Syntactic Complexity of Regular Languages

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Languages

- Alphabet Σ a finite set of letters
- Set of all words Σ^* free monoid generated by Σ
- Set of non-empty words Σ^+ free semigroup generated by Σ
- Empty word ε
- Language $L \subseteq \Sigma^*$
- The ε-function L^ε of a regular language L

$$L^\varepsilon = \begin{cases} \emptyset, & \text{if } \varepsilon \notin L; \\ \{\varepsilon\}, & \text{if } \varepsilon \in L. \end{cases}$$

Congruences on Σ^*

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- It is a congruence if it is both a left and a right congruence, or

$$x \sim y \Leftrightarrow uxv \sim uyv, \text{ for all } u, v \in \Sigma^*$$

Nerode Congruence on Σ^*

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- $x \sim_L y$ if and only if $L_x = L_y$
- Number of classes of $\sim_L$ = number of quotients of L
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- The quotient automaton of a regular language L is $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, where $Q = \{L_w \mid w \in \Sigma^*\}$, $\delta(L_w, a) = L_{wa}$, $q_0 = L_\varepsilon = L$, $F = \{L_w \mid \varepsilon \in L_w\}$.
- $\kappa(L)$=quotient complexity = state complexity

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- $\Sigma^*/\approx_L$ syntactic monoid of L
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- The transformation semigroup T_L of a quotient automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ of L:
- Set of transformations of states of $\mathcal{A}$ by non-empty words
- Syntactic semigroup isomorphic to transformation semigroup

Quotient Complexity vs Syntactic Complexity

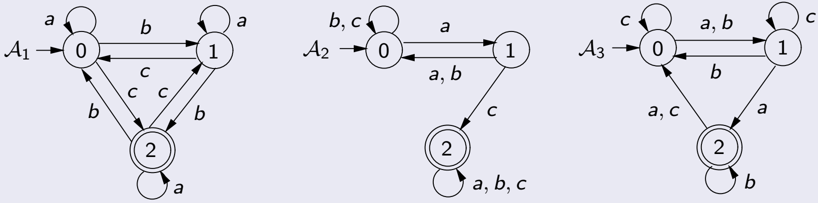

Figure: Automata with various syntactic complexities.

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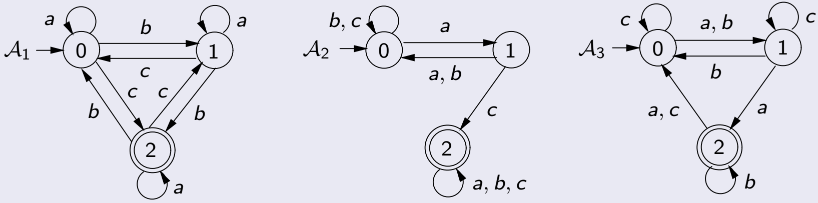

Figure: Automata with various syntactic complexities.

- $\sigma(L_1) = 3 \quad \sigma(L_2) = 9 \quad \sigma(L_3) = 27$
- Can we predict this?

Transformations of $Q = \{0, 1, \dots, n-1\}$

- A transformation $t = \begin{pmatrix} 0 & 1 & \cdots & n-2 & n-1 \\ i_0 & i_1 & \cdots & i_{n-2} & i_{n-1} \end{pmatrix}$
- The image of element i under transformation t is it
- The identity transformation maps each element to itself
- t contains a cycle $(i_1, i_2, \dots, i_k)$ of length k if there exist $i_1, \dots, i_k$ such that $i_1 t = i_2, i_2 t = i_3, \dots, i_{k-1} t = i_k, i_k t = i_1$
- A singular transformation, denoted by $\binom{i}{j}$, has $it = j$, and $ht = h$ for all $h \neq i$.
- For $i < j$, a transposition is the cycle (i, j)
- A constant transformation, $\binom{Q}{j}$, has $it = j$ for all i.

Generators

Theorem (Piccard, 1935)

The complete transformation monoid $\mathcal{T}_Q$ *on* $Q = \{0, 1, \dots, n-1\}$ *of size* n^n *can be generated by any cyclic permutation of* n *elements together with a transposition and a "returning" transformation* $r = \binom{n-1}{0}$*. In particular,* T_n *can be generated by* $c = (0, 1, \dots, n-1)$*,* $t = (0, 1)$ *and* $r = \binom{n-1}{0}$*.*

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Proposition

For any language L with $\kappa(L) = n > 1$, we have $n - 1 \leq \sigma(L) \leq n^n$.

Each state > 0 reached from the initial state, so at least $n - 1$

If $\Sigma = \{a\}$ and $L = a^{n-1}a^*$, then $\kappa(L) = n$, and $\sigma(L) = n - 1$

Special Quotients, $\kappa(L) = n$

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Theorem

1. *If L has $\emptyset$ or Σ^*, then $\sigma(L) \le n^{n-1}$.*
2. *If L has $\{\varepsilon\}$ or Σ^+, then $\sigma(L) \le n^{n-2}$.*
3. *If L is uniquely reachable, then $\sigma(L) \le (n-1)^n$.*
4. *If L_a is uniquely reachable, $a \in \Sigma$, then $\sigma(L) \le 1 + (n-2)^n$.*

Special Quotients, $\kappa(L) = n$

$\emptyset$	Σ^*	$\{\varepsilon\}$	Σ^+		L is ur	L_a is ur
$\surd$				n^{n-1}	$(n-1)^{n-1}$	$1+(n-3)^{n-2}$
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$\surd$	$\surd$	$\surd$	$\surd$	n^{n-4}	$(n-1)^{n-4}$	$1+(n-6)^{n-2}$

Proof of Special Quotient Theorem $n \geq 1$ $\kappa(L) = n$

Proof.

- Since $\emptyset_a = \emptyset$ for all $a \in \Sigma$, only $n-1$ states in the quotient automaton distinguish two transformations. n^{n-1}
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  If L has Σ^*, then $(\Sigma^*)_a = \Sigma^*$, for all $a \in \Sigma$.
- Since $\{\varepsilon\}_a = \emptyset$ for all $a \in \Sigma$, L has $\emptyset$ if L has $\{\varepsilon\}$. Two states that have image $\emptyset$. n^{n-2}
  Dually, $(\Sigma^+)_a = \Sigma^*$ for all $a \in \Sigma$.

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  Dually, $(\Sigma^+)_a = \Sigma^*$ for all $a \in \Sigma$.
- If L is uniquely reachable then $L_w = L$ implies $w = \varepsilon$, L does not appear, and there are $n-1$ choices. $(n-1)^n$
- If L_a is uniquely reachable, then so is L. Hence L never appears, and L_a appears only once. There can be at most $(n-2)^n$ other transformations. $1 + (n-2)^n$

Prefixes and Suffixes

- $w = uv$ u is a prefix of w
- $w = uv$ v is a suffix of w
- $w = uxv$ x is a factor of w

Convex Languages

- A language L is prefix-convex if u is a prefix of v, v is a prefix of w and $u, w \in L$ implies $v \in L$
- L is prefix-closed if u is a prefix of v and $v \in L$ implies $u \in L$
- L is converse prefix-closed if u is a prefix of v, and $u \in L$ implies $v \in L$ right ideal
- L is prefix-free if $u \neq v$ is a prefix of v and $v \in L$ implies $u \notin L$ prefix code

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- L is suffix-convex
- L is factor-convex
- L is bifix-convex

Ideals and Closed Languages

- right ideal $L = L\Sigma^*$
- left ideal $L = \Sigma^* L$
- 2-sided ideal $L = \Sigma^* L \Sigma^*$
- Ideals are complements of closed languages
  - right ideals are complements of prefix-closed languages
  - left ideals are complements of suffix-closed languages
  - 2-sided ideals are complements of factor-closed languages
- Since syntactic complexity is preserved under complementation, our proofs are in terms of ideals only.

Right Ideals and Prefix-Closed Languages

Theorem

Let $L \subseteq \Sigma^*$ and $\kappa(L) = n$. If L is a right ideal or a prefix-closed language, then $\sigma(L) \leq n^{n-1}$. Moreover, the bound is tight for

- $n = 1$ *if* $|\Sigma| \geq 1$
- $n = 2$ *if* $|\Sigma| \geq 2$
- $n = 3$ *if* $|\Sigma| \geq 3$
- $n \geq 4$ *if* $|\Sigma| \geq 4$

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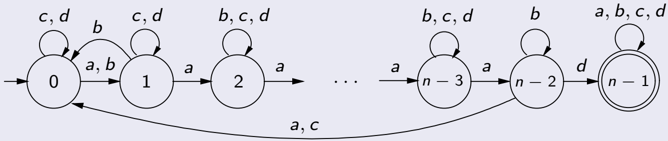
Proof.

Since L has Σ^*, $\sigma(L) \leq n^{n-1}$.
Next we show the bound is tight. □

Proof of Right Ideal Theorem, $n \leq 3$

- If $n = 1$, $L = a^*$ meets the bound.
- If $n = 2$, then $b^*a(a \cup b)^*$ meets the bound.
- If $n = 3$, then automaton on next slide with alphabet $\{a, c, d\}$ meets the bound.

Proof of Right Ideal Theorem, $n \geq 4$



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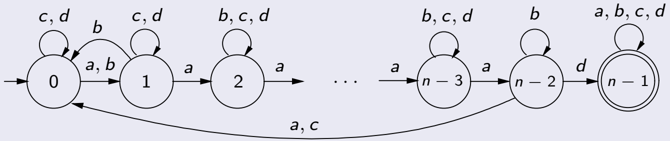

$$t = \begin{pmatrix} 0 & 1 & 2 & \cdots & n-3 & n-2 & n-1 \\ i_0 & i_1 & i_2 & \cdots & i_{n-3} & i_{n-2} & n-1 \end{pmatrix},$$

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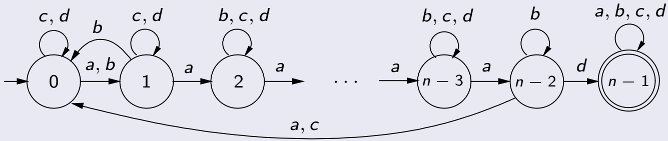

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Case 1

$i_k \neq n-1$ for all k, $0 \leq k \leq n-2$.
Since all the images of the first $n-1$ states are in the set $\{0, 1, \ldots, n-2\}$, t can be performed by $\{a, b, c\}$.

Case 2

- $i_h = n-1$ for some h, $0 \le h \le n-2$
- There exists j, $0 \le j \le n-2$ such that $i_k \ne j$ for all k
- Define i'_k for all $0 \le k \le n-2$ as follows:
- $i'_k = j$ if $i_k = n-1$, and $i'_k = i_k$ if $i_k \ne n-1$
- $s = \begin{pmatrix} 0 & 1 & 2 & \cdots & n-3 & n-2 & n-1 \\ i'_0 & i'_1 & i'_2 & \cdots & i'_{n-3} & i'_{n-2} & n-1 \end{pmatrix}$
- Let $r = (j, n-2)$ $\qquad \mathcal{A}_n$ can do s and r

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- Let $r = (j, n-2)$ $\quad$ $\mathcal{A}_n$ can do s and r

$t = srdr$

If $kt = n-1$, then $ks = j$, $jr = n-2$, $(n-2)d = n-1$, $(n-1)r = n-1$. If $kt = n-2$, then $ks = n-2$, $(n-2)r = j$, $jd = j$, and $jr = n-2$. If $kt = i_k < n-2$, then $k(srdr) = i_k$.

Syntactic Complexity Bounds for Right Ideals

$\|\Sigma\|$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	...	$n = n$
1	**1**	**1**	**2**	**3**	**4**	...	$\mathbf{n-1}$
2	–	**2**	**7**	**31**	**167**	...	?
3	–	–	**9**	**61**	**545**	...	?
4	–	–	–	**64**	**625**	...	$\mathbf{n^{n-1}}$

All the bounds are tight

The bounds for $n \leq 5$ were verified by a computer program

Left Ideals and Suffix-Closed Languages

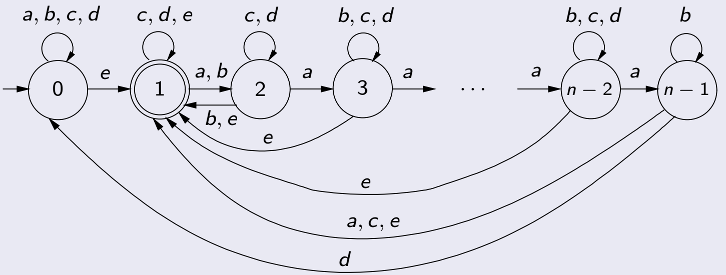
We provide strong support for the following conjecture:

Conjecture 1 *If L is a left ideal or a suffix-closed language with quotient complexity $\kappa(L) = n \geq 1$, then its syntactic complexity is less than or equal to $n^{n-1} + n - 1$.*

The bound is met with $|\Sigma| \geq 5$

Note the lack of symmetry between left and right ideals!

Automaton $\mathcal{B}_n$ with $n^{n-1} + n - 1$ Transformations



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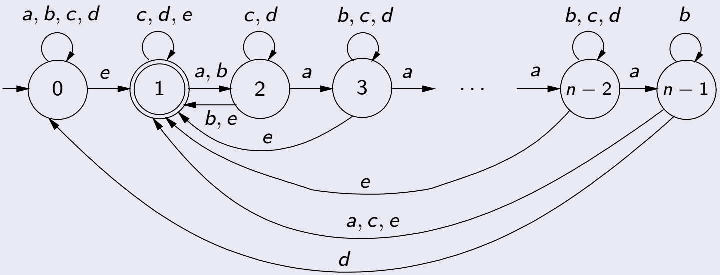

- $a = (1, \dots, n-1)$, $b = (1, 2)$, $c = \binom{n-1}{1}$, $d = \binom{n-1}{0}$, $e = \binom{Q}{1}$
- $\mathcal{B}_n$ is minimal
- $L(\mathcal{B}_n)$ is a left ideal

n^{n-1} Transformations

$$t = \begin{pmatrix} 0 & 1 & 2 & \cdots & n-3 & n-2 & n-1 \\ 0 & i_1 & i_2 & \cdots & i_{n-3} & i_{n-2} & i_{n-1} \end{pmatrix},$$

1. If $i_k \neq 0$ for all k, $1 \leq k \leq n-1$, t can be done by $\mathcal{B}_n$
2. If $i_h = 0$ for some h, $1 \leq h \leq n-1$, then there exists j, $1 \leq j \leq n-1$, $i_k \neq j$ for all k, $1 \leq k \leq n-1$.
   Let $i'_k = j$ if $i_k = 0$, and $i'_k = i_k$, otherwise, and let
   $s = \begin{pmatrix} 0 & 1 & 2 & \cdots & n-3 & n-2 & n-1 \\ 0 & i'_1 & i'_2 & \cdots & i'_{n-3} & i'_{n-2} & i'_{n-1} \end{pmatrix}$, $r = (j, n-1)$.
   $\mathcal{B}_n$ can do s and r; consider $srdr$.
   If $kt = 0$, then $ks = j$, $jr = n-1$, $(n-1)d = 0$, and $0r = 0$.
   If $kt = n-1$, then $ks = n-1$, $(n-1)r = j$, $jd = j$, and $jr = n-1$.
   If $0 < kt < n-1$, then $srdr$ maps k to kt.
   So $t = srdr$, and t can be performed by $\mathcal{B}_n$ as well.

$n-1$ Transformations

- Transformation $t = \binom{Q}{j}$ maps all the states to $j \neq 0$
- There are $n-1$ such transformations
- If $j=1$, then $t=e$; therefore t can be performed by $\mathcal{B}_n$
- Otherwise, let $s=(1,j)$
- s can be performed by $\mathcal{B}_n$
- Since $t=es$, t can also be performed by $\mathcal{B}_n$

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- If $\delta(0, w) = i \neq 0$, then $w = uev$
- But ue maps all the states to 1
- So there are no other transformations

Aperiodic Inputs

- Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be the quotient DFA of a left ideal.
- For $w \in \Sigma^*$, consider $q_0 = p_0, p_1, p_2 \dots$ where $p_i = \delta(q_0, w^i)$.
- We must have some i and $j > i$ such that $p_0, p_1, \dots, p_i, p_{i+1}, \dots p_{j-1}$ are distinct and $p_j = p_i$.
- The sequence $q_0 = p_0, p_1, \dots, p_i, p_{i+1}, \dots p_{j-1}$ of states with $p_j = p_i$ is called the behavior of w on $\mathcal{A}$
- $j - i$ is the period of that behavior.
- If the period of w is 1, then its behavior is aperiodic; otherwise, it is periodic.

Properties of Left Ideals

Lemma

If $\mathcal{A}$ is the quotient automaton of a left ideal L, then the behavior of every word $w \in \Sigma^*$ is aperiodic. Also, L does not have $\emptyset$.

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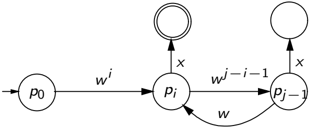

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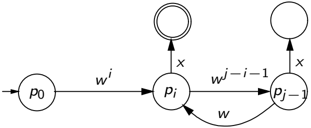


- If $w^i x \in L$ but $w^{j-1}x = w^{j-i-1}(w^i x) \notin L$, L is not left ideal
- If $w^{j-1}x \in L$ but $w^i x \notin L$, then $w^i x = w^j x = ww^{j-1} \notin L$

Left Ideals $\mathbf{n} \leq 3$

Theorem

If $1 \leq n \leq 3$ and L is a left ideal or a suffix-closed language with $\kappa(L) = n$, then $\sigma(L) \leq n^{n-1} + n - 1$. Moreover, the bound is tight for $n = 1$ if $|\Sigma| \geq 1$, for $n = 2$ if $|\Sigma| \geq 3$, and for $n = 3$ if $|\Sigma| \geq 4$.

$n < 3$

n=1: Here $L = \Sigma^*$. The bound is met by a^* over $\Sigma = \{a\}$.

n=2:

- Only $[1, 0]$ is ruled out by Lemma. The bound 3 holds.

$\mathbf{n} < \mathbf{3}$

n=1: Here $L = \Sigma^*$. The bound is met by a^* over $\Sigma = \{a\}$.

n=2:

- Only $[1, 0]$ is ruled out by Lemma. The bound 3 holds.
- We must have $\delta(0, a) = 1$ for some $a \in \Sigma$.

$\mathbf{n} < 3$

n=1: Here $L = \Sigma^*$. The bound is met by a^* over $\Sigma = \{a\}$.

n=2:

- Only $[1, 0]$ is ruled out by Lemma. The bound 3 holds.
- We must have $\delta(0, a) = 1$ for some $a \in \Sigma$.
- We cannot have $a : [1, 0]$, and so we have $a : [1, 1]$

$n < 3$

n=1: Here $L = \Sigma^*$. The bound is met by a^* over $\Sigma = \{a\}$.

n=2:

- Only $[1, 0]$ is ruled out by Lemma. The bound 3 holds.
- We must have $\delta(0, a) = 1$ for some $a \in \Sigma$.
- We cannot have $a : [1, 0]$, and so we have $a : [1, 1]$
- If $\Sigma = \{a\}$, then $L = aa^* = a^*a$ with $\sigma(L) = 1$.

$\mathbf{n} < 3$

n=1: Here $L = \Sigma^*$. The bound is met by a^* over $\Sigma = \{a\}$.

n=2:

- Only $[1, 0]$ is ruled out by Lemma. The bound 3 holds.
- We must have $\delta(0, a) = 1$ for some $a \in \Sigma$.
- We cannot have $a : [1, 0]$, and so we have $a : [1, 1]$
- If $\Sigma = \{a\}$, then $L = aa^* = a^*a$ with $\sigma(L) = 1$.
- If $\Sigma = \{a, b\}$, then we have three cases:
  1. If $b : [1, 1]$, then $L = \Sigma^*\Sigma$ with $\sigma(L) = 1$.
  2. If $b : [0, 0]$, then $L = \Sigma^*a$ with $\sigma(L) = 2$.
  3. If $b : [0, 1]$, then $L = \Sigma^*a\Sigma^*$ with $\sigma(L) = 2$.

$\mathbf{n < 3}$

n=1: Here $L = \Sigma^*$. The bound is met by a^* over $\Sigma = \{a\}$.

n=2:

- Only $[1, 0]$ is ruled out by Lemma. The bound 3 holds.
- We must have $\delta(0, a) = 1$ for some $a \in \Sigma$.
- We cannot have $a : [1, 0]$, and so we have $a : [1, 1]$
- If $\Sigma = \{a\}$, then $L = aa^* = a^*a$ with $\sigma(L) = 1$.
- If $\Sigma = \{a, b\}$, then we have three cases:
  1. If $b : [1, 1]$, then $L = \Sigma^*\Sigma$ with $\sigma(L) = 1$.
  2. If $b : [0, 0]$, then $L = \Sigma^*a$ with $\sigma(L) = 2$.
  3. If $b : [0, 1]$, then $L = \Sigma^*a\Sigma^*$ with $\sigma(L) = 2$.
- If $\Sigma = \{a, b, c\}$, $L = \Sigma^*a(a \cup b)^*$ meets the bound 3.

$\mathbf{n = 3}$

n=3:

- For $|\Sigma| = 1$, $L = a^*aa$ and $\sigma(L) = 2$

$n = 3$

n=3:

- For $|\Sigma| = 1$, $L = a^*aa$ and $\sigma(L) = 2$
- For $|\Sigma| = 2$, $\sigma(L) \leq 7$; $a : [001]$, $b : [122]$ meet this bound

$n = 3$

n=3:

- For $|\Sigma| = 1$, $L = a^*aa$ and $\sigma(L) = 2$
- For $|\Sigma| = 2$, $\sigma(L) \leq 7$; $a : [001]$, $b : [122]$ meet this bound
- For $|\Sigma| = 3$, $\sigma(L) \leq 9$; and $\mathcal{B}_3$ restricted to inputs $b : [0, 2, 1]$, $d : [0, 1, 0]$ and $e : [1, 1, 1]$ meets this bound

$\mathbf{n = 3}$

n=3:

- For $|\Sigma| = 1$, $L = a^*aa$ and $\sigma(L) = 2$
- For $|\Sigma| = 2$, $\sigma(L) \le 7$; $a : [001]$, $b : [122]$ meet this bound
- For $|\Sigma| = 3$, $\sigma(L) \le 9$; and $\mathcal{B}_3$ restricted to inputs $b : [0, 2, 1]$, $d : [0, 1, 0]$ and $e : [1, 1, 1]$ meets this bound
- For $|\Sigma| = 4$, a and b of $\mathcal{B}_3$ coincide; omit a.

$\mathbf{n = 3}$

n=3:

- For $|\Sigma| = 1$, $L = a^*aa$ and $\sigma(L) = 2$
- For $|\Sigma| = 2$, $\sigma(L) \leq 7$; $a : [001]$, $b : [122]$ meet this bound
- For $|\Sigma| = 3$, $\sigma(L) \leq 9$; and $\mathcal{B}_3$ restricted to inputs $b : [0, 2, 1]$, $d : [0, 1, 0]$ and $e : [1, 1, 1]$ meets this bound
- For $|\Sigma| = 4$, a and b of $\mathcal{B}_3$ coincide; omit a.
- Next table shows $\mathcal{B}_3$ with $3^2 + 2 = 11$ transformations. We show that 11 is indeed the maximal bound.

$n = 3$ with $|\Sigma| = 4$

Table: The eleven transformations of automaton $\mathcal{B}_3$ of a left ideal.

	b	c	d	e	bb	bd	cb	db	eb	bdb	cbd
0	0	0	0	1	0	0	0	0	2	0	0
1	2	1	1	1	1	0	2	2	2	0	0
2	1	1	0	1	2	1	2	0	2	2	0

$\mathbf{n = 3}$ continued, periodic behaviors

- $(p_0, p_1; p_2 = p_0)$, $(p_0, p_1, p_2; p_3 = p_0)$, $(p_0, p_1, p_2; p_3 = p_1)$

$\mathbf{n} = 3$ continued, periodic behaviors

- $(p_0, p_1; p_2 = p_0)$, $(p_0, p_1, p_2; p_3 = p_0)$, $(p_0, p_1, p_2; p_3 = p_1)$
- Ruled out by Lemma: $[1, 0, 0]$, $[1, 0, 1]$, $[1, 0, 2]$, $[1, 2, 0]$, $[1, 2, 1]$, $[2, 0, 0]$, $[2, 1, 0]$, $[2, 2, 0]$, $[2, 0, 1]$, and $[2, 2, 1]$

$\mathbf{n = 3}$ continued, periodic behaviors

- $(p_0, p_1; p_2 = p_0)$, $(p_0, p_1, p_2; p_3 = p_0)$, $(p_0, p_1, p_2; p_3 = p_1)$
- Ruled out by Lemma: $[1, 0, 0]$, $[1, 0, 1]$, $[1, 0, 2]$, $[1, 2, 0]$, $[1, 2, 1]$, $[2, 0, 0]$, $[2, 1, 0]$, $[2, 2, 0]$, $[2, 0, 1]$, and $[2, 2, 1]$
- Not ruled out by Lemma and not in Table: $[1, 1, 0]$, $[1, 1, 2]$, $[1, 2, 2]$, $[2, 0, 2]$, $[2, 1, 1]$, and $[2, 1, 2]$

$\mathbf{n = 3}$ continued, periodic behaviors

- $(p_0, p_1; p_2 = p_0)$, $(p_0, p_1, p_2; p_3 = p_0)$, $(p_0, p_1, p_2; p_3 = p_1)$
- Ruled out by Lemma: $[1,0,0]$, $[1,0,1]$, $[1,0,2]$, $[1,2,0]$, $[1,2,1]$, $[2,0,0]$, $[2,1,0]$, $[2,2,0]$, $[2,0,1]$, and $[2,2,1]$
- Not ruled out by Lemma and not in Table: $[1,1,0]$, $[1,1,2]$, $[1,2,2]$, $[2,0,2]$, $[2,1,1]$, and $[2,1,2]$
- $t_1 : [1,1,0]$ and $cb : [0,2,2]$ yield $t_1 cb : [2,2,0]$
  $t_2 : [1,1,2]$ and $db : [0,2,0]$ yield $t_2 db : [2,2,0]$
  $t_3 : [1,2,2]$ and $d : [0,1,0]$ yield $t_3 d : [1,0,0]$
  $t_4 : [2,0,2]$ and $c : [0,1,1]$ yield $t_4 c : [1,0,1]$
  $t_5 : [2,1,1]$ and $bdb : [0,0,2]$ yield $t_5 bdb : [2,0,0]$
  $t_6 : [2,1,2]$ and $bd : [0,0,1]$ yield $t_6 bd : [1,0,1]$

$\mathbf{n} = 3$ continued, periodic behaviors

- $(p_0, p_1; p_2 = p_0)$, $(p_0, p_1, p_2; p_3 = p_0)$, $(p_0, p_1, p_2; p_3 = p_1)$
- Ruled out by Lemma: $[1, 0, 0]$, $[1, 0, 1]$, $[1, 0, 2]$, $[1, 2, 0]$, $[1, 2, 1]$, $[2, 0, 0]$, $[2, 1, 0]$, $[2, 2, 0]$, $[2, 0, 1]$, and $[2, 2, 1]$
- Not ruled out by Lemma and not in Table: $[1, 1, 0]$, $[1, 1, 2]$, $[1, 2, 2]$, $[2, 0, 2]$, $[2, 1, 1]$, and $[2, 1, 2]$
- $t_1 : [1, 1, 0]$ and $cb : [0, 2, 2]$ yield $t_1cb : [2, 2, 0]$
  $t_2 : [1, 1, 2]$ and $db : [0, 2, 0]$ yield $t_2db : [2, 2, 0]$
  $t_3 : [1, 2, 2]$ and $d : [0, 1, 0]$ yield $t_3d : [1, 0, 0]$
  $t_4 : [2, 0, 2]$ and $c : [0, 1, 1]$ yield $t_4c : [1, 0, 1]$
  $t_5 : [2, 1, 1]$ and $bdb : [0, 0, 2]$ yield $t_5bdb : [2, 0, 0]$
  $t_6 : [2, 1, 2]$ and $bd : [0, 0, 1]$ yield $t_6bd : [1, 0, 1]$
- Conflicts are independent of the set of accepting states

$\mathbf{n = 3}$ continued, periodic behaviors

- $(p_0, p_1; p_2 = p_0)$, $(p_0, p_1, p_2; p_3 = p_0)$, $(p_0, p_1, p_2; p_3 = p_1)$
- Ruled out by Lemma: $[1,0,0]$, $[1,0,1]$, $[1,0,2]$, $[1,2,0]$, $[1,2,1]$, $[2,0,0]$, $[2,1,0]$, $[2,2,0]$, $[2,0,1]$, and $[2,2,1]$
- Not ruled out by Lemma and not in Table: $[1,1,0]$, $[1,1,2]$, $[1,2,2]$, $[2,0,2]$, $[2,1,1]$, and $[2,1,2]$
- $t_1 : [1,1,0]$ and $cb : [0,2,2]$ yield $t_1cb : [2,2,0]$
  $t_2 : [1,1,2]$ and $db : [0,2,0]$ yield $t_2db : [2,2,0]$
  $t_3 : [1,2,2]$ and $d : [0,1,0]$ yield $t_3d : [1,0,0]$
  $t_4 : [2,0,2]$ and $c : [0,1,1]$ yield $t_4c : [1,0,1]$
  $t_5 : [2,1,1]$ and $bdb : [0,0,2]$ yield $t_5bdb : [2,0,0]$
  $t_6 : [2,1,2]$ and $bd : [0,0,1]$ yield $t_6bd : [1,0,1]$
- Conflicts are independent of the set of accepting states
- Conflicts above are disjoint pairs

$\mathbf{n = 3}$ continued, periodic behaviors

- $(p_0, p_1; p_2 = p_0)$, $(p_0, p_1, p_2; p_3 = p_0)$, $(p_0, p_1, p_2; p_3 = p_1)$
- Ruled out by Lemma: $[1,0,0]$, $[1,0,1]$, $[1,0,2]$, $[1,2,0]$, $[1,2,1]$, $[2,0,0]$, $[2,1,0]$, $[2,2,0]$, $[2,0,1]$, and $[2,2,1]$
- Not ruled out by Lemma and not in Table: $[1,1,0]$, $[1,1,2]$, $[1,2,2]$, $[2,0,2]$, $[2,1,1]$, and $[2,1,2]$
- $t_1 : [1,1,0]$ and $cb : [0,2,2]$ yield $t_1 cb : [2,2,0]$
  $t_2 : [1,1,2]$ and $db : [0,2,0]$ yield $t_2 db : [2,2,0]$
  $t_3 : [1,2,2]$ and $d : [0,1,0]$ yield $t_3 d : [1,0,0]$
  $t_4 : [2,0,2]$ and $c : [0,1,1]$ yield $t_4 c : [1,0,1]$
  $t_5 : [2,1,1]$ and $bdb : [0,0,2]$ yield $t_5 bdb : [2,0,0]$
  $t_6 : [2,1,2]$ and $bd : [0,0,1]$ yield $t_6 bd : [1,0,1]$
- Conflicts are independent of the set of accepting states
- Conflicts above are disjoint pairs
- At most one from each pair, so no more than 11

Syntactic Complexities for Left Ideals

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$		$n = n$
$\lvert\Sigma\rvert = 1$	**1**	**1**	**2**	**3**	**4**	...	$\mathbf{n-1}$
$\lvert\Sigma\rvert = 2$	–	**2**	**7**	17	34	...	?
$\lvert\Sigma\rvert = 3$	–	**3**	**9**	25	65	...	?
$\lvert\Sigma\rvert = 4$	–	–	**11**	64	453	...	?
$\lvert\Sigma\rvert = 5$	–	–	–	67	629	...	$n^{n-1} + n - 1$

Two-Sided Ideals and Factor-Closed Languages

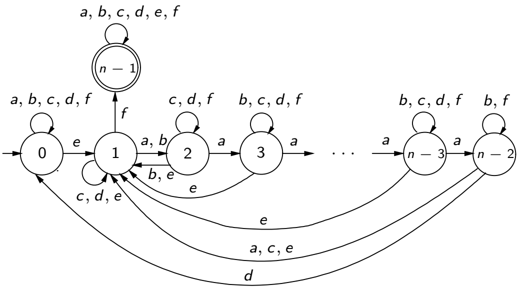
Conjecture 2. *If L is a two-sided ideal or a factor-closed language with $\kappa(L) = n \geq 2$, then $\sigma(L) \leq n^{n-2} + (n-2)2^{n-2} + 1$.*

Two-Sided Ideals and Factor-Closed Languages

Conjecture 2. *If L is a two-sided ideal or a factor-closed language with $\kappa(L) = n \geq 2$, then $\sigma(L) \leq n^{n-2} + (n-2)2^{n-2} + 1$.*

- For $n = 2$ and $\Sigma = \{a, b\}$, $\Sigma^* a \Sigma^*$ meets the bound 2
- For $n = 3$ and $\Sigma = \{a, b, c\}$, $(b \cup c \cup ac^*b)^* ac^* a \Sigma^*$ works
- For $n \geq 4$, use $\mathcal{C}_n = (Q, \Sigma, \delta, 0, \{n-1\})$, where $Q = \{0, \ldots, n-1\}$, $\Sigma = \{a, b, c, d, e, f\}$, and δ is on next slide
- For $n = 4$, a and b coincide, so $|\Sigma| = 5$

Automaton $\mathcal{C}_n$



- $a = (1, 2, \ldots, n-2)$, $b = (1, 2)$, $c = \binom{n-2}{1}$, $d = \binom{n-2}{0}$, $\delta(i, e) = 1$ for $i = 0, \ldots, n-2$, $\delta(n-1, e) = n-1$, $f = \binom{1}{n-1}$
- DFA $\mathcal{C}_n$ is minimal and $L = L(\mathcal{C}_n)$ is a two-sided ideal

Syntactic Complexities for Two-Sided Ideals

$\|\Sigma\|$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$		$n=n$
1	**1**	**1**	**2**	**3**	**4**	...	$\mathbf{n-1}$
2	–	**2**	5	11	19	...	?
3	–	–	6	16	47	...	?
4	–	–	–	23	90	...	?
5	–	–	–	25	147	...	?
6	–	–	–	–	150	...	$f(n)$

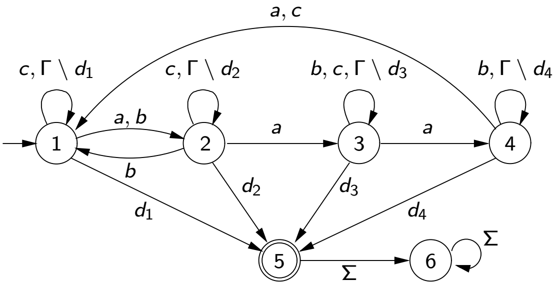
Prefix-Free Regular Languages

Theorem

If L is regular and prefix-free with $\kappa(L) = n \geq 2$, then $\sigma(L) \leq n^{n-2}$. Moreover, this bound is tight for $n = 2$ if $|\Sigma| \geq 1$, for $n = 3$ if $|\Sigma| \geq 2$, for $n = 4$ if $|\Sigma| \geq 4$, and for $n \geq 5$ if $|\Sigma| \geq n + 1$.

Prefix-Free Witness with 1,296 Transformations

$$\Sigma = \{a, b, c\} \cup \Gamma \qquad \Gamma = \{d_1, \ldots, d_{n-2}\}$$



Suffix-Free Regular Languages

Notation change: $Q = \{1, 2, \dots, n\}$.

$G_n = \{t \in \mathcal{T}_Q \mid 1 \notin \text{img } t, nt = n, \text{and } 1t = n \text{ or } 1t \neq it \text{ for } i \neq 1\}$.

Let $g(n) = |G_n|$. G_n is not a semigroup for $n \geq 3$:
$s = [2, 3, 3, \dots, 3, n] \in G_n$ but $s^2 = [3, 3, 3, \dots, 3, n] \notin G_n$.

Proposition

If L is a regular language with $\kappa(L) = n$, then the following hold:

1. *If L is suffix-free, then T_L is a subset of G_n.*
2. *If L is suffix-free and $n \geq 2$, then*

$$\sigma(L) \leq g(n) = (n-1)^{n-2} + (n-2)^{n-1}$$

3. *If L has 1 final quotient, and $T_L \subseteq G_n$, then L is suffix-free.*

Suffix-Free Regular Languages

$$P_n = \{t \in G_n \mid \text{ for all } i,j \in Q, i \neq j, \text{ we have } it = jt = n \text{ or } it \neq jt\}.$$

Proposition

For $n \geq 3$, $P_n \subseteq G_n$ is a semigroup, and

$$p(n) = |P_n| = \sum_{k=1}^{n-1} C_k^{n-1}(n-1-k)!\,C_{n-1-k}^{n-2}.$$

Suffix-Free Regular Languages

Proposition

When $n \geq 3$, the semigroup P_n can be generated by the following set I_n of transformations of Q: $I_3 = \{a, b\}$, where $a = [3, 2, 3]$ and $b = [2, 3, 3]$; $I_4 = \{a, b, c\}$, where $a = [4, 3, 2, 4]$, $b = [2, 4, 3, 4]$, $c = [2, 3, 4, 4]$; for $n \geq 5$, $I_n = \{a_0, \dots, a_{n-1}\}$, where

$$\begin{array}{lcllllllll} a_0 & = & [\ n, & 3, & 2, & 4, & \dots, & n-1, & n\ ], \\ a_1 & = & [\ n, & 3, & 4, & \dots, & n-1, & 2, & n\ ], \\ a_i & = & [\ 2, & \dots, & i, & n, & i+1, & \dots, & n\ ], \end{array}$$

for $i = 2, \dots, n-1$. That is, $a_0 = \binom{1}{n}(2,3)$, $a_1 = \binom{1}{n}(2, 3, \dots, n-1)$, and $ja_i = j+1$ for $j = 1, \dots, i-1$, $ia_i = n$, and $ja_i = j$ for $j = i+1, \dots, n$.

Suffix-Free Regular Languages

Proposition

For $n \geq 5$, let $\mathcal{A}_n = \{Q, \Sigma, \delta, 1, F\}$ be the DFA with alphabet $\Sigma = \{a_0, a_1, \ldots, a_{n-1}\}$, where each a_i defines a transformation as above, and $F = \{2\}$. Then $L = L(\mathcal{A}_n)$ has quotient complexity $\kappa(L) = n$, and syntactic complexity $\sigma(L) = p(n)$. Moreover, L is suffix-free.

Suffix-Free Regular Languages

Proposition

For $n \geq 5$, let $\mathcal{A}_n = \{Q, \Sigma, \delta, 1, F\}$ be the DFA with alphabet $\Sigma = \{a_0, a_1, \ldots, a_{n-1}\}$, where each a_i defines a transformation as above, and $F = \{2\}$. Then $L = L(\mathcal{A}_n)$ has quotient complexity $\kappa(L) = n$, and syntactic complexity $\sigma(L) = p(n)$. Moreover, L is suffix-free.

Conjecture 3 (Suffix-Free Regular Languages). *If L is a suffix-free regular language with $\kappa(L) = n \geq 2$, then $\sigma(L) \leq p(n)$ and this is a tight bound.*

Proved for $n \geq 4$

Bifix-Free Regular Languages

$$H_n = \{t \in G_n \mid (n-1)t = n\} \quad h(n) = |H_n|$$

Proposition

If L is a regular language with quotient complexity n and syntactic semigroup T_L, then the following hold:

1. *If L is bifix-free, then T_L is a subset of H_n.*
2. *If L is bifix-free and $n \geq 3$, then*

$$\sigma(L) \leq h(n) = (n-1)^{n-3} + (n-2)^{n-2}.$$

3. *If L has 1 accepting quotient, $T_L \subseteq H_n$, then L is bifix-free.*

Bifix-Free Regular Languages

$$R_n = \{t \in H_n \mid it = jt = n \text{ or } it \neq jt \text{ for all } 1 \le i,j \le n\}.$$

Proposition

For $n \geq 3$, $R_n \subseteq H_n$ is a semigroup, and its cardinality is

$$r(n) = |R_n| = \sum_{k=0}^{n-2} \left(C_k^{n-2}\right)^2 (n-2-k)!$$

Bifix-Free Regular Languages

$$R_n = \{t \in H_n \mid it = jt = n \text{ or } it \neq jt \text{ for all } 1 \leq i, j \leq n\}.$$

Proposition

For $n \geq 3$, $R_n \subseteq H_n$ is a semigroup, and its cardinality is

$$r(n) = |R_n| = \sum_{k=0}^{n-2} \left(C_k^{n-2}\right)^2 (n-2-k)!$$

Conjecture 4 (Bifix-Free Regular Languages). *If L is a bifix-free regular language with $\kappa(L) = n \geq 2$, then $\sigma(L) \leq r(n)$ and this is a tight bound.*

Proved for $n \leq 5$

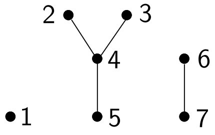
Summary for Prefix-, Suffix-, and Bifix-Free Languages

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$\|\Sigma\| = 1$	**1**	**2**	**3**	**4**	**5**
$\|\Sigma\| = 2$	$*$	**3/3**/$*$	**11/11/7**	**49/49/20**	?
$\|\Sigma\| = 3$	$*$	$*$	**14/13**/$*$	**95/61/31**	?
$\|\Sigma\| = 4$	$*$	$*$	**16**/ $*$ / $*$	**110/67/32**	?
$\|\Sigma\| = 5$	$*$	$*$	$*$	**119**/73/**33**	?
$\|\Sigma\| = 6$	$*$	$*$	$*$	**125**/ ? /**34**	? /501/ ?
...					
$n^{n-2}/p(n)/r(n)$	**1/1/1**	**3/3/2**	**16/13/7**	**125**/73/**34**	**1296**/501/209
Suf-free : $g(n)$	1	3	17	145	1,649
Bif-free : $h(n)$	1	2	7	43	381

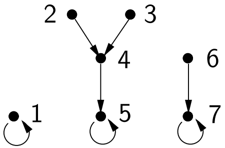
Star-Free Languages

- $\emptyset$, $\{\varepsilon\}$, $\{a\}$, $a \in \Sigma$ are star-free
- If K and L are star-free, then so are
  - $\overline{L}$
  - $K \cup L$
  - KL

Aperiodic Transformations



(a)



(b)

Convert forest into a directed graph. This graph defines

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 4 & 5 & 5 & 7 & 7 \end{pmatrix}$$

Thus there is a one-to-one relation between aperiodic transformations of a set of n elements and forests with n nodes.

Aperiodic Transformations Bound

Proposition

The syntactic complexity $\sigma(L)$ of a star free language L satisfies $\sigma(L) \leq (n+1)^{n-1}$.

Monotonic Automata

A DFA $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$ is monotonic if there exists a total order $\le$ on Q such that, for each $a \in \Sigma$, we have $p < q$ implies $\delta(p, a) \le \delta(q, a)$.

Theorem

Every monotonic DFA is permutation-free. The number $f(n)$ of monotonic transformations of $Q = \{1, \dots, n\}$ is

$$f(n) = \sum_{k=1}^{n} C_{k-1}^{n-1} C_k^n = C_n^{2n-1}.$$

Partially Monotonic Automata

A partial transformation may be undefined for some arguments.
Partially monotonic - monotonic where defined.

Example

	a	b	c	d	e	f	g
1	1	1	2	2	—	1	1
2	3	3	2	—	1	1	2
3	3	—	3	3	2	1	3

Partially Monotonic Automata

Replace dashes by a new state 4, add singular transformation f:

Example

	a	b	c	d	e	f	g
1	1	1	2	2	4	1	1
2	3	3	2	4	1	1	2
3	3	4	3	3	2	1	3
4	4	4	4	4	4	1	4

Generates 41 transformations

Summary for Star-Free Languages

$\|\Sigma\|$ / n	1	2	3	4	5	6
1	**1**	**1**	**2**	**3**	**5**	**6**
2	$*$	**2**	**7**	?	?	?
3	$*$	**3**	**9**	?	?	?
4	$*$	$*$	**10**	?	?	?
5	$*$	$*$	$*$	?	?	?
6	$*$	$*$	$*$	?	?	?
7	$*$	$*$	$*$	47	?	?
8	$*$	$*$	$*$	?	?	?
9	$*$	$*$	$*$	?	196	?
...						
mon. C_n^{2n-1}	1	3	10	35	126	462
part.mon.		3	10	41	196	1007
aper. $(n+1)^{n-1}$	1	2	16	125	1,296	16,807

Conclusions

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- Despite the fact that syntactic congruence has left-right symmetry, there are significant differences between left and right ideals, and between prefix and suffix-free languages.
- The major open problems are the upper bound for left ideals, suffix-free languages, and star-free languages.
- Although star-free languages meet (almost) all the quotient complexity bounds of regular languages, their syntactic complexity is much smaller.

References

- Brzozowski, Ye. Left, Right, 2-Sided Ideals: DLT 2011

References

- Brzozowski, Ye. Left, Right, 2-Sided Ideals: DLT 2011
- Brzozowski, Li, Ye. Prefix-, Suffix-, Bifix-Free: DCFS 2011

References

- Brzozowski, Ye. Left, Right, 2-Sided Ideals: DLT 2011
- Brzozowski, Li, Ye. Prefix-, Suffix-, Bifix-Free: DCFS 2011
- Brzozowski, Li. Star-Free: In preparation