The Tarski alternative and the Garden-of-Eden theorem

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Introduction

- The discovery of the Banach-Tarski paradox and the study of the axiomatic properties of the Lebesgue integral originated an area of research merging measure theory with group theory.
- In 1929 John von Neumann defined amenable groups and proved that abelian groups are amenable.
- The Tarski alternative specifies that amenable groups are precisely those that disallow the Banach-Tarski paradox.
- A surprising link with E.F. Moore's Garden-of-Eden theorem was established by the work of Ceccherini-Silberstein *et al.* (1999) and Bartholdi (2007).

The Banach-Tarski paradox (1924)

A closed ball U in the 3-dimensional Euclidean space can be decomposed into two disjoint subsets X, Y, both of which are piecewise congruent to U.

Recall that two subsets A, B of the Euclidean space are piecewise congruent if they can be decomposed as $A = \bigsqcup_{i=1}^{n} A_i$, $B = \bigsqcup_{i=1}^{n} B_i$, with A_i congruent to B_i for each i.

The reasons behind the paradox

At the root of the Banach-Tarski paradox lies the Hausdorff phenomenon: The sphere S^2 can be decomposed into four disjoint parts A, B, C, Q such that:

- A, B, and C are congruent to each other,
- A is congruent to $B \cup C$, and
- Q is countable.

In turn, the Hausdorff phenomenon is made possible by a series of facts:

- The axiom of choice.
- The group of rotations of the 3-dimensional space has a free subgroup on two generators.
  This does not happen with the rotations on the plane.
- The pieces of the decomposition are not Lebesgue measurable.

Notation

Let X be a set.

- $\mathcal{PF}(X)$ is the family of finite subsets of X.
- For $f, g : X \to \mathbb{R}$ we write $f \geq g$ if $f(x) \geq g(x)$ for all $x \in X$.
- $\ell^\infty(X)$ is the space of bounded real-valued functions defined on X, with the norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$.
  (We consider X as a discrete topological space.)

Let G be a group.

- $L_g : G \to G$ is the left multiplication: $L_g(g') = gg'$ for every $g' \in G$.
- For every set Q, G acts on the left on Q^G by

$$gf = f \circ L_{g^{-1}} \, ,$$

i.e., $(gh)f = g(hf)$ and $1_G f = f$ for every $g, h \in G$, $f \in Q^G$.

Amenable groups

von Neumann, 1929

A group is **amenable**
if it admits a finitely additive probability measure μ
such that $\mu(gA) = \mu(A)$ for every $g \in G$, $A \subseteq G$.

Remarks on the definition of amenable group

- As we consider discrete groups, the probability measure shall be defined on every subset of the group.
- For the same reason, we cannot ask more than finite additivity.
- Left-invariance can be replaced by right-invariance, and yield the same definition.
  In fact, bi-invariance can be obtained, *i.e.*, $\mu(gA) = \mu(Ag) = \mu(A)$.
- This is **not** true for monoids! Non-commutative monoids can be "left-amenable" without being "right-amenable".
- Finite groups are amenable, with $\mu(A) = |A|/|G|$.

Means

A **mean** on a set X is a linear map $m : \ell^\infty(X) \to \mathbb{R}$ such that:

1. $m(1) = 1$.
2. If $f \geq 0$ then $m(f) \geq 0$.

The set $\mathcal{M}(X)$ of means on X is a compact convex subset of $(\ell^\infty(X))^*$ for the weak-$*$ topology, which is the coarsest topology that makes the evaluations $\phi \mapsto \phi(x)$ continuous.

Every mean has operator norm 1, *i.e.*, $\sup_{\|f\|_\infty = 1} |m(f)| = 1$.

If $X = G$ is a group, then G acts on $\mathcal{M}(G)$ via

$$gm(f) = m(f \circ L_g) = m(g^{-1}f) \;\; \forall g \in G \; \forall m \in \mathcal{M}(G) \, .$$

m is left-invariant if $gm = m$ for every $g \in G$.

The mean-measure duality

Let X be a set.

- If m is a mean on X, then

$$\mu(A) = m(\chi_A)$$

is a finitely additive probability measure on X.

- If μ is a finitely additive probability measure on X, then

$$m(f) = \int_X f \, d\mu \quad = \mathbb{E}_\mu(f)$$

is a mean on X.

- The two operations above are each other's inverse.
- $gm = m$ if and only if $g\mu = \mu$, where $g\mu(A) = \mu(g^{-1}A)$.

Closure properties of the class of amenable groups

A subgroup of an amenable group is amenable.

- If $G = \bigsqcup_{j \in J} Hj$ define $\mu_H(A)$ as $\mu\left(\bigsqcup_{j \in J} Aj\right)$.

A quotient of an amenable group is amenable.

- Put $\mu_{G/K}(A) = \mu(\rho^{-1}(A))$ where $\rho : G \to G/K$ is the canonical homomorphism.

An extension of an amenable group by an amenable group is amenable.

- Let m_K, $m_{G/K}$ be left-invariant means on $K \lhd G$ and G/K.
- If $f \in \ell^\infty(K)$, then $\tilde{f}(Kg) = m_K(g^{-1}f\big|_K)$ belongs to $\ell^\infty(G/K)$.
- Then $m(f) = m_{G/K}(\tilde{f})$ is a left-invariant mean on G.

A direct product of finitely many amenable groups is amenable.

- This is not true for infinite products!

A group whose subgroups of finite index are all amenable, is amenable.

Abelian groups are amenable

Let G be a group.

- The space $\mathcal{M}(G)$ of means on G, with the weak-$*$ topology, is Hausdorff, convex and compact.
- The transformations $m \mapsto gm$ are affine, *i.e.*, for every $g \in G$, $m_1, m_2 \in \mathcal{M}(G)$, $t \in (0,1)$,

$$g(tm_1 + (1-t)m_2) = t(gm_1) + (1-t)(gm_2) \; .$$

Suppose G is abelian.

- Then the transformations $m \mapsto gm$ commute with each other.
- By the Markov-Kakutani fixed point theorem, there exists a mean m such that $gm = m$ for every $g \in G$.

Corollary: solvable groups are amenable.

The free group is not amenable

Let $G = \mathbb{F}_2$ be the free group on two generators a, b.
Let $w = w_1 \dots w_\ell$ be the writing of g as a reduced word. Define:

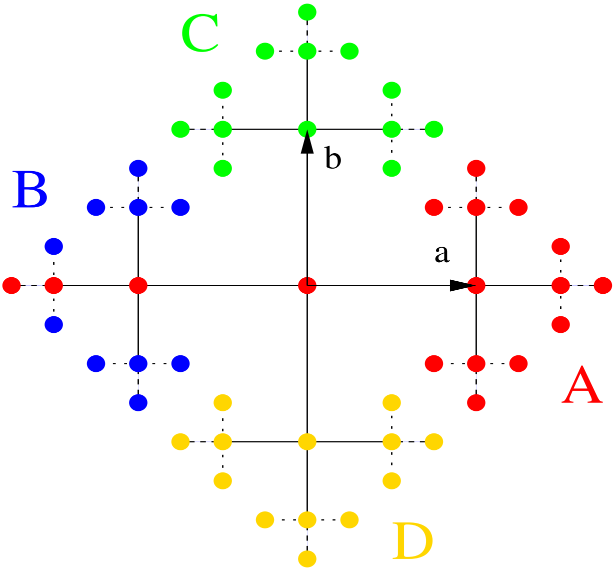
- $A = \{g \in G \mid w_1 = a\} \cup \{a^{-n} \mid n \in \mathbb{N}\}$.
- $B = \{g \in G \mid w_1 = a^{-1}\} \setminus \{a^{-n} \mid n \in \mathbb{N}\}$.
- $C = \{g \in G \mid w_1 = b\}$.
- $D = \{g \in G \mid w_1 = b^{-1}\}$.

Then

$$\begin{aligned} G &= A \sqcup B \sqcup C \sqcup D \\ &= A \sqcup aB \\ &= C \sqcup bD , \end{aligned}$$

and a left-invariant finitely additive probability measure on $\mathbb{F}_2$ cannot exist.

A paradoxical decomposition of $\mathbb{F}_2$



Paradoxical groups

Let G be a group.

- A paradoxical decomposition is a partition

$$G = \bigsqcup_{i=1}^{n} A_i$$

  together with $\alpha_1, \ldots, \alpha_n \in G$ such that, for some $k \in (1, n)$,

$$G = \bigsqcup_{i=1}^{k} \alpha_i A_i = \bigsqcup_{i=k+1}^{n} \alpha_i A_i \; .$$

- G is paradoxical if it admits a paradoxical decomposition.

Equivalently, one can give a partition $G = \bigsqcup_{i=1}^{k} A_i \alpha_i = \bigsqcup_{i=k+1}^{n} A_i \alpha_i$.

Examples of paradoxical groups

- The free group on two generators is paradoxical.
- Every group with a paradoxical subgroup is paradoxical.
  - If $H = \bigsqcup_{i=1}^{n} A_i$ and $G = \bigsqcup_{j \in J} Hj$ then $G = \bigsqcup_{i=1}^{n} A_i J$.
- In particular, every group with a free subgroup on two generators is paradoxical.
- The converse of the previous point is not true!
  (von Neumann's conjecture; disproved by Ol'shanskii, 1980)
- In fact, there exist paradoxical groups where every element has finite order. (Adian, 1983)

The Tarski alternative

Let G be a group. **Exactly one of the following happens.**

1. G is amenable.
2. G is paradoxical.

Why is this an alternative?

Characterizations of paradoxical groups

Let G be a group. The following are equivalent.

1. G has a paradoxical decomposition.
2. There exists $K \in \mathcal{PF}(G)$ such that $|KF| \geq 2|F|$ for every $F \in \mathcal{PF}(G)$.
   Equivalently: $H \in \mathcal{PF}(G)$ s.t. $|FH| \geq 2|F|$ for every $F \in \mathcal{PF}(G)$.
3. G has a bounded propagation 2:1 compressing map.
   That is: G has a map $\phi : G \to G$ such that, for a finite set S,
   1. $\phi(g)^{-1}g \in S$ for every $g \in G$, and
   2. $|\phi^{-1}(g)| = 2$ for every $g \in G$.

Proof

Point 1 implies point 3.

- Let $G = \bigsqcup_{i=1}^{n} A_i = \bigsqcup_{r=1}^{k} A_r \alpha_r = \bigsqcup_{s=k+1}^{n} A_s \alpha_s$.
- Put $S = \{\alpha_1^{-1}, \ldots, \alpha_n^{-1}\}$.
- If $g = a_r \alpha_r = a_s \alpha_s$ put $\phi(a_r) = \phi(a_s) = g$.

Point 3 implies point 1.

- For every $g \in G$ sort $\phi^{-1}(g) = \{g_1, g_2\}$.
- If $S = \{s_1, \ldots, s_k\}$ and $\phi(g)^{-1} g = s_i$, put g_1 in A_i and g_2 in A_{i+k}.
- Then $G = \bigsqcup_{i=1}^{2k} A_i$ is a paradoxical decomposition.

Proof (cont.)

Point 3 implies point 2.

- FS contains at least the two ϕ-preimages of each $x \in F$.

Point 2 implies point 3.

- Consider the bipartite graph $(G, G, \mathcal{E})$ with

$$\mathcal{E} = \{(g, h) \in G \times G \mid h \in Kg\}$$

- For every $F \in \mathcal{PF}(G)$, $x \in F$ there are at least $2|F|$ y's such that $(x, y) \in \mathcal{E}$.
- For every $F \in \mathcal{PF}(G)$, $y \in F$ there are at least $|F|/2$ x's such that $(x, y) \in \mathcal{E}$.
- Then ϕ exists by the Hall harem theorem.

The Følner conditions

Let G be a group. The following are equivalent.

1. For every $K \in \mathcal{PF}(G)$ and every $\varepsilon > 0$ there exists $F \in \mathcal{PF}(G)$ such that
$$\frac{|kF \setminus F|}{|F|} < \varepsilon \;\; \forall k \in K \, .$$
2. There exists a net $\mathcal{F} = \{F_i\}_{i \in I}$ of finite nonempty subsets of G such that
$$\lim_{i \in I} \frac{|gF_i \setminus F_i|}{|F_i|} = 0 \;\; \forall g \in G \, .$$
   Such $\mathcal{F}$ is called a left Følner net.

In fact, if point 1 holds:

- Set $I = \mathcal{PF}(G) \times \mathbb{N}$ with $(K_1, n_1) \leq (K_2, n_2)$ iff $K_1 \subseteq K_2$ and $n_1 \leq n_2$.
- For $i = (K, n)$ define F_i so that $|kF_i \setminus F_i| < |F_i|/n|K|$ for every $k \in K$.

Similar conditions hold with right, instead of left, multiplication.

Proof of the Tarski alternative

Either G satisfies the Følner conditions, or it does not.
If it does:

- For every $i \in I$ define $m_i(f) = \frac{1}{|F_i|} \sum_{x \in F_i} f(x)$.
- m_i is a mean and $\lim_{i \in I}(gm_i - m_i) = 0$ in $(\ell^\infty(G))^*$ for every $g \in G$.
- Every limit point m of $\{m_i\}_{i \in I}$ satisfies $gm = m$ for every $g \in G$.

If it does not:

- Choose $K_0 \in \mathcal{PF}(G)$, $\varepsilon_0 > 0$, and $k_0 \in K_0$ such that
$$|k_0 F \setminus F| > \varepsilon_0 |F| \;\; \forall F \in \mathcal{PF}(G) \, .$$
- Set $K_1 = K_0 \cup \{1_G\}$. Then $F \subseteq K_1 F$ and $K_1 F \setminus F = K_0 F \setminus F$.
- As $1_G \in K_1$, $|K_1 F \setminus F| = |K_1 F| - |F|$.
- But then, $|K_1 F| \geq |F| + |k_0 F \setminus F| \geq (1 + \varepsilon_0)|F|$ for every finite F.
- Put then $K = K_1^n$ with $(1 + \varepsilon_0)^n \geq 2$.

The Ornstein-Weiss lemma

Let G be an amenable group.
Let $\phi : \mathcal{PF}(G) \to \mathbb{R}$ be a subadditive, left-invariant map, *i.e.*:

1. For every $U, V \in \mathcal{PF}(G)$, $\phi(U \cup V) \leq \phi(U) + \phi(V)$.
2. For every $g \in G$, $U \in \mathcal{PF}(G)$, $\phi(gU) = \phi(U)$.

Then for every left Følner net $\mathcal{F} = \{F_i\}_{i \in I}$,

$$L = \lim_{i \in I} \frac{\phi(F_i)}{|F_i|}$$

exists and does not depend on the choice of $\mathcal{F}$.

Entropy

Let G be an amenable group. For $E \in \mathcal{PF}(G)$ let

$$\pi_E(c) = c|_E \ .$$

By the Ornstein-Weiss lemma, the entropy

$$h(X) = \lim_{i \in I} \frac{\log |\pi_{F_i}(X)|}{|F_i|} \ ,$$

of $X \subseteq A^G$, where $\{F_i\}_{i \in I}$ is a left Følner sequence on G, is well defined and does not depend on $\{F_i\}_{i \in I}$.

Growth rate

Let G be a finitely generated group, *i.e.*, every $g \in G$ can be seen as a word on the elements of some $S \in \mathcal{PF}(G)$ and their inverses.

- The length of $g \in G$ w.r.t. S is the minimum length of a word determining g.
- Let $D_{n,S}$ be the disk of radius n, *i.e.*, set of elements of G with length at most n w.r.t. S. Call $\gamma_S(n) = |D_{n,S}|$ the growth function.
- If S' is another finite set of generators for G, then

$$\frac{1}{C} \cdot \gamma_S\left(\frac{n}{C}\right) \leq \gamma_{S'}(n) \leq C \cdot \gamma_S(C \cdot n)$$

  for a suitable $C > 0$ and for every n large enough.
- The growth rate of G,

$$\lambda = \lim_{n \to \infty} \sqrt[n]{\gamma_S(n)} \, ,$$

  is thus well defined, and does not depend on S.

Growth rate and amenability

G is of subexponential growth if $\lambda = 1$.

- If G is of exponential growth, then $\{D_{n,S}\}_{n \geq 0}$ does not contain any Følner subsequence.
- If G is of subexponential growth, then $\{D_{n,S}\}_{n \geq 0}$ does contain a Følner subsequence.
- If G is of polynomial growth, then $\{D_{n,S}\}_{n \geq 0}$ is a Følner sequence.
- However, there do exist amenable groups of exponential growth.

Incidentally:
A group whose finitely generated subgroups are all amenable, is amenable.

Cellular automata

A cellular automaton (CA) on a group G is a triple $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ where

- Q is a finite set of states.
- $\mathcal{N} = \{n_1, \ldots, n_k\} \subseteq G$ is a finite neighborhood index.
- $f : Q^k \to Q$ is a finitary local function

The local function induces a global function $F : Q^G \to Q^G$ via

$$\begin{aligned} F(c)(x) &= f\left(c(x \cdot n_1), \ldots, c(x \cdot n_k)\right) \\ &= f\left(c \circ L_x|_{\mathcal{N}}\right) \end{aligned}$$

The same rule induces a function over patterns with finite support:

$$f(p) : E \to Q \ , \quad f(p)(x) = f\left(p \circ L_x|_{\mathcal{N}}\right) \ \ \forall p : E\mathcal{N} \to Q$$

In a Garden of Eden

Let $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ be a CA.

- A Garden of Eden (GoE) for $\mathcal{A}$ is a configuration c such that
$$F_{\mathcal{A}}^{-1}(c) = \emptyset.$$
- An orphan for $\mathcal{A}$ is a pattern p such that
$$f^{-1}(p) = \emptyset.$$

By compactness of Q^G, a CA has a GoE if and only if it has an orphan.

Not injectivity, but almost

- Two configurations are almost equal if they differ only on finitely many points.
- A cellular automaton is pre-injective if any two almost equal configurations with the same image are equal.
- Two distinct patterns $p, p' : E \to Q$ are mutually erasable for a CA with global rule F, if any two configurations c, c' with

$$c|_E = p \,, \quad c'\big|_E = p' \,, \quad \text{and} \quad c|_{G \setminus E} = c'\big|_{G \setminus E}$$

satisfy $F(c) = F(c')$.

A CA is pre-injective if and only if it does not have m.e. patterns.

The Garden-of-Eden theorem (Moore, 1962)

If a d-dimensional cellular automaton
has two mutuably erasable patterns,
then it also has an orphan pattern.

Notably, the converse was proved by Myhill the same year.
This means that:

cellular automata on an infinite space
behave, with regard to surjectivity,
more or less as they were finitary functions.

Not completely: XOR with right neighbor is surjective but not injective.

Balancedness

A cellular automaton $\mathcal{A}$ is balanced if for any given shape E, every pattern $p : E \to Q$ has the same number of preimages.

- For elementary 1D CA:
  every contiguous block has four preimages.
- For 2D CA with Moore neighborhood:
  every square pattern of side ℓ has $|Q|^{4\ell+4}$ preimages.
- A balanced CA has no orphans.

Theorem (Maruoka and Kimura, 1976)

A surjective CA on $\mathbb{Z}^d$ is balanced.

The Tarski alternative from the CA point of view

Let G be a group. The following are equivalent.

1. G is amenable.
2. Every surjective CA on G is pre-injective.
   (Ceccherini-Silberstein *et al.*, 1999; Bartholdi, 2007)
3. Every surjective CA on G is balanced.
   (Bartholdi, 2010)

Some notation and a lemma

Let G be a group, $E \in \mathcal{PF}(G)$.

- $B^{+E} = \{z \in G \mid zE \cap B \neq \emptyset\} = BE^{-1}$.
- $B^{-E} = \{z \in G \mid zE \subseteq B\}$.
- If $E = D_r$ we write B^{+r} and B^{-r} instead.

Lemma (Ceccherini-Silberstein, Machì and Scarabotti, 1999)
Let G be a finitely generated amenable group, $q \geq 2$, and $n > r > 0$.
For $L = D_n$ there exist $m > 0$ and $B \in \mathcal{PF}(G)$ such that:

1. There exist $g_1, \ldots, g_m \in G$ such that $g_i L \subseteq B$ for every i, and $g_i L \cap g_j L = \emptyset$ for every $i \neq j$.
2. $(q^{|L|} - 1)^m \cdot q^{|B| - m|L|} < q^{|B^{-r}|}$.

In the next slides, unless stated differently, we will suppose $\mathcal{N} = D_r$.

The Moore-Myhill theorem for amenable groups

Suppose G is amenable.
Then every surjective CA on G is pre-injective.

- Define a relation on Q^B by saying that $p_1 \sim p_2$ if they are equal or mutually erasable on each copy of L, and equal elsewhere.
- There are at most $(|Q|^{|L|} - 1)^m \cdot |Q|^{|B| - m|L|}$ classes, and each element of the same class has same image.
- By the lemma, at least one $p : B^{-r} \to Q$ must be orphan.

And every pre-injective CA on G is surjective.

- If no two patterns on B^{+r} are m.e., then there are at least as many non-GoE patterns on B than patterns on B^{-r}.
- Then either there are no GoE at all, or it is impossible to satisfy the lemma.

No Moore's theorem for the free group!

Let $\mathcal{A}$ be the majority CA on the free group.
Then $\mathcal{A}$ is clearly not pre-injective. However:

- For $g \neq 1$, $g = s_1 \cdots s_n$ let $\phi(g) = s_1 \cdots s_{n-1}$.
- Given $c : \mathbb{F}_2 \to Q$, set $e(1) = 0$ and $e(g) = c(\phi(g))$ otherwise.
- Then each $g \in G$ has at least three neighbors j with $e(j) = c(g)$.

No Myhill's theorem for the free group!

Let $Q = \{1, u, v, uv\}$ be the Klein group and let

$$f(q_1, q_a, q_b, q_{a^{-1}}, q_{b^{-1}}) = p_u(q_a) \cdot p_v(q_b) \cdot p_u(q_{a^{-1}}) \cdot p_v(q_{b^{-1}})$$

where $p_u(u) = p_u(uv) = p_v(v) = p_v(uv) = u$, $p_i(x) = 1$ otherwise.

- Suppose c and e have same image, but differ in finitely many points. Define $d : G \to Q$ by $d(g) = c(g) \cdot e(g)$. Then $F(d) = 1$.
- Let g be a point of maximal length where $c(g) \neq e(g)$. Then $d(g)$ is either u, v, or uv.
- If it is u or uv, choose $h \in \{ga, ga^{-1}\}$ so that it has length greater than g. Then $F(d)(h) = u$, impossible.
- If it is v, choose $h \in \{gb, gb^{-1}\}$ so that it has length greater than g. Then $F(d)(h) = u$, impossible.

No Moore's theorem for paradoxical groups!

Let G be a non-amenable group, ϕ a bounded propagation 2:1 compressing map with propagation set S.
Define on S a total ordering $\preceq$.
Define a CA $\mathcal{A}$ on G by $Q = (S \times \{0,1\} \times S)$, $\mathcal{N} = S$, and

$$f(u) = \begin{cases} (p, \alpha, q) & \text{if } \exists (s,t) \in S \times S \text{ minimal } \mid u_s = (s, \alpha, p), u_t = (t, \beta, q \\ q_0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{A}$ is surjective.

- For $j \in G$ it is $j = \phi(js) = \phi(jt)$ for exactly two $s, t \in S$ with $s \prec t$.
- If $c(j) = (p, \alpha, q)$ put $e(js) = (s, \alpha, p)$ and $e(jt) = (t, 0, q)$.
- Then $F_{\mathcal{A}}(e) = c$.

However, $\mathcal{A}$ is not pre-injective.

- In the construction above we can always replace $(t, 0, q)$ with $(t, 1, q)$.

Surjective CA on amenable groups are balanced

The following proof is due to Jarkko Kari.

- Let $L' = L^{-r}$. Suppose $p : L' \to Q$ satisfies $|f^{-1}(p)| \leq |Q|^{|L|-|L'|} - 1$.
- Then there are at most

$$\left(|Q|^{|L|-|L'|} - 1 \right)^m \cdot |Q|^{|B|-m|L|}$$

  patterns on B that are mapped to p on each copy of L'.
- But $\left(|Q|^{|L|-|L'|} - 1 \right) \leq |Q|^{-|L'|} \left(|Q|^{|L|} - 1 \right)$.
- So the number of said patterns is at most

$$|Q|^{-m|L'|} \cdot \left(|Q|^{|L|} - 1 \right)^m \cdot |Q|^{|B|-m|L|} < |Q|^{|B^{-r}|-m|L'|} .$$

- The right-hand side is the number of patterns on B^{-r} that coincide with p on each copy of L': some of which must be orphan.

A surjective, non-balanced CA (Guillon, 2011)

Let G be a non-amenable group, ϕ a bounded propagation 2:1 compressing map with propagation set S.
Define on S a total ordering $\preceq$.
Define a CA $\mathcal{A}$ on G by $Q = (S \times \{0,1\} \times S) \sqcup \{q_0\}$, $\mathcal{N} = S$, and

$$f(u) = \begin{cases} q_0 & \text{if } \exists s \in S \mid u_s = q_0, \\ (p, \alpha, q) & \text{if } \exists (s,t) \in S \times S \mid s \prec t, u_s = (s, \alpha, p), u_t = (t, 1, q), \\ q_0 & \text{otherwise.} \end{cases}$$

(Due to ϕ being 2:1, if a pair (s,t) as above exists, it is unique.)
Then $\mathcal{A}$, although clearly non-balanced, is surjective.

- For $j \in G$ it is $j = \phi(js) = \phi(jt)$ for exactly two $s, t \in S$ with $s \prec t$.
- If $c(j) = q_0$ put $e(js) = e(jt) = (s, 0, s)$.
- If $c(j) = (p, \alpha, q)$ put $e(js) = (s, \alpha, p)$ and $e(jt) = (t, 1, q)$.
- Then $F_{\mathcal{A}}(e) = c$.

Conclusions and open questions

- Amenable groups are the obstacle to the Banach-Tarski paradox.
- The Tarski alternative can be expressed in terms of finite sets.
- Moore's Garden-of-Eden theorem characterizes amenable groups.
- Is Myhill's theorem characteristic to amenable groups as well?

Thank you for attention!

Any questions?