Noether's Theorem

Past, present, and a possible future

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Introduction

- In the last three centuries, analytical mechanics has provided profound concepts and powerful tools for the study of physical systems.
- Of these, Noether's theorem establishes a key link between symmetries of the dynamics and conserved quantities.
- But at least since the last century, the study of abstract dynamics has taken an ever more important role.
- Can we adapt the results from the former to work for the latter?
- What about Noether's theorem in particular?

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The principle of conservation of energy

In an isolated physical system,
no matter what transformations take place within it,
there is a quantity called energy
which does not change with time.

- This is a principle so strong, that if a change in energy is recorded, we look for the loss!
- And physicists have in several occasions stretched the definition of energy, to match it with new evidence.
- But how far can we stretch it without shredding it?
- Is energy meaningful for a discrete system?
- And for a Turing machine?

Emmy Noether (1882–1935)

- Daughter of mathematician Max Noether.
- Student of Felix Klein, David Hilbert, and Hermann Minkowski.
- PhD 1907 at Erlangen supervised by Paul Gordan.
- Professor at Göttingen University (1915–1933) and Bryn Mawr College.
- Fundamental contributions in abstract algebra.

Noether's theorem: The popular form

If a physical system is invariant
with respect to a group of transformations,
then there is a quantity conserved along the motion.

Noether's theorem: The fine print

What is "physical?"

- Noether's theorem is a statement in analytical mechanics.
- In analytical mechanics, the trajectories of a system described by some variables q are those such that the value of the action functional

$$\int L(t, q, \dot{q})\, dt$$

  for a suitable Lagrangian function L, is an extremal.
- Extremality leads to the well-known Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Noether's theorem holds for systems that admit a Lagrangian.

Noether's theorem: More fine print

Which groups are "good"?

- Consider a class of transformations $h = \{h^s\}_{s \in \mathbb{R}}$ of a set S that satisfy

$$h^{s+t} = h^s \circ h^t \quad \forall t \in \mathbb{R}$$

- Then h^0 is the identity and $(h^s)^{-1} = h^{-s}$.
- We then say that h is a one-parameter group of transformations.
- The trajectories of Lagrangian systems have some level of continuity.
- To be sure to preserve this, transformations should be smooth.

Noether's theorem: The rigorous form

If a Lagrangian system is invariant
with respect to a one-parameter group of smooth transformations,
then there is a quantity conserved along the motion.

Examples

If a system is invariant for time translations

- then the conserved quantity is energy.

If a system is invariant for space translations in a direction

- then the conserved quantity is momentum in the given direction.

If a system is invariant for space rotations around an axis

- then the conserved quantity is angular momentum relative to that axis.

Noether's theorem: The original setting

Noether's original article deals with variational problems where

$$J = \int_{t_0}^{t_1} L(t, q, \dot{q})\,dt$$

is perturbated as

$$L \mapsto L_\varepsilon = L + \varepsilon\eta(t) \ , \ \eta \in C^2 \ , \ \eta(t_0) = \eta(t_1) = 0$$

It is then well known that

$$J \mapsto J_\varepsilon = \int_{t_0}^{t_1} \sum_i \psi_i(t, q, \dot{q}) \frac{\partial q_i}{\partial t}\,dt$$

where the ψ_i are the Lagrangian expressions.

Noether's theorems: The original form

Theorem 1

1. If J is invariant with respect to a transformation group depending on ρ real parameters, then ρ linearly independent combinations of the Lagrange expressions become divergences.
2. The converse also holds.
3. The theorem remains true for the limit case $\rho \to \infty$.

Theorem 2

1. If J is invariant with respect to a transformation group depending on ρ functions and their derivatives up to order σ, then ρ identity relations between the Lagrange expressions and their derivatives up to order σ hold.
2. The converse also holds.

Notes on Noether's original theorems

They link physics with group theory.

- Noether was a master of algebra and group theory.
- Her theorems are grounded in Lie theory.
- The paper was in honour of Felix Klein, who in his Erlangen program had suggested reducing geometry to algebra.

They are **much** more general than the standard form.

- That comes as a corollary, for $\rho = 1$, when considering the corresponding variational problem

$$\left[\frac{dJ_\varepsilon}{d\varepsilon}\right]_{\varepsilon=0} = 0$$

. . . and what about discrete systems?

- Is the continuity requirement necessary?
- And if it is, in which sense?
- What kinds of transformation groups will be allowed?
- Can one define an **energy** for a discrete system?

Cellular automata

A cellular automaton (CA) on a regular lattice $\mathcal{L}$ is a triple $\langle S, \mathcal{N}, f \rangle$ where

1. S is a finite set of states
2. $\mathcal{N} = \{\nu_1, \ldots, \nu_N\}$ is a finite neighborhood index on $\mathcal{L}$
3. $f : S^N \to S$ is the local function

The local function induces a global function on $S^{\mathcal{L}}$

$$G(c)(z) = f(c(z + \nu_1), \ldots, c(z + \nu_N))$$

The next value of a configuration c at site z depends on the current value of $z + \mathcal{N}$ by

$$c_z^{t+1} = f\left(c_{z+\nu_1}^t, \ldots, c_{z+\nu_N}^t\right)$$

Variations on a theme

Reversible cellular automata (RCA)

- Global function is bijective.
- It is then ensured that converse is a CA rule.
- Reversibility decidable in dimension 1, **undecidable** in greater.

Second order cellular automata

- The global law has the form

$$c_z^{t+1} = f\left(c_{z+\nu_1}^t, \ldots, c_{z+\nu_N}^t ; \mathbf{c_z^{t-1}}\right)$$

The Ising spin glass model on the plane

Description:

- Universe: square grid.
- Entities: magnetic dipoles.
- Grid links: ferromagnetic bonds between dipoles.
- A link is excited if orientation of dipoles is opposite.
- A link is relaxed if orientation of dipoles is same.

Update alternatively on even- and odd-indexed cells:

- If as many excited as relaxed: flip node.
- Otherwise: do nothing.

A conservation law for a class of CA

Consider a class of cellular automata of the form

$$\begin{cases} \hat{\sigma}_i^{t+1} & = & \sigma_i^t \\ \sigma_i^{t+1} & = & \hat{\sigma}_i^t + A_i^t - 2\hat{\sigma}_i^t A_i^t \end{cases}$$

where:

- i varies in a lattice $\mathcal{I}$.
- For every $i \in \mathcal{I}$ exists $\mathcal{N}_i \subseteq \mathcal{I}$ so that $\forall i, j$, $i \in \mathcal{N}_j$ iff $j \in \mathcal{N}_i$.
- σ_i^t and $\hat{\sigma}_i^t$ are Boolean, and A_i^t is a function of the σ_j^t for $j \in \mathcal{N}_i$.

Pomeau, 1984: if

$$A_i^t = \begin{cases} 1 & \text{if } \sum_{j \in \mathcal{N}_i} \sigma_j^t = q_i \,, \\ 0 & \text{otherwise} \end{cases}$$

then

$$\Phi^t = \sum_{i \in \mathcal{I}, j \in \mathcal{N}_i} \sigma_i^t \hat{\sigma}_j^t - \sum_{i \in \mathcal{I}} (\sigma_i^t + \hat{\sigma}_i^t) q_i$$

satisfies $\Phi^t = \Phi^{t+1}$.

A general form for 1D CA conservation laws

Consider a class of 1D **reversible** CA with the following properties:

1. $\mathcal{N} = \{0, 1\}$.
2. $f(x, x) = x$ for every state x.

Note that:

- Every 1D RCA can be written in this form.
- For such CA, if $f(a, b) = f(c, d) = x$, then $f(a, d) = f(c, b) = x$.

Boykett, Kari and Taati, 2008: **every** conservation law for such RCA is a sum of **independent noninteracting flows**.
(The proof actually holds for a broader class.)

Pros and cons of previous work

Other authors look for conserved quantities

- but don't care **the reasons why** quantities are conserved!

What it takes to be the energy

The total energy of a system may be defined as

1. A real-valued function of the system's state,
2. which is additive,
3. and is a generator of the dynamics.

Additivity: Prerequisites

For a function of the state to be additive:

1. It must be meaningful to subdivide the system into subsystems so that:
   - Each system has its own state.
   - The state of the whole system is a composition (*e.g.*, Cartesian product) of the states of the subsystem.
2. The function must be well-defined on each substate.
3. The value of the function on the whole system is a composition (*e.g.*, sum) sum of its values on the subsystems.

Additivity: The fine print

If there are no interactions between subsystems...

- ...then definition poses no problem but is vacuous.

If there are interactions...

- ...then one will have some uncertainty about the actual value of the energy.
- However, such interaction usually grows like the boundary of the subsystems, and vanish in the limit of arbitrarily large blocks.

Generator of the dynamics: what is it?

By the expression "generator of the dynamics" we mean

- a function of the system's state
- whose knowledge allows reconstructing the system's dynamics
- in an explicit form
- up to an isomorphism.

Generator of the dynamics: the case of the Hamiltonian

In classical physics, the dynamics of a system may be described by a function $H = H(q, p)$ of the state variables and momenta.

- A state is a pair (q, p).
- The dynamics is described by Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} \; ; \; \dot{p} = -\frac{\partial H}{\partial q}$$

- Evaluating H on a single pair only yields a real number.
- But repeated samplings in the proximity of (q, p) provide a sense of direction of the state.
- But this is precisely what Hamilton's equations do!

So H is a generator of the dynamics in the sense stated before.

Exercise

Consider a two-dimensional CA with neighborhood index $\mathcal{N} = \{n_1, \ldots, n_r\}$ and local function f.
Define

$$E(c_0, c_1) = \sum_{x \in \mathbb{Z}^2} \eta_{c_0,c_1}(x)$$

where

$$\eta_{c_0,c_1} = \begin{cases} 0 & \text{if } c_1(x) = f(c_0(x+n_1), \ldots, c_0(x+n_r)) \,, \\ 1 & \text{otherwise} \,. \end{cases}$$

Question: is E a suitable candidate for energy?

A candidate to Ising energy

Let c be an Ising configuration which is finite in the following sense:

all the points far enough from the center have the same value.

As an energy for configuration c we propose

the total number of excited bonds.

This is surely an invariant.
But does it respect our definition for an energy?

The principle of virtual displacements

Problem:

given a system with an energy function,
how do we find the successor of a given state?

Heuristics:

1. Guess the next state.
2. Compare energies of guessed and current state.
3. If energy is the same: Add to a list.
4. Else: Use difference of energies to estimate how far the true next state is far from the guessed one.

A procedure for the next state

1. We take all the 32 configurations of the form

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & n & 0 & 0 & \cdots \\
\cdots & 0 & w & c & e & 0 & \cdots \\
\cdots & 0 & 0 & s & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

   where c is a cell in the past and n, s, w, e its neighbors in the present.
2. For each cell, we propose as a next state either the same state or the other.
3. We check the local values of the "energy" for each case.
4. If it is the same, we flip.

Table of values

c $nswe$	H f	c'
0 0000	0 0	0
0 0001	1 0	0
0 0010	1 0	0
0 0011	2 1	1
0 0100	1 0	0
0 0101	2 1	1
0 0110	2 1	1
0 0111	3 0	0

c $nswe$	H f	c'
0 1000	1 0	0
0 1001	2 1	1
0 1010	2 1	1
0 1011	3 0	0
0 1100	2 1	1
0 1101	3 0	0
0 1110	3 0	0
0 1111	4 0	0

c $nswe$	H f	c'
1 0000	4 0	1
1 0001	3 0	1
1 0010	3 0	1
1 0011	2 1	0
1 0100	3 0	1
1 0101	2 1	0
1 0110	2 1	0
1 0111	1 0	1

c $nswe$	H f	c'
1 1000	3 0	1
1 1001	2 1	0
1 1010	2 1	0
1 1011	1 0	1
1 1100	2 1	0
1 1101	1 0	1
1 1110	1 0	1
1 1111	0 0	1

An energy for the Ising model!

The quantity we have defined has all the right to be called energy.
It is a real-valued function of the state.
It is additive.

- The value c' given the patch $\begin{smallmatrix} & n & \\ w & c & e \\ & s & \end{smallmatrix}$ does not depend on any of the other sites.
- In fact, the energy spanning $\begin{smallmatrix} & & u & \\ & n & a & r \\ w & c & e & \\ & s & & \end{smallmatrix}$ is precisely the sum of the two energies spanning $\begin{smallmatrix} & n & \\ w & c & e \\ & s & \end{smallmatrix}$ and $\begin{smallmatrix} & u & \\ n & a & r \\ & e & \end{smallmatrix}$ respectively.

And it is a generator of the dynamics.

- The next state can be found via the principle of virtual displacements.

. . . or is it?

Our energy function is a generator of the dynamics.

- But it generates the dynamics of **one** system!
- How can we know that it is not an artifact?
- What about other systems with other energy functions?

A class of Ising models

Let us introduce a variant in the form of antiferromagnetic bonds:

- An antiferromagnetic bond between antiparallel spins is relaxed.
- An antiferromagnetic bond between parallel spins is excited.

This yields 16 local dynamics—and many more global ones.

Our goal is to prove the following:

1. No two non-isomorphic dynamics have the same energy function.
2. Every dynamics is specified by at least one energy function.

Sanity check

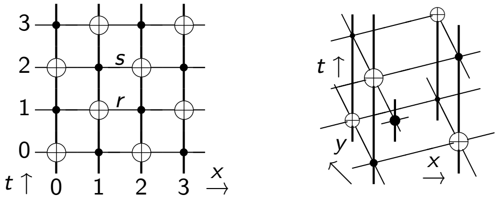
www.ioc.ee/ silvio/nrg/2x2b

- On a 2×2 periodic structure, compute the list of successors of states according to given bonds configurations.
- Regroup bond structures according to energy functions.
- Check that structures in same group have same look-up table.

www.ioc.ee/ silvio/nrg/bonds_evo.py

- Dynamics and energy functions for all 16 local cases on a 4×4 torus are explicitly tabulated.
- Dynamics are regrouped according to energy functions.
- If two dynamics in the same group have different sequences of next states, then the conjecture is disproved.

Space-time diagram visualization



- Spins are represented by thick vertical lines.
- Bonds are represented by thin horizontal lines
- Gates are represented by $\oplus$.
- State is defined between integer steps.
- Bonds nature may change at half-integer steps.

What about conservation of energy?

If the bonds never change

- then energy is conserved "for free".

If some bonds change at some moment

- then there will be some configurations for which the energy will not be preserved.

We resume the above as follows:

For the generalized Ising system,
the number of excited bonds,
is the quantity that is conserved **because**
the dynamics is time-invariant.

Conclusions and future work

Conclusions

- Noether's theorem is a wonderful result of classical mechanics.
- We have shown that certain aspects of it also apply to certain discrete dynamics.
- In both cases, second order appears to have a role.

Future work

- What is the role of second order in the emergence of symmetries?
- Can we define **momentum** for an Ising spin system?

Thank you for attention!

Any questions?