A categorical model of the Fusion Calculus

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Adding the missing notion of state to behaviour functors obtained from nice labelled transition systems, in order to get a working categorical semantics

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How to uglify your semantics – but only a bit, and for good reasons

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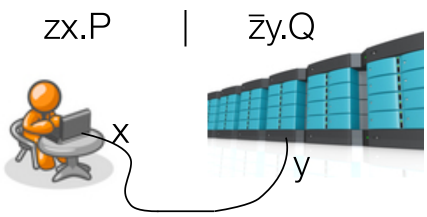
Handshakes, negotiations

- Distributed programming often involve *negotiations, handshakes…*
    - “Connect port 127.0.0.1:26451 to port 163.10.72.61:25”   (TCP)
    - “let us choose a cypher supported by both” (SSL, IPSec)
    - “we will use the following encoding”
- Basic concept: **communication via unification**
- Quite different from usual one-way communication (like CCS and π-calculus)
- In this talk: a categorical model for the Fusion calculus
- Methodology is more general

4

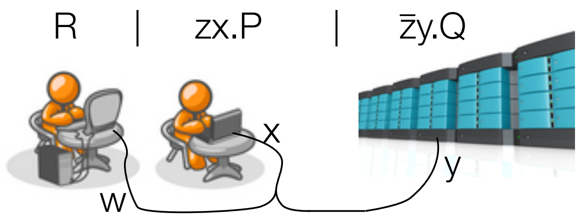
Communication via unification



- **Symmetric**: unification affects *both* agents (not only one side)
- P and Q keep using x and y for the same “wire” (i.e., substitution is not (always) performed)
- Action is **locally performed** by two agents

5

Communication via unification



- **Symmetric**: unification affects *both* agents (not only one side)
- P and Q keep using x and y for the same “wire” (i.e., substitution is not (always) performed)
- Action is **locally performed** by two agents, but has **global effect**

6

The Fusion Calculus (Parrow and Victor, 1998)

- **Syntax** (up-to alpha-equivalence):

x is bound

$$P, Q ::= \mathbf{0} \mid zx.P \mid \bar{z}x.P \mid P|Q \mid (x)P$$

- **Semantics**: labelled transitions of the form

fusion: "from now on, consider x and y equal"

$$\alpha ::= xy \mid \bar{x}y \mid x(z) \mid \bar{x}(z) \mid \mathbf{1} \mid x{=}y$$

- transitions **do not** keep track of equivalences between names: fusions caused by communications are *exposed* to the environment as labels; e.g.

$$Com\ \frac{P \xrightarrow{ux} P',\ Q \xrightarrow{\bar{u}y} Q'}{P|Q \xrightarrow{\{x=y\}} P'|Q'}$$

The Fusion Calculus – Labelled transition system

$$Pref\ \frac{-}{\alpha.P \xrightarrow{\alpha} P} \qquad Com\ \frac{P \xrightarrow{ux} P',\ Q \xrightarrow{\bar{u}y} Q'}{P|Q \xrightarrow{\{x=y\}} P'|Q'} \qquad Par\ \frac{P \xrightarrow{\alpha} P'}{P|Q \xrightarrow{\alpha} P'|Q}$$

$$Open\ \frac{P \xrightarrow{ux} P',\ u \notin \{x, \bar{x}\}}{(x)P \xrightarrow{u(x)} P'} \qquad Pass\ \frac{P \xrightarrow{\alpha} P',\ x \notin n(\alpha)}{(x)P \xrightarrow{\alpha} (x)P'} \qquad Scope\ \frac{P \xrightarrow{\{x=y\}} P'}{(x)P \xrightarrow{\mathbf{1}} P'\{y/x\}}$$

$$Congr\ \frac{P \equiv P'\ P' \xrightarrow{\alpha} Q'\ Q' \equiv Q}{P \xrightarrow{\alpha} Q} \qquad (P|Q)|R \equiv P|(Q|R) \quad P|Q \equiv Q|P \quad P|\mathbf{0} \equiv P$$

$$(x)\mathbf{0} \equiv \mathbf{0} \quad (x)(y)P \equiv (y)(x)P \qquad P|(x)P \equiv (x)(P|Q) \text{ if } x \notin fn(P)$$

Notice the Congruence rule - terms are taken up to congruence

The Fusion Calculus – Bisimulation

A fusion bisimulation *is a symmetric relation* $\mathcal{S}$ *between processes such that whenever* $(P, Q) \in \mathcal{S}$, *if* $P \xrightarrow{\alpha} P'$ *with* $bn(\alpha) \cap fn(Q) = \emptyset$, *then* $Q \xrightarrow{\alpha} Q'$ *and*

- *if* α *is a communication action:* $(P', Q') \in \mathcal{S}$;
- *if* α *is a fusion:* $(P'\sigma, Q'\sigma) \in \mathcal{S}$, *for some* σ *agreeing with* α.

P *and* Q *are* fusion bisimilar *if* $(P, Q) \in \mathcal{S}$ *for some fusion bisimulation* $\mathcal{S}$.

A hyperbisimulation *is a substitution-closed fusion bisimulation, i.e., an* $\mathcal{S}$ *such that* $(P, Q) \in \mathcal{S}$ *implies* $(P\sigma, Q\sigma) \in \mathcal{S}$ *for any substitution* σ. P *and* Q *are* hyper-equivalent, *written* $P \sim Q$, *if they are related by a hyperbisimulation.*

- Ad hoc treatment for fusions: substitution must be performed at each step. Does not fall in the usual "categorical" bisimilarity definition

- Bisimilarity is **not** a congruence; closure under substitutions is needed

In this talk:
A categorical presentation of Fusion

- We want to give a categorical presentation of Fusion calculus:
- Find a category *C* with two endofunctors Σ,*B* : *C* → *C* such that
  - object of processes (up-to alpha) is the initial Σ-algebra *Proc*
  - transition relations of processes are B-coalgebras
  - B has final coalgebra ν B (and is weak pullback preserving)
- We obtain:
  - alternative (more explicit) labelled operational semantics
  - characterization of hyperequivalence as (categorical) B-bisimulation
  - the unique map ⟦ ⟧ : *Proc* → ν B is a fully abstract semantics (by [TP97])
- Also: a methodology for making explicit in the categorical semantics the "hidden states" of operational semantics

10

"Home is where the syntax lives"

- Fusion processes have "lambda"-like binders => standard approach in category of presheaves [FPT99]:
  - $Set^{\mathbb{F}}$ = category of presheaves over $\mathbb{F}$, (skeleton) category of finite sets and functions

$$\Sigma_F : Set^{\mathbb{F}} \to Set^{\mathbb{F}}$$
$$\Sigma_F(A) = 1 + N \times N \times A + N \times N \times A + A \times A + \delta A$$
$$(\Sigma_F(A))_n = \overbrace{1}^{\mathbf{0}} + \overbrace{n \times n \times A_n}^{xy.a} + \overbrace{n \times n \times A_n}^{\bar{x}y.a} + \overbrace{A_n \times A_n}^{a|b} + \overbrace{A_{n+1}}^{(x)a}$$

  - Proc is the initial Σ_F-algebra, that is $Proc_m = \{P \mid fn(P) \subseteq m\}$
- Thus we use $Set^{\mathbb{F}}$ as ambient category

11

What about structural congruence?

- The presheaf approach allows only for *free* algebras (with binders), and does not deal with equations (like e.g. structural congruences)
- (But neither the theory of universal semantics à la Plotkin-Turi does)
- **"Solution"**: define a different but equivalent presentation of the semantics, without congruence
- Does not work always – we need a more general theory for universal semantics, covering also equations and coequations - future work…

12

The Fusion Calculus – Labelled transition system without congruence rule

$$Pref\ \frac{-}{\alpha.P \xrightarrow{\alpha} P}$$

$$Com\ \frac{P \xrightarrow{ux} P',\ Q \xrightarrow{\bar{u}y} Q'}{P|Q \xrightarrow{\{x=y\}} P'|Q'} \qquad Par_l\ \frac{P \xrightarrow{\alpha} P'}{P|Q \xrightarrow{\alpha} P'|Q} \qquad Par_r\ \frac{P \xrightarrow{\alpha} P'}{Q|P \xrightarrow{\alpha} Q|P'}$$

$$Open\ \frac{P \xrightarrow{ux} P',\ u \notin \{x,\bar{x}\}}{(x)P \xrightarrow{u(x)} P'} \qquad Pass\ \frac{P \xrightarrow{\alpha} P',\ x \notin n(\alpha)}{(x)P \xrightarrow{\alpha} (x)P'} \qquad Scope\ \frac{P \xrightarrow{\{x=y\}} P'}{(x)P \xrightarrow{1} P'\{y/x\}}$$

$$Close_l\ \frac{P \xrightarrow{u(x)} P',\ Q \xrightarrow{\bar{u}y} Q'}{P|Q \xrightarrow{1} P'\{y/x\}|Q'} \qquad Close_r\ \frac{P \xrightarrow{ux} P',\ Q \xrightarrow{\bar{u}(y)} Q'}{P|Q \xrightarrow{1} P'|Q'\{x/y\}}$$

$$Close\ \frac{P \xrightarrow{u(x)} P',\ Q \xrightarrow{\bar{u}(x)} Q'}{P|Q \xrightarrow{1} (x)(P'|Q')}$$

The new *Close* rules allow to get rid of structural equivalence

Behaviour functor

- Usually, of the form $BX = \wp_f(Act \times X)$ where *Act* is fixed
- *B*-coalgebras correspond to LTSs: given (A, α:A→*B*(A))

$$P \xrightarrow{a} Q \iff (a, Q) \in \alpha(P)$$

- Each *B* induces *coalgebraic B-bisimulation:* two coalgebras A_1, A_2 are B-bisimilar if there exist R and f_1,f_2 jointly monic such that

$$\begin{array}{ccccc} A_1 & \xleftarrow{f_1} & R & \xrightarrow{f_2} & A_2 \\ \downarrow \alpha_1 & & \exists \vdots \gamma & & \downarrow \alpha_2 \\ BA_1 & \longleftarrow & BR & \longrightarrow & BA_2 \end{array}$$

First attempt

- In our case we could define

$$LX \triangleq \overbrace{N \times N \times X}^{xy} + \overbrace{N \times N \times X}^{\bar{x}y} + \overbrace{N \times \delta X}^{x(y)} + \overbrace{N \times \delta X}^{\bar{x}(y)} + \overbrace{X}^{1} + \overbrace{N \times N \times X}^{x=y}$$
$$BX \triangleq \tilde{K}(LX)$$

where $\tilde{K}$ is Freyd's "finite powerobject" endofunctor (preserves weak pullbacks, needed for universal semantics construction – usual pointwise finite powerset does not preserve weak pullbacks)

- Does **not** work: *B*-bisimilarity ≠ hyperequivalence
- Problem is: *B*-bisimilarity says nothing about substitutions after fusions.
- We have to "correct" *B*.

The approach – in a nutshell

- When a local action has a global effect, a *global state* is hidden somewhere
  - semantics (intentionally) does not mention state => simpler rules, labels
  - at the price of a more complex bisimulation
  - labels do not induce right behavior functor
- Solution:

  1. define the object of states
  2. build it into the naive behaviour functor, using the *state monad*
  3. unravel the coalgebras into "stateful" LTS, to get a simple notion of bisimulation
  4. (same happens for categorical rules)

1. Finding the State

- In Fusion: *state* = *equivalence between names*
- Presheaf of states is *E*

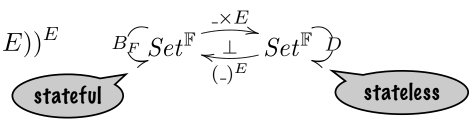
$$E_n \triangleq \{coeq(f,g) \mid k \in \mathbb{F}, f, g : k \rightrightarrows n\}$$

- $E_n \ni e = \{x_1=y_1, \ldots, x_k=y_k\}$
- But also: (class of) surjective substitution $e: n \twoheadrightarrow m$
- Has nice properties (e.g. finitary), and operations (e.g. union is pushout, restriction is composition)

2. Adding state to behaviour functor

- In general, state can be added by lifting along the *(global) state monad*

$$B_F X \triangleq (D(X \times E))^E$$



- In our case:

$$LX \triangleq \overbrace{N \times N \times X}^{xy} + \overbrace{N \times N \times X}^{\bar{x}y} + \overbrace{N \times \delta X}^{x(y)} + \overbrace{N \times \delta X}^{\bar{x}(y)} + \overbrace{X}^{\mathbf{1}} + \overbrace{N \times N \times X}^{x=y}$$
$$DX \triangleq \tilde{K}(LX)$$
$$B_F X = (\tilde{K}(L(X \times E)))^E$$

- B_F seems cumbersome (exponents in presheaf categories)…

3. Simplifying coalgebras

- **Proposition**: B_F-coalgebras (A, α:A→B_F(A)) correspond to (A, β:A×E→*D*(A×E)) such that β is a *D*-coalgebra structure

$$\frac{A \longrightarrow B_F(A) = D(A \times E)^E}{A \times E \longrightarrow D(A \times E)}$$

- A "coalgebra" (A, β:A×E→*D*(A×E)) correspond to "indexed LTS" (A,⟶) where
  - nodes: elements of ∫A×E = {(n,P,e) | (P,e) ∈ A_n×E_n}
  - labels: elements of ∫*Act* = {(n,a) | a ∈ Act_n}
  - subject to some conditions (e.g. closure under substitutions)
- We can denote transitions as

$$(P, e) \xrightarrow{\alpha}_n (Q, d)$$

a "stateful" LTS

From original LTS to indexed LTS

- Original LTS does not mention state => it is not an indexed LTS
- Define $\mathcal{L}_F$ = (Proc, ⟶) as

$$\text{for } \alpha = xy, \bar{x}y, 1: \quad (P, e) \xrightarrow{\alpha}_n (Q, e) \iff P[e] \xrightarrow{\alpha[e]} Q[e]$$

$$\text{for } \alpha = x, \bar{x}: \quad (P, e) \xrightarrow{\alpha}_n (Q, e+1) \iff P[e] \xrightarrow{\alpha[e](z)} Q[e+1];$$

$$(P, e) \xrightarrow{x=y}_n (Q, e \cup x{=}y) \iff P[e] \xrightarrow{(x=y)[e]} Q[e]$$

state is updated

- **Proposition**: $\mathcal{L}_F$ is an indexed LTS and

$$P \xrightarrow{\alpha} Q \iff (P, \emptyset) \xrightarrow{\alpha}_n (Q, e)$$

3 bis. Indexed bisimulation

- A B_F-bisimulation can be presented as a family of symmetric relations

$$\{R_n \subseteq (A_n \times E_n) \times (A_n \times E_n)\}_{n \in N}$$

such that, if $(P, e) R_n (Q, d)$ then

$$(P, e) \xrightarrow{\alpha}_n (P', e') \Rightarrow \exists (Q', d') \in A_{n'} \times E_{n'} s.t.\ (Q, d) \xrightarrow{\alpha}_n (Q', d') \text{ and } (P', e') R_{n'} (Q', d')$$

and vice versa, and closed under substitution:

$$\text{for all } \sigma : n \to m : (P[\sigma], e[\sigma]) R_m (Q[\sigma], d[\sigma])$$

- **Theorem**: P, Q ∈ *Proc*$_n$ are hyperequivalent iff (P,∅) and (Q,∅) are B_F-bisimilar

  Proof hint: substitution-closed bisimulations correspond to indexed families of relations above.

4. Categorical rules for stateful behaviour functors

- Following [TP97,FT01], we have to define

$$\mathcal{S}_X : \Sigma(X \times BX) \longrightarrow BTX \qquad \text{in } \mathit{Set}^{\mathbb{F}}$$

(where *T* is the monad of free Σ-algebras).

- Unfolding *B* and by adjunction, this is equivalent to:

constructor's arguments

their stateful behaviour

$$\mathcal{S}_X : \Sigma(X \times BX) \times E \longrightarrow D(TX \times E)$$

current state

result: labelled transitions with new state

Categorical rules for the Fusion calculus

- expanding Σ, this is the product of the interpretation of each constructor

$$\begin{aligned}
\mathcal{S}_X^{\mathbf{0}} :&\qquad E \longrightarrow D(TX \times E)\\
\mathcal{S}_X^{\mathrm{in}} :&\quad N \times N \times X \times BX \times E \longrightarrow D(TX \times E)\\
\mathcal{S}_X^{\mathrm{out}} :&\quad N \times N \times X \times BX \times E \longrightarrow D(TX \times E)\\
\mathcal{S}_X^{\mathrm{par}} :&X \times BX \times X \times BX \times E \longrightarrow D(TX \times E)\\
\mathcal{S}_X^{\mathrm{res}} :&\quad N \times \delta X \times \delta BX \times E \longrightarrow D(TX \times E)
\end{aligned}$$

each defined by collecting relevant rules, taking care of the state; e.g. for |

$$\rho^{Par}, \rho^{Com}, \rho^{Close_l}, \rho^{Close_r}, \rho^{Close} : X \times BX \times X \times BX \times E \longrightarrow \tilde{K}(\mathit{Act} \times TX \times E)$$

$$\begin{aligned}
\rho_n^{Com}(P, \beta, Q, \gamma, e) \;=\; & \{(x{=}y, P'|Q', e_1 \cup e_2 \cup \{x{=}y\}) \mid \text{for some } z, w \in n, e(z) = e(w) :\\
& ((zx, P', e_1) \in \beta_n(id_n, e) \wedge (\bar{w}y, Q', e_2) \in \gamma_n(id_n, e)) \vee\\
& ((\bar{z}x, P', e_1) \in \beta_n(id_n, e) \wedge (wy, Q', e_2) \in \gamma_n(id_n, e))\}
\end{aligned}$$

Categorical rules – in rule format

- Categorical rules can be written as "stateful LTS"

$$\frac{(P, e) \xrightarrow{ux}_n (P', e_1) \quad (Q, e) \xrightarrow{\bar{w}y}_n (Q', e_2)}{(P|Q, e) \xrightarrow{x=y}_n (P'|Q', e_1 \cup e_2 \cup \{x = y\})} e \vdash u = w$$

- (actually $e_1 = e_2 = e$)

Categorical rules for the Fusion calculus

- **Theorem**: there exists a (necessarily unique)

$$[\![\ ]\!] : Proc \to \nu B_F$$

which is both compositional and fully abstract, that is, for all P,Q ∈ $Proc_n$

$[\![P]\!]_n = [\![Q]\!]$ iff P and Q are B_F-bisimilar (iff are hyperequivalent)

Proof sketch: follows from general results [TP97], by proving that *Proc* can be given a B_F-coalgebra structure, νB_F a Σ-algebra structure, and that B_F is finitary and preserves weak pullbacks.

Conclusions

- Often calculi for distributed systems hide some notion of global state
  - omitted in the LTSs (“stateless”: simpler relation, actions, rules)
  - comes out in bisimulation, and yields unsatisfactory behaviour functor
- State can be added systematically using the state monad
  - Resulting “stateful” coalgebras (LTSs) can be quite simplified
  - For Fusion: state = equivalence of names; yields correct behaviour functor
- **To Do**:
  - other examples (Explicit Fusion, Mobile Ambients, in particular)
  - fit into some theory of *modular mathematical operational semantics*

Thanks!