A Set that is Streamless and Not Provably Noetherian

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Overview

- Topic: constructive ‘finiteness’ of sets $A \subseteq \mathbb{N}$
- Prerequisite: elementary intuitionistic reasoning
- Prerequisite: elementary recursion theory
- Definition of streamless set
- Definition of noetherian set
- Comparing ‘streamless’ to ‘noetherian’
- Conjecture by Coquand and Spiwack

Finiteness

- Ubiquitous:
  - Reasoning about termination
  - Reasoning using fairness (‘eventually’)
  - Infinite combinatorics (PHP, Ramsey, Higman, ...)
  - Recently: initial algebra of a certain functor having the Cantor space as final co-algebra (Escardo, Bauer).
- Classically: surprisingly unproblematic (not FO-def.)
- Constructively: the obvious ‘comprehensive list of elements’ often inadequate (f.e. fairness, ‘Aussonderung’)

Finiteness, variants

- Knowing all the finitely many elements of $A \subseteq \mathbb{N}$
- Knowing the exact number of elements of (undecidable) A
- Knowing an upper bound on the number of elements of A
- Not knowing an upper bound, yet knowing that A is finite (!)
- Less attractive: doubly negated variants

Streamless and Noetherian

- For $A \subseteq \mathbb{N}$, for our purposes, streams $s : \mathit{str}\,A := \mathbb{N} \to A$
- For $A \subseteq \mathbb{N}$, lists $\ell : \mathit{list}\,A$ as usual $(\langle\rangle, ::)$
- For both lists and streams, *dup* for having duplicates
- Streamless $A := \forall s : \mathit{str}\,A.\ \mathit{dup}\,s$
- Noetherian $A := \mathit{Acc}_A\langle\rangle$, where:

$$\frac{\mathit{dup}\,\ell}{\mathit{Acc}_A\,\ell} \qquad \frac{\forall a : A\ \mathit{Acc}_A\,a{::}\ell}{\mathit{Acc}_A\,\ell}$$

Noetherian vs. Streamless

Let A be noetherian, that is, $Acc_A\langle\rangle$. Prove by induction that $Acc_A\,\ell$ implies $dup\,s$ for all $s : str\,A$ extending $reversed\,\ell$:

$$\frac{dup\,\ell}{Acc_A\,\ell} \qquad \frac{\forall a : A\ Acc_A\ a{::}\ell}{Acc_A\,\ell}$$

Let A be streamless: $\forall s : str A.\ dup\,s$. How to prove $Acc_A\langle\rangle$?

- By classical logic (and dependent choice)
- By bar induction (*dup* is the bar)

NB Bar induction fails in recursive analysis (by the Kleene tree)

Elementary Recursion Theory

- Kleene-brackets (universal machine): $\{\cdot\}\cdot$
- Church's Thesis: every stream over $\mathbb{N}$ has a Kleene-index

$$CT := \forall s : str\,\mathbb{N}.\ \exists i : \mathbb{N}.\ \forall n : \mathbb{N}.\ s(n) = \{i\}n$$

- Halting set $H := \{n : \mathbb{N} \mid \{n\}n\downarrow\}$
- Bitstring *b approximates H* means:

$$k \in H \iff b_k = 1, \quad \text{for all } k < lth(b)$$

- Bitstrings are encoded as natural numbers

Streamless But Not Provably Noetherian

- Define:

$$A := \{b \in \mathbb{N} \mid CT \wedge b \text{ approximates } H\}$$

- Classically: A empty
- Constructively: empty bitstring $\in A \iff CT$
- NB1: if s stream over A, then CT
- NB2: if $a, b \in A$ and $lth(a) \leq lth(b)$, then $a \preceq b$

Streamless A

Define partial recursive $\varphi(x, y)$ as follows:
Compute $\{x\}0, \ldots, \{x\}(y+1)$ and decode these as bitstrings.
Let $b = \{x\}n$ be the first of these having maximal length.

$$\varphi(x, y) \simeq \begin{cases} \uparrow & \text{if } b_y = 1 \\ 0 & \text{if } b_y = 0 \end{cases}$$

provided $lth(b) > y$, otherwise put $\varphi(x, y) = 0$ (irrelevant).
By the S-n-m Theorem there exists a total recursive f such that $\{f(x)\}y \simeq \varphi(x, y)$.
If s is a stream over A, then s has Kleene-index i and there is a duplicate among $s(0), \ldots, s(f(i)+1)$. Details on blackboard.

Not Provable: Noetherian A

We prove $Acc_A\langle\rangle \implies \neg CT$. Assume $Acc_A\langle\rangle \wedge CT$ and let S be the set of all lists of bitstrings containing some bitstring twice or more. Then, S is closed under the rules defining $Acc_A \subseteq list\ A$:

$$\frac{dup\,\ell}{\ell \in S} \qquad \frac{\forall a : A\ a{::}\ell \in S}{\ell \in S}$$

For the left rule this is obvious. For the right rule, assume $\forall a : A\ a{::}\ell \in S$ for some $\ell : list\ A$. Let b be the longest bitstring in ℓ. Let bi be b extended by $i = 0, 1$. By construction we have that $bi{::}\ell \in S$ implies $\ell \in S$. By contraposition we get that $\ell \notin S$ implies $bi{::}\ell \notin S$, so $bi \notin A$, $i = 0, 1$, as $\forall a : A\ a{::}\ell \in S$. Having CT (only needed for $\ell = \langle\rangle$), this is absurd (details on blackboard). Hence $\neg\ell \notin S$ and so $\ell \in S$, as this is decidable. Now $Acc_A\langle\rangle$ implies $\langle\rangle \in S$, absurd, so $Acc_A\langle\rangle \implies \neg CT$.