Y_2)}\ >$$

Logical rules

Logical rules

$$\frac{X \vdash A, Y \quad X, B \vdash Y}{X, A \rightarrow B \vdash Y} \rightarrow_L \qquad \frac{X, A \vdash B}{X \vdash Y, A \rightarrow B} \rightarrow_R$$

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$$\frac{A \vdash B, Y}{X, A \prec B \vdash Y} \prec_L \qquad \frac{X \vdash A, Y \quad X, B \vdash Y}{X \vdash A \prec B, Y} \prec_R$$

Uustalu's Example Revisited

Using cut:

$$\dfrac{\dfrac{\dfrac{}{p \vdash q, p}\;Id \qquad \dfrac{}{q \vdash q}\;Id}{p \vdash q, p \mathrel{-\!\!<} q}\;{\mathrel{-\!\!<}}_R \qquad \dfrac{\dfrac{\dfrac{}{p \mathrel{-\!\!<} q, r \vdash p \mathrel{-\!\!<} q}\;Id \qquad \dfrac{}{p \mathrel{-\!\!<} q, r \vdash r}\;Id}{p \mathrel{-\!\!<} q, r \vdash (p \mathrel{-\!\!<} q) \wedge r}\;\wedge_R}{p \mathrel{-\!\!<} q \vdash r \to ((p \mathrel{-\!\!<} q) \wedge r)}\;\to_R}{p \vdash q, r \to ((p \mathrel{-\!\!<} q) \wedge r)}\;\text{cut}$$

Using LBiInt_1 without cut:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash q, p}\;Id \qquad \dfrac{}{p, q \vdash q}\;Id}{p \vdash q, p \mathrel{-\!\!<} q}\;{\mathrel{-\!\!<}}_R}{(p < q), r \vdash p \mathrel{-\!\!<} q}\;< \qquad \dfrac{}{(p < q), r \vdash r}\;Id}{(p < q), r \vdash (p \mathrel{-\!\!<} q) \wedge r}\;\wedge_R}{p < q \vdash r \to ((p \mathrel{-\!\!<} q) \wedge r)}\;\to_R}{p \vdash q, r \to ((p \mathrel{-\!\!<} q) \wedge r)}\;s_L$$

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- Completeness: by embedding Rauszer's G1 into LBiInt_1

General Contraction and Weakening

Lemma

Contraction and weakening on structures admissible:

$$\frac{X , Y , Y \vdash Z}{X , Y \vdash Z}\ gc_L \qquad \frac{X \vdash Y , Y , Z}{X \vdash Y , Z}\ gc_R$$

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Proof.

By induction on the size of Y. □

Atomic Cuts

We transform

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\text{id} \quad \cdots \quad \dfrac{}{p \vdash p}\text{id}}{X_1 \vdash Y_1, p}\,\vdots\,\theta \qquad \begin{matrix}\pi \\ p, X_2 \vdash Y_2\end{matrix}}{X_1, X_2 \vdash Y_1, Y_2}\text{cut}$$

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into:

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{p, X_2 \vdash Y_2}}{p \vdash X_2 > Y_2} > \quad \cdots \quad \dfrac{\overset{\pi}{p, X_2 \vdash Y_2}}{p \vdash X_2 > Y_2} >}{X_1 \vdash Y_1, (X_2 > Y_2)}\,\vdots\,\theta[p/X_2 > Y_2]}{X_1, X_2 \vdash Y_1, Y_2}s_R$$

General Cuts: $A \to B$

We transform

$$
\dfrac{
\begin{array}{c}
\pi_1 \\
\dfrac{X_1', A \vdash B}{X_1' \vdash Y_1', A \to B}\,{\to_R} \\
\vdots\;\theta_1 \\
X_1 \vdash Y_1, A \to B
\end{array}
\qquad
\begin{array}{c}
\dfrac{\begin{array}{cc}\pi_2 & \pi_3 \\ X_2' \vdash A, Y_2' & B, X_2' \vdash Y_2'\end{array}}{A \to B, X_2' \vdash Y_2'}\,{\to_L} \\
\vdots\;\theta_2 \\
A \to B, X_2 \vdash Y_2
\end{array}
}{X_1, X_2 \vdash Y_1, Y_2}\;\text{cut}
$$

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$$
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\pi_1 \\ X_1', A \vdash B\end{array}}{X_1' \vdash Y_1', A \rightarrow B} \rightarrow_R \\
\vdots\ \theta_1 \\
X_1 \vdash Y_1, A \rightarrow B
\end{array}
\qquad
\begin{array}{c}
\dfrac{\begin{array}{c}\pi_2 \\ X_2' \vdash A, Y_2'\end{array} \qquad \begin{array}{c}\pi_3 \\ B, X_2' \vdash Y_2'\end{array}}{A \rightarrow B, X_2' \vdash Y_2'} \rightarrow_L \\
\vdots\ \theta_2 \\
A \rightarrow B, X_2 \vdash Y_2
\end{array}
}{X_1, X_2 \vdash Y_1, Y_2} \text{ cut}
$$

into:

$$
\dfrac{
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_2 \\ X_2' \vdash A, Y_2'\end{array} \qquad \dfrac{\begin{array}{c}\pi_1 \\ X_1', A \vdash B\end{array} \qquad \begin{array}{c}\pi_3 \\ B, X_2' \vdash Y_2'\end{array}}{X_1', A, X_2' \vdash Y_2'} \text{ cut}}{X_1', X_2', X_2' \vdash Y_2', Y_2'} \text{ cut}}{X_1', X_2' \vdash Y_2'} \ gc_L, gc_R \\
\vdots\ \theta_2[A \rightarrow B / X_1'] \\
\dfrac{X_1', X_2 \vdash Y_2}{X_1' \vdash Y_1', (X_2 > Y_2)} > \\
\vdots\ \theta_1[A \rightarrow B / X_2 > Y_2] \\
X_1 \vdash Y_1, (X_2 > Y_2)
\end{array}
}{X_1, X_2 \vdash Y_1, Y_2} \ s_R
$$

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$$\cfrac{\cfrac{\cfrac{(X < Y, A \to B), X, A \vdash B}{(X < Y, A \to B), X \vdash Y, A \to B}\ \to_R}{X, X \vdash Y, Y, A \to B, A \to B}\ s_L}{X \vdash Y, A \to B}\ gc_L, gc_R \quad \rightsquigarrow \quad \cfrac{(X < Y, A \to B), \{X\}, A \vdash B}{X \vdash Y, A \to B}\ \to_{R2}$$

$\{X\} = \{A \mid X = (A, Y)$ for some A and $Y\}$

LBiInt$_2$ Rules

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$$\frac{X, A \to B \vdash A, Y \quad X, A \to B, B \vdash Y}{X, A \to B \vdash Y} \to_L \qquad \frac{X \vdash Y, A \to B, B}{X \vdash Y, A \to B} \to_{R1}$$

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$$\frac{X, A \prec B, A \vdash Y}{X, A \prec B \vdash Y}\ \prec_{L1} \qquad \frac{X \vdash A, A \prec B, Y \quad X, B \vdash A \prec B, Y}{X \vdash A \prec B, Y}\ \prec_R$$

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$$\frac{A \vdash B, \{Y\}, (X, A \prec B > Y)}{X, A \prec B \vdash Y} \prec_{L2} \qquad \frac{(X < Y, A \to B), \{X\}, A \vdash B}{X \vdash Y, A \to B} \to_{R2}$$

Uustalu's Example Revisited

Using LBiInt$_1$:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{p \vdash q, p}\,Id
          \qquad
          \dfrac{}{p, q \vdash q}\,Id
        }{p \vdash q, p \mathrel{-\!\!<} q}\,{-\!\!<}_R
      }{(p < q), r \vdash p \mathrel{-\!\!<} q}\,<
      \qquad
      \dfrac{}{(p < q), r \vdash r}\,Id
    }{(p < q), r \vdash (p \mathrel{-\!\!<} q) \wedge r}\,\wedge_R
  }{p < q \vdash r \to ((p \mathrel{-\!\!<} q) \wedge r)}\,\to_R
}{p \vdash q, r \to ((p \mathrel{-\!\!<} q) \wedge r)}\,s_L
$$

Using LBiInt$_2$:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{p \vdash q, \cdots, p}\,Id
        \qquad
        \dfrac{}{p, q \vdash q, \cdots}\,Id
      }{p \vdash q, \cdots, p \mathrel{-\!\!<} q}\,{-\!\!<}_R
    }{(p < q, \cdots), p, r \vdash p \mathrel{-\!\!<} q}\,<
    \qquad
    \dfrac{}{(p < q, \cdots), p, r \vdash r}\,Id
  }{(p < q, \cdots), p, r \vdash (p \mathrel{-\!\!<} q) \wedge r}\,\wedge_R
}{p \vdash q, r \to ((p \mathrel{-\!\!<} q) \wedge r)}\,\to_{R2}
$$

Save/Restore

- LBiInt$_1$ vs LBiInt$_2$:

Lose context:

$$\frac{X, A \vdash B}{X \vdash Y, A \rightarrow B} \rightarrow_R$$

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- Restore context:

$$\frac{X_2 \vdash Y_2, \{Y_1\}}{X_1, (X_2 < Y_2) \vdash Y_1} < \quad \{Y_1\} \not\subseteq \{Y_2\}$$

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- LBiInt$_2$ completeness via translation from GBiInt
- GBiInt recompute rule $\rightsquigarrow$ pair of LBiInt$_2$ save/restore rules

Saturation

Definition

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6. If $A \rightarrow B \in \{X\}$ then $A \in \{Y\}$ or $B \in \{X\}$
7. If $A {-\!\!<} B \in \{Y\}$ then $A \in \{Y\}$ or $B \in \{X\}$
8. If $A \rightarrow B \in \{Y\}$ then $B \in \{Y\}$
9. If $A {-\!\!<} B \in \{X\}$ then $A \in \{X\}$

Blocking

Definition

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Definition

A LBiInt$_2$ rule ρ is applicable to a sequent $\gamma_0 = X_0 \vdash Y_0$ if for every premise $X_i \vdash Y_i$ of ρ, $\{X_i\} \not\subseteq \{X_0\}$ or $\{Y_i\} \not\subseteq \{Y_0\}$.

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Corollary

Only jump and return rules are applicable to saturated sequents.

Proof Search Strategy

Function Prove
Input: sequent γ_0
Output: *true* (i.e. γ_0 is derivable) or *false* (i.e. γ_0 is not derivable)

1. If *id* is applicable to γ_0 then return *true*

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4. Else return *false*.

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Example

$C \vdash B, A \rightarrow B$

$(C < B, A \rightarrow B), C, A \vdash B$

$D \vdash E, ((C < B, A \rightarrow B), C, A > B)$

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$C \vdash B, A \to B$

$(C < B, A \to B), C, A \vdash B$

$D \vdash E, ((C < B, A \to B), C, A > B)$

Lemma

Every LBiInt$_2$-derivation of a linear end-sequent contains only linear sequents.

Lists

Definition (Linear Sequent to List)

$$
\begin{array}{lcl}
list(\Gamma \vdash \Delta) & = & \langle \Gamma, \Delta \rangle \\
list((X < Y), \Gamma \vdash \Delta) & = & list(X \vdash Y) \leq \langle \Gamma, \Delta \rangle \\
list(\Gamma \vdash \Delta, (X > Y)) & = & list(X \vdash Y) \geq \langle \Gamma, \Delta \rangle
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$$list(C \vdash B, A \rightarrow B) = \langle \{C\}, \{B, A \rightarrow B\} \rangle$$

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list((C < B, A \rightarrow B), C, A \vdash B) & = & list(C \vdash B, A \rightarrow B) \leq \langle \{C, A\}, \{B\} \rangle \\
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Remark

Lists encoded in sequents $\rightsquigarrow$ *branches in counter-model.*

List Operations

Corollary

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Example

$$\frac{(C < B, A \to B), C, A \vdash B}{C \vdash B, A \to B} \to_{R2} \qquad \frac{\langle \{C\}, \{B, A \to B\} \rangle \;\; \leq \;\; \langle \{C, A\}, \{B\} \rangle}{\langle \{C\}, \{B, A \to B\} \rangle}$$

Termination

Lemma (Bounded Lists)

Let $X \vdash Y$ be any sequent encountered during proof search. Using jump rules, list$(X \vdash Y)$ can be extended at most $\mathcal{O}(m^2)$ times.

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  - Eventually no subformulae can be added to any node
  - Return rules are blocked

Countermodel Construction

- Want to show: if A valid then $\emptyset \vdash A$ derivable

Countermodel Construction

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  - Then we have a proper counter-model

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- By contrapositive, if A is valid, then $\mathit{Prove}(\emptyset \vdash A) = \mathit{true}$

Restart Example

$$
\dfrac{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0 \qquad \vdots}{(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot} \to_L
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$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0 \qquad \vdots$$
$$\to_L$$
$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot$$
$$\to_{R2}$$
$$\vdots \qquad p_0 \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0$$
$$\prec_R$$
$$\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0$$
$$<$$
$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0 \qquad \vdots$$
$$\to_L$$
$$(< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot$$
$$\to_{R2}$$
$$\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot$$
$$\to_{R1}$$
$$\vdash p_1, ((\top \prec p_0) \to \bot) \to \bot$$

$I \quad \langle \{(\top \prec p_0) \to \bot\}, \{\bot\} \rangle$

$\uparrow \; \leq$

$C \quad \langle \{\}, \{p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot\} \rangle$

Restart Example

$$
\dfrac{
  \vdots
  \qquad
  \dfrac{
    \dfrac{
      (< p_1, ((\top{-\!\!<}p_0) \to \bot) \to \bot, \bot, \top{-\!\!<}p_0), p_0, (\top{-\!\!<}p_0) \to \bot \vdash \bot, \top{-\!\!<}p_0 \qquad \vdots
    }{
      (< p_1, ((\top{-\!\!<}p_0) \to \bot) \to \bot, \bot, \top{-\!\!<}p_0), p_0, (\top{-\!\!<}p_0) \to \bot \vdash \bot
    }\ {\to_L}
  }{
    p_0 \vdash p_1, ((\top{-\!\!<}p_0) \to \bot) \to \bot, \bot, \top{-\!\!<}p_0
  }\ {\to_{R2}}
}{
  \vdash p_1, ((\top{-\!\!<}p_0) \to \bot) \to \bot, \bot, \top{-\!\!<}p_0
}\ {-\!\!<_R}
$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \vdash p_1, ((\top{-\!\!<}p_0) \to \bot) \to \bot, \bot, \top{-\!\!<}p_0
      }{
        \color{blue}{(< p_1, ((\top{-\!\!<}p_0) \to \bot) \to \bot, \bot), (\top{-\!\!<}p_0) \to \bot \vdash \bot, \top{-\!\!<}p_0}
      }\ {<} \qquad \vdots
    }{
      (< p_1, ((\top{-\!\!<}p_0) \to \bot) \to \bot, \bot), (\top{-\!\!<}p_0) \to \bot \vdash \bot
    }\ {\to_L}
  }{
    \vdash p_1, ((\top{-\!\!<}p_0) \to \bot) \to \bot, \bot
  }\ {\to_{R2}}
}{
  \vdash p_1, ((\top{-\!\!<}p_0) \to \bot) \to \bot
}\ {\to_{R1}}
$$

I $\quad \langle \{(\top{-\!\!<}p_0) \to \bot\}, \{\bot, \top{-\!\!<}p_0\} \rangle$

$\uparrow \ \leq$

C $\quad \langle \{\}, \{p_1, ((\top{-\!\!<}p_0) \to \bot) \to \bot, \bot\} \rangle$

Restart Example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \vdots
      \qquad
      \dfrac{
        \dfrac{
          \dfrac{
            (< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0), p_0, (\top -\!\!< p_0) \to \bot \vdash \bot, \top -\!\!< p_0
            \qquad \vdots
          }{
            (< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0), p_0, (\top -\!\!< p_0) \to \bot \vdash \bot
          }\ \to_L
        }{
          p_0 \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0
        }\ \to_{R2}
      }{}
    }{
      \color{blue}{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0}
    }\ -\!\!<_R
  }{
    (< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot, \top -\!\!< p_0
  }\ <
  \qquad \vdots
}{
  \dfrac{
    \dfrac{
      (< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot
    }{
      \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot
    }\ \to_{R2}
  }{
    \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot
  }\ \to_{R1}
}\ \to_L
$$

(I) $\langle \{\}, \{p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0\} \rangle$

Restart Example

$$\dfrac{(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0), p_0, (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot, \top \mathrel{-\!\!<} p_0 \qquad \vdots}{(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0), p_0, (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot} \to_L$$

$$\dfrac{(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0), p_0, (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot}{p_0 \vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0} \to_{R2}$$

$$\dfrac{\vdots \qquad p_0 \vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0}{\vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0} \mathrel{-\!\!<}_R$$

$$\dfrac{\vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0}{(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot), (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot, \top \mathrel{-\!\!<} p_0} <$$

$$\dfrac{(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot), (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot, \top \mathrel{-\!\!<} p_0 \qquad \vdots}{(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot), (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot} \to_L$$

$$\dfrac{\dfrac{(< p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot), (\top \mathrel{-\!\!<} p_0) \to \bot \vdash \bot}{\vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot} \to_{R2}}{\vdash p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot} \to_{R1}$$

(I) $\langle \{p_0\}, \{p_1, ((\top \mathrel{-\!\!<} p_0) \to \bot) \to \bot, \bot, \top \mathrel{-\!\!<} p_0\} \rangle$

Restart Example

$$\dfrac{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0), p_0, (\top -\!\!< p_0) \to \bot \vdash \bot, \top -\!\!< p_0 \qquad \vdots}{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0), p_0, (\top -\!\!< p_0) \to \bot \vdash \bot} \to_L$$

$$\dfrac{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0), p_0, (\top -\!\!< p_0) \to \bot \vdash \bot}{p_0 \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0} \to_{R2}$$

$$\dfrac{\vdots \qquad p_0 \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0}{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0} -\!\!<_R$$

$$\dfrac{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0}{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot, \top -\!\!< p_0} <$$

$$\dfrac{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot, \top -\!\!< p_0 \qquad \vdots}{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot} \to_L$$

$$\dfrac{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot}{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot} \to_{R2}$$

$$\dfrac{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot}{\vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot} \to_{R1}$$

$$I \quad \langle \{p_0, (\top -\!\!< p_0) \to \bot\}, \{\bot\} \rangle$$

$$C \leq I$$

$$C \quad \langle \{p_0\}, \{p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0\} \rangle$$

Restart Example

$$
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    \vdots
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      \dfrac{
        \dfrac{
          \color{blue}{(< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0), p_0, (\top -\!\!< p_0) \to \bot \vdash \bot, \top -\!\!< p_0}
          \quad \vdots
        }{
          (< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0), p_0, (\top -\!\!< p_0) \to \bot \vdash \bot
        }\ \to_L
      }{
        p_0 \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0
      }\ \to_{R2}
    }{}
  }{
    \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0
  }\ -\!\!<_R
}{}
$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0
      }{
        (< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot, \top -\!\!< p_0
      }\ <
      \quad \vdots
    }{
      (< p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot), (\top -\!\!< p_0) \to \bot \vdash \bot
    }\ \to_L
  }{
    \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot
  }\ \to_{R2}
}{
  \vdash p_1, ((\top -\!\!< p_0) \to \bot) \to \bot
}\ \to_{R1}
$$

I : $\langle \{p_0, (\top -\!\!< p_0) \to \bot\}, \{\bot, \top -\!\!< p_0\} \rangle$

$\uparrow \leq$

C : $\langle \{p_0\}, \{p_1, ((\top -\!\!< p_0) \to \bot) \to \bot, \bot, \top -\!\!< p_0\} \rangle$

Restart Example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \vdots
      \qquad
      \dfrac{
        \dfrac{
          (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0
          \qquad \vdots
        }{
          (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0), p_0, (\top \prec p_0) \to \bot \vdash \bot
        }\to_L
      }{
        p_0 \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0
      }\to_{R2}
    }{
      \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0
    }\prec_R
  }{
    (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot, \top \prec p_0
  }<
  \qquad \vdots
}{
  \dfrac{
    \dfrac{
      (< p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot), (\top \prec p_0) \to \bot \vdash \bot
    }{
      \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot
    }\to_{R2}
  }{
    \vdash p_1, ((\top \prec p_0) \to \bot) \to \bot
  }\to_{R1}
}\to_L
$$

C $\quad \langle \{p_0, (\top \prec p_0) \to \bot\}, \{\bot, \top \prec p_0\} \rangle$

$\uparrow \; \leq$

C $\quad \langle \{p_0\}, \{p_1, ((\top \prec p_0) \to \bot) \to \bot, \bot, \top \prec p_0\} \rangle$

Conclusion and Further Work

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  - Bridge gap between $LBiInt_1$ and $LBiInt_2$
  - Generalise LBiInt to other similar logics (KtS4, S5, bi-Lambek, $\cdots$)
  - Explore connection with type theory